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Trapping in complex networks

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Abstract – We investigate the trapping problem in Erdős-Rényi (ER) and scale-free (SF) networks. We calculate the evolution of the particle density $\rho(t)$ of random walkers in the presence of one or multiple traps with concentration c. We show using theory and simulations that in ER networks, while for short times $\rho(t) \propto \exp(-Act)$, for longer times $\rho(t)$ exhibits a more complex behavior, with explicit dependence on both the number of traps and the size of the network. In SF networks we reveal the significant impact of the trap's location: $\rho(t)$ is drastically different when a trap is placed on a random node compared to the case of the trap being on the node with the maximum connectivity. For the latter case we find $\rho(t) \propto \exp\left[-At/N^{\frac{\gamma-2}{\gamma-1}}\langle k\rangle\right]$ for all $\gamma > 2$, where γ is the exponent of the degree distribution $P(k) \propto k^{-\gamma}$.

Introduction. – The properties of the random walk greatly vary depending on the dimension and the structure of the medium in which it is confined [1–4], where a particularly interesting medium for the study of the random walk are complex networks [5–9]. Networks describe systems from various fields, such as communication (*e.g.* the Internet), the social sciences, transportation, biology, and others. Many of these networks are scale free (SF) [10–13]. This class of networks is defined by a broad degree distribution, such as a power law $P(k) \propto k^{-\gamma}$ ($k \geqslant m$), where γ is a parameter which controls the broadness of the distribution.

Trapping is a random-walk problem in which traps are placed in random locations, absorbing all walkers that visit them. This problem was shown to yield different results over different geometries, dimensions and time regimes [2,3,14–17]. The main property of interest during such a process is the survival probability $\rho(t)$, which denotes the probability that a particle survives after t steps. The problem was studied in regular lattices and in fractal spaces [2,14–19] and recently, in small-world networks [6].

In this letter we study the problem of trapping in networks. This is a model for the propagation of information in certain communication networks. This follows since in some cases data packets traverse the network in a random fashion (for example, in wireless sensor networks [20], *ad hoc* networks [21] and peer-to-peer networks [22]). A malfunctioning node in which information is lost (*e.g.*, a router which cannot transmit data due to some failure) acts just like a trap in the model. This model can also be applied to loss of information in messages over communication systems, *e.g.* in the case of e-mail messages, where a malfunctioning e-mail server acts as a node absorbing, but not transmitting, all e-mail messages it receives. Furthermore, our model may be relevant in social systems, where some information may initially spread randomly, but in later stages it might be held by certain individuals.

We study the survival probability $\rho(t)$ of random walkers on random regular networks (networks in which all nodes have equal degree), Erdős-Rényi (ER) networks (a simple model for random networks in which all links exist with the same probability [23–25]), and SF networks. We derive analytical expressions for $\rho(t)$ for a wide range of trap degrees and concentrations and highlight the role of the network structure, obtaining new scaling relations for the survival probability and average trapping time which are absent in lattices. Our analytical predictions are confirmed with Monte Carlo simulations.

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Methods. – To perform Monte Carlo simulations, we generate ER networks by considering all pairs of nodes and linking a pair with probability p. The construction of an SF network follows the Molloy-Reed scheme [26]. Each node i is assigned a number of links taken from the distribution $P(k) \propto k^{-\gamma}$ and then open links are connected randomly. The value of k is taken to be between m (typically 1–3) and $k_{max} = N-1$ (no upper cutoff value is imposed). We perform simulations only on the largest cluster of the network. Starting from a fixed density of particles initially placed in random nodes, particles hop with equal probability to one of their nearest neighbors. Certain nodes are randomly chosen to serve as traps. These are perfect traps; if a particle falls on it then it is trapped and removed from the network. In the case of multiple traps, $n = cN$ traps are placed in the network, where c is the trap concentration.

