Anomalous fluctuations in tracer concentration in stratified media with random velocity fields

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We find a quantitative law that describes the nature of the anomalous fluctuations in tracer concentration when diffusion occurs in a stratified medium, with drift velocities which are constant in the x direction, but vary randomly from layer to layer in the y direction. We find that the fluctuations of the tracer concentration $P=P(x,t)$ are described by an extremely broad histogram of the form $n(\log_{10}P)\sim A/(\log_{10}P)^3\exp[-B/(\log_{10}P)^2]$. Moreover, the relative fluctuations δP increase exponentially with x. Our results are supported by numerical calculations based on the method of exact enumeration.

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How are the laws of diffusion modified when a tracer particle diffuses in a layered medium? Such a question arises when one seeks to apply the laws of physics to practical situations, such as the nature of underground water transport in stratified media, particularly the diffusion of solute (tracer particles) in groundwater flow—a quite complex phenomena related to water quality and pollution [1]. In the experiments, a tracer solute is introduced in the flow at a specific point, and its concentration is measured at different observation points [2].

We use a simple model, due to Matheron and de Marsily (MM) [3], to study the fluctuation in tracer concentration in the presence of random velocity fields. The MM model is a two-dimensional stratified system consisting of distinct layers with different transport properties in each layer. The layers consist of contiguous rows of the same orientation parallel to the x axis. A random velocity $\pm v_0$ is assigned to each layer. To study the tracer diffusion, a walker starts at the origin and at each step moves either in the $\pm y$ directions with probability $\frac{1}{4}$ or along the direction of the bias ($\pm x$ direction) with probability $\frac{1}{2}$ (Fig. 1). It is known [4,5] that the second moment of the displacement scales faster than linearly with time: $\langle x^2\rangle\sim t^{3/2}$. This phenomenon, called superdiffusive transport, is of considerable current interest in a variety of fields [6].

Here we show that $P(x,t)$, the probability to find the tracer with a given x coordinate at time t in a given configuration of bias velocities, varies remarkably from layer to layer. Also, we find that the relative fluctuation in $P(x,t)$ increases exponentially with x, in contrast to a homogeneous system where $P(x,t)$ has no fluctuations at all. The *fluctuations* of $P=P(x,t)$ (for a given x and t) can be described by the histogram $n(\log_{10}P)$, which gives the number of times the values of $P(x,t)$ are between $\log_{10}P$ and $\log_{10}P+d\log_{10}P$ and by the moments $\langle P^q(x,t)\rangle$ with $q>0$; here the brackets denote an average over different velocity configurations [7]. We find both an anomalously broad histogram and that the average moments cannot be described by a single exponent

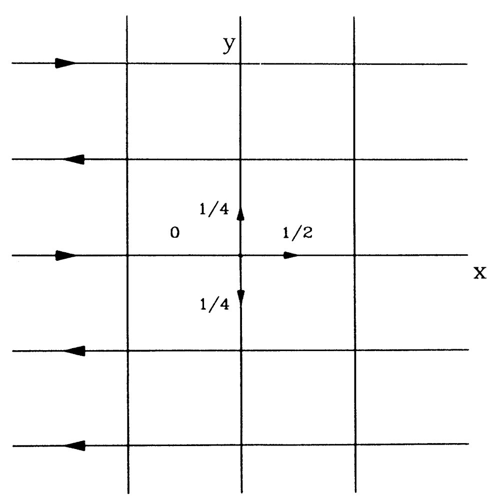

FIG. 1. The random stratified medium on the square lattice, indicating the hopping rules for a typical lattice site.

but rather display the characteristic of multifractality: $\langle P^q \rangle \sim \langle P \rangle^{\tau(q)}$ and $\tau(q) \sim q^{2/3}$ [8].

First, we calculate the moments of $P(x,t)$ from which we derive the corresponding histograms. We follow Zumofen, Klafter, and Blumen [9] and make the assumption that (a) $x(t)$ is Gaussian distributed when averaged over velocity configurations for which the walker visits a fixed number of layers R,

$$\langle P(x,t|R) \rangle = \left[\frac{R}{2\pi C} \right]^{1/2} \frac{1}{t} e^{-x^2 R/2Ct^2} , \tag{1}$$

and (b) for a given time t, the probability that the walker has visited exactly R layers is given by [10]

$$\psi(t|R) = c_1 t R^{-3} \sum_{j=1}^{\infty} \left[\frac{2\pi^2 (2j+1)^2 c_2 t}{R^2} - 1 \right] \times e^{-(2j+1)^2 \pi^2 c_2 t/R^2} . \tag{2}$$

In (1) and (2), C, c_1, and c_2 denote constants. Making use of these expressions, we write the qth moment of the probability density as [11]

$$\langle P^q(x,t) \rangle = \int_0^{\infty} \langle P^q(x,t|R) \rangle \psi(t|R) dR . \tag{3}$$

For t/R^2 large, we can consider only the first term of the sum in (2). Using the method of steepest descent to evaluate the convolution integral in Eq. (3), we see that the principal contribution to the integral comes from $R \sim t/q^{1/3} x^{2/3}$. The first moment of $P(x,t)$ is [12]

$$\langle P(x,t) \rangle \sim t^{-3/4} e^{-Cu^{4/3}}, \quad u \equiv \frac{x}{t^{3/4}} \gg 1 . \tag{4}$$

Using (3), we find that the leading term of the qth moment is

$$\langle P^q(x,t) \rangle \sim e^{-q^{2/3} u^{4/3}} . \tag{5}$$

In terms of the average probability density

$$\langle P^q(x,t) \rangle \sim \langle P(x,t) \rangle^{\tau(q)} , \tag{6a}$$

where for $q > 0$

$$\tau(q) \sim q^{\gamma} \quad (\gamma = \tfrac{2}{3}) . \tag{6b}$$

This nonlinear dependence of $\tau(q)$ on q implies that the moments cannot be described by a single exponent but instead show multifractal behavior [13], and it is not valid for very small q (i.e., as $q \to 0$) since the steepest-descent method used here fails in that range, as can be easily seen from Eq. (3).

To test our prediction (6), we use the exact enumeration method [14] to calculate numerically $P(x,t)$ for a *given* random configuration of velocities, in a system composed of $L = 196$ layers with periodic boundary conditions along the y direction. Values of $P(x,t)$ for fixed x and t were recorded, and we average over 75 000 configurations of velocities. Figure 2(a) tests (6) for the choice $x = 45$, $t = 96$; plotted as the logarithm of $\log_{10}\langle |P^q| \rangle$ as a function of $\log_{10} q$ for $q = 1$ to 8. The slope of this plot gives $\gamma = 0.693$, which is close to the prediction $\gamma = \frac{2}{3}$ from (6). The analysis of Fig. 2(a) was repeated for a range of values of $u \equiv x/t^{3/4}$, and the resulting predictions for $\gamma(u)$ are plotted in Fig. 2(b). We see that $\gamma \approx \frac{2}{3}$ for $u > 1$, the regime where the asymptotic expression for $\langle P(x,t) \rangle$ is valid.