Results. – Assume the network has N nodes, average degree $\langle k \rangle$ and n traps. How does ρ change as t increases to $t+1$ (*i.e.*, after each particle has moved once)? Denote the traps by $(i_1, i_2, \ldots, i_n)$ and define $k_n \equiv k_{i_1} + k_{i_2} + \ldots + k_{i_n}$ as the total number of links emanating from all traps. If at time t a given particle is not on a trap, but will hop on any of these k_n links on its next step, it will be trapped at time $t+1$. We approximate the probability for the particle to hop on any of these k_n links to be proportional to their relative number in the network, that is, $A\frac{k_n}{N\langle k \rangle}$, where $N\langle k \rangle$ is the total number of links in the network, and $A = O(1)$ is the proportion constant which we will study later. In continuous time, this results in the equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho A \frac{k_n}{N\langle k \rangle} \tag{1}$$

whose solution is

$$\rho(t) = \rho_0 \exp[-Atk_n/(N\langle k \rangle)]. \tag{2}$$

Surprisingly, although eq. (2) is based on a rather simple approximation, we show below that it predicts very accurately the survival probability for various network models, time scales, and trap concentrations. In fact, eq. (2) can be seen as a special case of the theory developed earlier in [15,17,27,28], where it was shown, that for a d-dimensional lattice, the survival probability decays as a stretched exponential $\rho \sim e^{-\alpha t^{\beta}}$ with $\beta = \frac{d}{d+2}$. Since networks have infinite dimension, $d/(d+2) \to 1$ to recover the exponential decay we predict. Note that the average time before trapping is $\mathcal{O}(N)$, as expected from first-passage time considerations [29] (see also below).

A necessary condition for the above approximation to hold is that the number of links between the traps is negligible. For ER networks where links exist independently of one another the probability that all links emanating from the traps connect to non-traps nodes is $[(N-n)/(N-1)]^{k_n}$. Since $\langle k_n \rangle = n\langle k \rangle$, and $[1-n/(N-1)]^{n\langle k \rangle} \approx 1 - n^2\langle k \rangle/N$ (for $n \ll N$), we expect that as long as $n \ll \sqrt{N/\langle k \rangle}$, this condition is satisfied.

In the following, we will apply eq. (2) to specific network topologies. In random regular networks, where each node has exactly k neighbors, we use eq. (2) by substituting $\langle k \rangle = k$ and $k_n = nk$:

$$\rho = \exp[-Ant/N] = \exp[-Act] \text{ (regular, } n \text{ traps)}, \tag{3}$$

(without loss of generality, we set $\rho(0) = 1$). For other networks one has to take into account the distribution of degrees. Thus, in order to average $\rho(t)$ over all networks in the ensemble, we need to condition ρ on k_n:

$$\rho(t) = \sum_k P\{k_n = k\} \exp[-Atk/(N\langle k \rangle)] \quad \text{(non-regular, } n \text{ traps)}. \tag{4}$$

Consider ER networks with one trap ($n = 1$): $P\{k_n = k\} = e^{-\langle k \rangle}\langle k \rangle^k/k!$ is the degree distribution (a Poissonian) [23–25]. Thus,

$$\begin{aligned} \rho(t) &= \sum_{k=1}^{\infty} e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!} \exp\left[-\frac{Atk}{N\langle k \rangle}\right] \\ &= e^{-\langle k \rangle} \sum_{k=1}^{\infty} \frac{\left[\langle k \rangle \exp\left[-\frac{At}{N\langle k \rangle}\right]\right]^k}{k!} \\ &\approx \exp\left[-\langle k \rangle \left(1 - \exp\left[-\frac{At}{N\langle k \rangle}\right]\right)\right] \\ &\quad \text{(ER, one trap, random } k\text{)}, \end{aligned} \tag{5}$$

where we start the summation from $k = 1$, since we do not place a trap on an isolated node ($k = 0$). However, when evaluating the sum, we assume that the probability for $k = 0$ is negligible, which is justified whenever $\langle k \rangle$ is large enough, which we assume henceforth. Also, in the simulations, we consider only the largest connected cluster, which by definition contains no isolated nodes.

The same approach can be applied to the case of multiple traps, by realizing that (neglecting links between the traps) the sum of links emanating from the traps is a sum of Poisson variables with mean $\langle k \rangle$, which is itself a Poissonian with mean $n\langle k \rangle$:

$$\begin{aligned} \rho(t) &= \sum_{k=n}^{\infty} e^{-n\langle k \rangle} \frac{(n\langle k \rangle)^k}{k!} \exp\left[-\frac{Atk}{N\langle k \rangle}\right] \\ &\approx \exp\left[-n\langle k \rangle \left(1 - \exp\left[-\frac{At}{N\langle k \rangle}\right]\right)\right] \\ &\quad \text{(ER, } n \text{ traps, random } k\text{)}. \end{aligned} \tag{6}$$

The agreement of eq. (6) with simulation results is evident from fig. 1. Note that the survival probability in eq. (6) does not solely depend on the trap concentration $c \equiv n/N$, but on both n and N, except for the short time limit ($t \ll N\langle k \rangle$), when $1 - \exp\left[-\frac{At}{N\langle k \rangle}\right] \approx \frac{At}{N\langle k \rangle}$ and $\rho \approx \exp[-Ant/N] = \exp[-Act]$. For long times

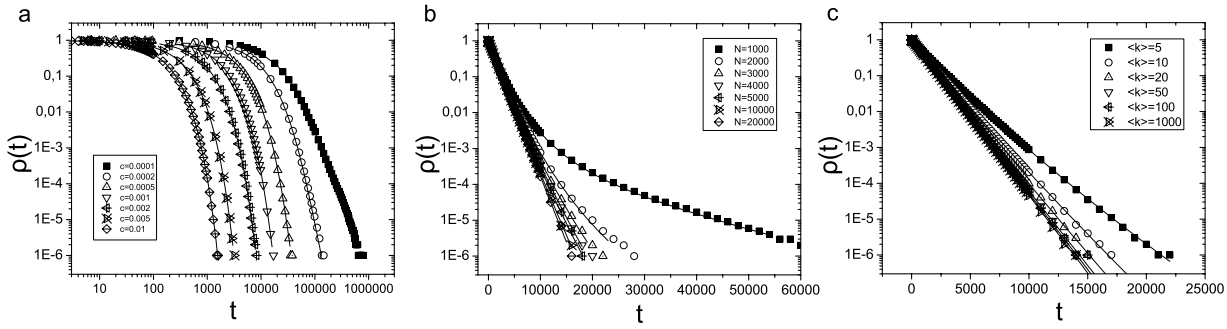

Fig. 1: Trapping in ER networks. (a) Particle density $\rho(t)$ *vs.* t (measured in Monte Carlo steps). The network parameters are: $N = 10000$ and $\langle k\rangle = 10$. Traps are placed with a concentration c on random nodes of the network. All results are averaged on at least 5000 runs, each with a different configuration of the network. Solid lines represent fitting with eq. (6) (with the number of traps $n = cN$). (b) $\rho(t)$ for fixed trap concentration $c = 0.001$, average degree $\langle k\rangle = 10$, and different system sizes. (c) $\rho(t)$ for fixed trap concentration $c = 0.001$, system size $N = 10000$, and different average degrees.

$(t \gg N\langle k\rangle)$, due to the exponential dependence on t, the main contribution to the survival probability comes from configurations in which k_n is small, the probability of which depends on n alone. On the other hand, the probability that the particle falls into the trap still depends on the total number of links $N\langle k\rangle$. Thus, the survival probability depends on both n and N independently. It can also be seen that particles survive longer as the network becomes smaller (fig. 1(b)) and sparser (fig. 1(c)).

Even though scale-free networks are highly heterogeneous and thus the approximate approach is expected to yield less accurate results, nevertheless it is still quite useful. The degree distribution is $P(k) = Ck^{-\gamma},\ k \geqslant m$, where C is a normalization factor. Thus, for a single trap,

$$\rho = \sum_k Ck^{-\gamma} \exp\left[-Atk/(N\langle k\rangle)\right] \ (\text{SF},\ \text{one trap}). \qquad (7)$$

Since this does not lead to a closed-form formula, we focus on the case where the degree of the trap k is fixed. We expect

$$\rho = \exp\left[-Akt/(N\langle k\rangle)\right] \ (\text{SF},\ \text{one trap},\ \text{fixed}\ k). \qquad (8)$$

Interestingly, simulations show a distinct behavior for $m < 3$, and $m \geqslant 3$ (fig. 2). While in the case of $m \geqslant 3$ the simulations agree with the theory (eq. (8)), as is evident by the collapse of all curves with the same kt; for $m < 3$ the decay of $\rho(t)$ is slower than exponential. Note that in contrast to ER networks, $\rho(t)$ is larger for the denser networks (smaller γ). Thus, whereas ER networks become less robust as links are added, SF networks gain robustness. This is a fundamental difference between ER and SF networks revealed by our results.

When the degree of the trap is allowed to vary, we consider the long-time regime. As in ER networks, the main contribution comes from configurations in which the degree of trap is minimal, *i.e.*, $k = m$. Thus we expect

$$\rho(t) \approx \exp\left[-Amt/(N\langle k\rangle)\right]$$
$$(\text{SF},\ \text{one trap},\ \text{random}\ k, m \geqslant 3, t \gg N\langle k\rangle) \qquad (9)$$

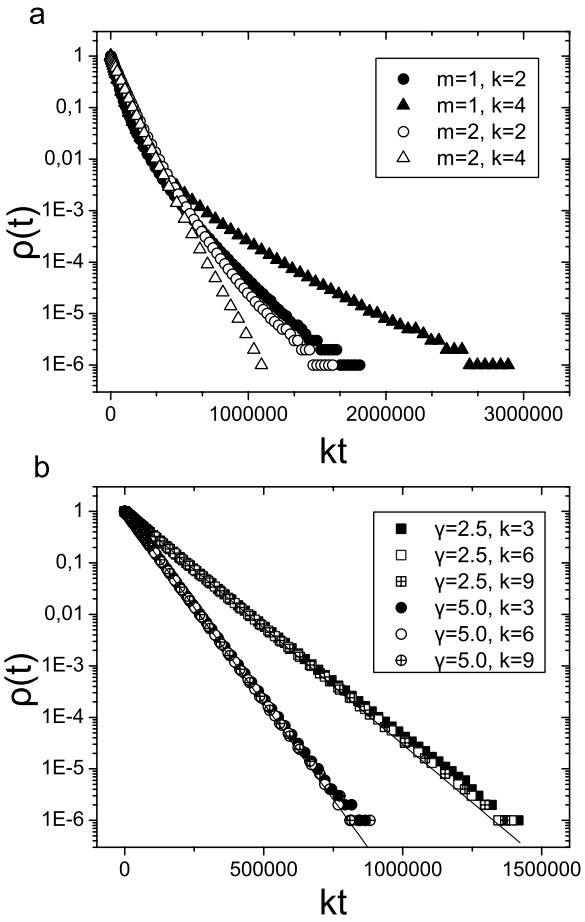

Fig. 2: Trapping in scale-free networks with a single trap on a node with fixed degree k. (a) Particle density $\rho(t)$ *vs.* kt, for SF networks with $N = 10000$, $\gamma = 2.5$, $m = 1, 2$, and different trap degrees. (b) Same as (a), but for $m = 3$ ($\gamma = 2.5, 5$). In this case, all curves collapse, in agreement with eq. (8).

which agrees with simulations (see fig. 3(a)). For SF networks with many traps, a simple generalization of eq. (9) (replacing m by nm) is not applicable, and we report only the numerical results (fig. 3(b)). Here,

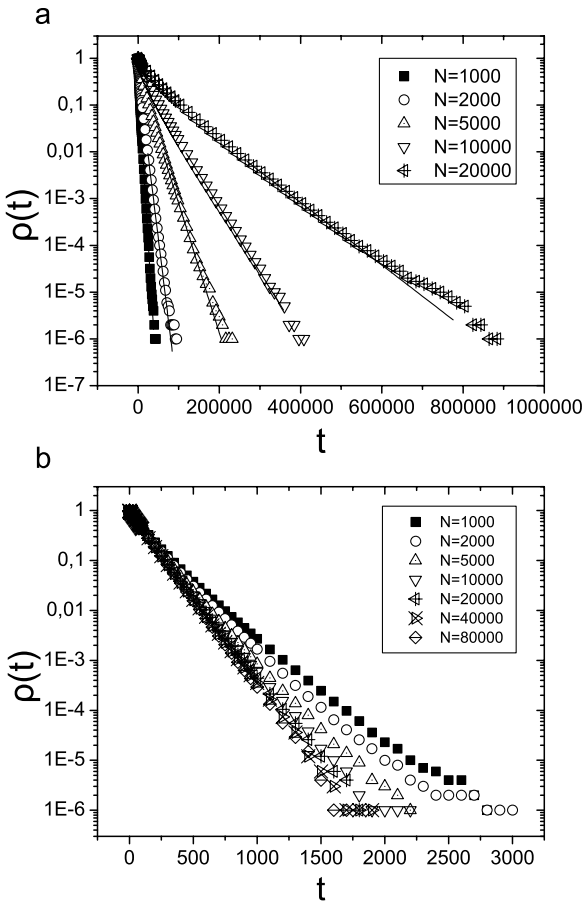

Fig. 3: Trapping in scale-free networks with $m = 3$. (a) Particle density $\rho(t)$ *vs.* t, for SF networks with $\gamma = 2.5$, a single trap on a random node, and different system sizes. Solid lines represent fitting to exponential decay in the long-time regime, eq. (9). (b) Particle density $\rho(t)$ *vs.* t, for SF networks with $\gamma = 2.5$, traps with concentration $c = 0.01$ placed on random nodes, and different system sizes.

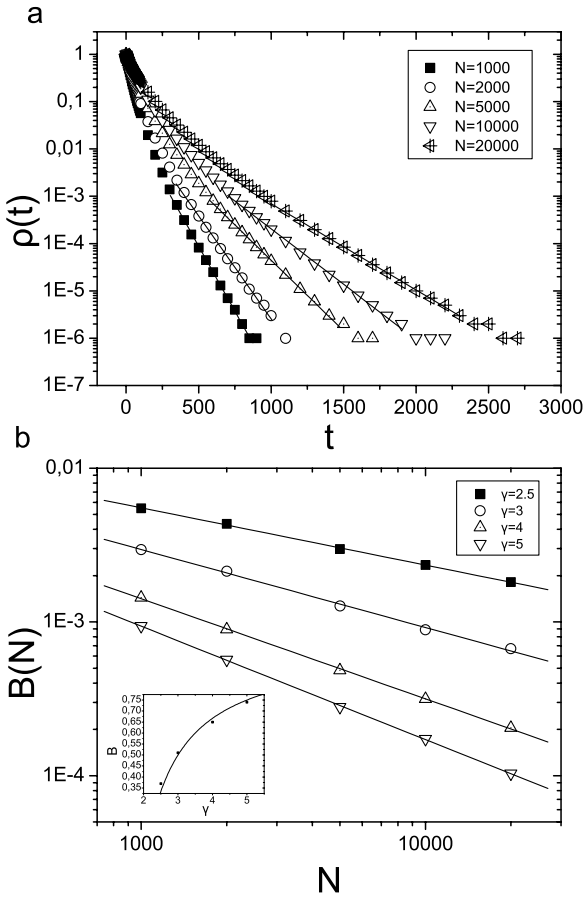

Fig. 4: Trapping in SF networks after failure of the most connected node. (a) Particle density $\rho(t)$ *vs.* t, in SF networks with $\gamma = 2.5$ and $m = 3$, for different system sizes. One trap is placed on the node with the maximum degree. Solid lines represent fitting to an exponential decay $\rho \sim e^{-Bt}$ in the long-time regime. (b) The exponent B *vs.* N, for $\gamma = 2.5$ plotted in (a), as well as for $\gamma = 3, 4, 5$. It can be seen that $B \sim N^{-\beta}$ with $\beta \approx \frac{\gamma-2}{\gamma-1}$ (inset), in agreement with eq. (10).

similarly to ER networks, the smaller networks are more robust.

SF networks exhibit nodes of particular importance which have many connections and play a special role in transport [30]. Thus, it is interesting to study a failure in the node of highest degree (the hub) [31], which results in trapping of incoming particles. The maximum degree K in SF networks scales like $K \approx mN^{\frac{1}{\gamma-1}}$ (for $\gamma > 2$) [32]. Substituting $k = K$ in eq. (8), we find

$$\rho(t) = \exp[-AtK/(N\langle k\rangle)] \approx \exp\left[-Amt/(N^{\frac{\gamma-2}{\gamma-1}}\langle k\rangle)\right]$$
$$(\text{SF, trap on the hub}, m \geqslant 3, t \gg N\langle k\rangle) \qquad (10)$$

and the average time before trapping is thus $t_{\rm tr} \sim N^{\beta}$, where $\beta = \frac{\gamma-2}{\gamma-1} < 1$ (see fig. 4). Realistic SF networks have $2 < \gamma < 3$ [10,33,34], so that $0 < \beta < 1/2$. This implies that real-world networks are ultra-prone to failure in their highly-connected nodes. This is an even stronger effect compared to the targeted removal of high-degree nodes [31]; whereas a failure of only one hub induces a significant decrease in the trapping time, a finite concentration of hubs has to be removed to fragment the network.

The dramatic decrease in the time before failure is not limited to placing the trap precisely on the node with the maximal degree. It can be proven that whenever we either i) choose the node with maximal degree out of n random nodes, when n/N is finite, or ii) choose one of the n-th nodes of highest degree when $n = O(1)$, the probability of the trap degree to exceed $K = mN^{\frac{1}{\gamma-1}}$ is finite. Thus, in these cases, the trap will be attached to a sufficient number of links for the scaling $t_{\rm tr} \sim N^{\frac{\gamma-2}{\gamma-1}}$ to appear.

The value of γ for which SF networks are equivalent to ER networks is a topic of recent interest [35]. Our results suggest that SF networks are equivalent to ER only when γ is infinite, since only when $\gamma \to \infty$, does $\beta \to 1$, as for homogenous ER networks. For ER networks, the degree distribution is a Poissonian with

variance equals to the mean $\langle k\rangle$. Consequently, the typical maximal degree is roughly $K \approx \langle k\rangle + \sqrt{\langle k\rangle}$. This yields $\rho \approx \exp\left[-\frac{At}{N}\left(1+\frac{1}{\sqrt{\langle k\rangle}}\right)\right]$, such that the typical time is $t_{\rm tr} \sim N$ as before.

In the following, we study the behavior of the prefactor A. For fully connected network and large N, $A \to 1$ (see, *e.g.*, [36]). For sparse networks where the particle might be far from the trap, A is less than unity, reflecting the fact that the probability to follow a link to the trap is somewhat less than $k_n/(N\langle k\rangle)$. To find the value of A, we first point out that the trapping problem is a special case of a first-passage time problem [3,4,8] (since $\rho(t) = 1 - \sum_{t'=0}^{t} F(t)$, where $F(t)$ is the probability to reach the trap for the first time at time t). To calculate the first-passage time in networks, Baronchelli and Loreto [37] used an approximate method that exploits the small-world nature of most networks [33,38]. In theory, using the adjacency matrix one can calculate the transition probability matrix of the random walker, from which the first-passage time can be easily obtained via consecutive powers of the matrix (see below). However, this is not feasible for large networks, and thus the original random-walk process was reduced to a random walk between the network layers [37]. Given the trap, the number of nodes n_ℓ that are in distance ℓ from it is calculated. Then, a matrix B of size $\ell_{\max} \times \ell_{\max}$ is constructed, in which $B_{\ell,\ell'}$ is the probability of a random walker in layer ℓ to jump into a node in layer ℓ'. For most real and model networks, $\ell_{\max} \sim \log N$ such that the size of the problem is reduced exponentially. Define the number of links that connect layers ℓ and $\ell+1$ by s_ℓ, the number of links within layer ℓ by o_ℓ, and the sum of degrees of nodes in layer ℓ by $m_\ell = (s_\ell + s_{\ell-1} + 2o_\ell)$. Since the random walker jumps into each link with equal probability, the only non-zero elements are: $B_{\ell,\ell+1} = s_\ell/m_\ell$, $B_{\ell,\ell-1} = s_{\ell-1}/m_\ell$, and $B_{\ell,\ell} = 2o_\ell/m_\ell$. To represent the trap, $B_{0,\ell} = 0$ for all ℓ (since the trap forms layer $\ell = 0$). Since the probability for a random particle to start in a node of layer ℓ is $n_\ell/(N-1)$, the first-passage time probability $F(t)$ is given by [37]

$$F(t) = \sum_{\ell=1}^{\ell_{\max}} \frac{n_\ell}{N-1} (B^t)_{\ell,0}. \qquad (11)$$

In [37], the matrix B was constructed for ER networks, and it was found that $F(t) \propto \exp[-Akt/N\langle k\rangle]$, with the value of A determined numerically. In the following, we extend this approach to study random regular and SF networks which were not previously studied, and the behavior of A for a concentration of traps and different average degrees in ER networks.

To construct the transition probability matrix for n traps in ER networks, one has to calculate the number of nodes within distance ℓ from *each* of the n nodes. This is easily accomplished by setting $n_0 = n$ and using the same formulae as in [37], $n_{\ell+1} = \left(N - \sum_{k=0}^{\ell} n_k\right)[1-(1-p)^{n_\ell}]$,

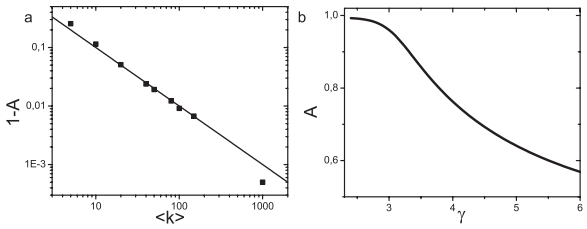

Fig. 5: An analysis of the prefactor A. (a) Plot of $1-A$ *vs.* $\langle k\rangle$ in ER networks, where A is the fitting parameter in eq. (5). Here $N = 10000$ and one trap was placed randomly (black squares). The solid line corresponds to the theoretical result $A = 1 - 1/\langle k\rangle$ derived in the text. (b) A *vs.* γ for SF networks with $N = 10^8$ and $m = 3$, when the trap is located at the hub (theory only).

with $p = \langle k\rangle/(N-1)$ being the independent probability of a link to exist. The number of links connecting the different layers is the same as in [37]: $s_\ell = n_\ell\left(N - \sum_{k=0}^{\ell} n_k\right)p$, and $o_\ell = n_\ell(n_\ell - 1)p/2$, from which the matrix B is constructed. We then calculate $F(t)$ from eq. (11), and A by fitting $F(t)$ to an exponential. We find that the change in A for $n > 1$ is minor (of the order of $\mathcal{O}(N^{-1})$) and proportional to n: $A(n) - A(n=1) \propto n$.

We next study the dependence of A on the average degree (for a single trap). Applying the above method for ER networks with different average degrees, we find that $A \approx 1 - 1/\langle k\rangle$. This is also confirmed by simulations (fig. 5(a)). For random regular networks, we derived the transition matrix B, from which we found that $A \approx 1 - 1/(k-1)$. Curiously, both results can be written as $A \approx 1 - 1/(\kappa - 1)$, where $\kappa - 1 \equiv \langle k^2\rangle/\langle k\rangle - 1$ is the branching factor of the network (since in ER networks $\kappa = \langle k\rangle + 1$, and in regular networks $\kappa = k$). A qualitative explanation of this relation (which can also be recast as $A \approx 1 - p_c$, where p_c is the percolation threshold [32]) is still lacking.

For SF networks, we calculate A for the case when the trap is located at the node of maximal degree, by using [39] for the number of nodes in layer ℓ (n_ℓ), and the number of links emanating from layer ℓ into itself (o_ℓ) and into layer $\ell+1$ (s_ℓ). As before, construction of the transition matrix B, application of eq. (11), and fitting to an exponential are used to calculate A for different values of γ (fig. 5(b)). Using this method we can predict A for very large N's in which simulations are not possible. Here the relation $A \approx 1 - 1/(\kappa - 1)$ [32] is valid up to $\gamma \lesssim 4$.

Conclusions. – We have studied the trapping problem on regular, ER, and SF networks using theory and simulations and developed a simple theory to account for the behavior of the survival probability in a variety of conditions. In ER networks we find that the trapping process exhibits a non-exponential behavior which depends on both the number of traps and the size of the network. For SF networks we find anomalous behavior for networks with small minimal degree, expressed as deviations from the theory. We also find that as opposed to ER networks,

particles survive for longer times in denser SF networks. Finally, when the trap is placed in one of the network hubs, we find a new scaling with the system size. The average time before trapping decreases dramatically in comparison to random failure or to ER networks. This is true for all values of γ, suggesting that the equivalence of SF and ER networks for $\gamma > 4$ does not exist for the trapping problem.

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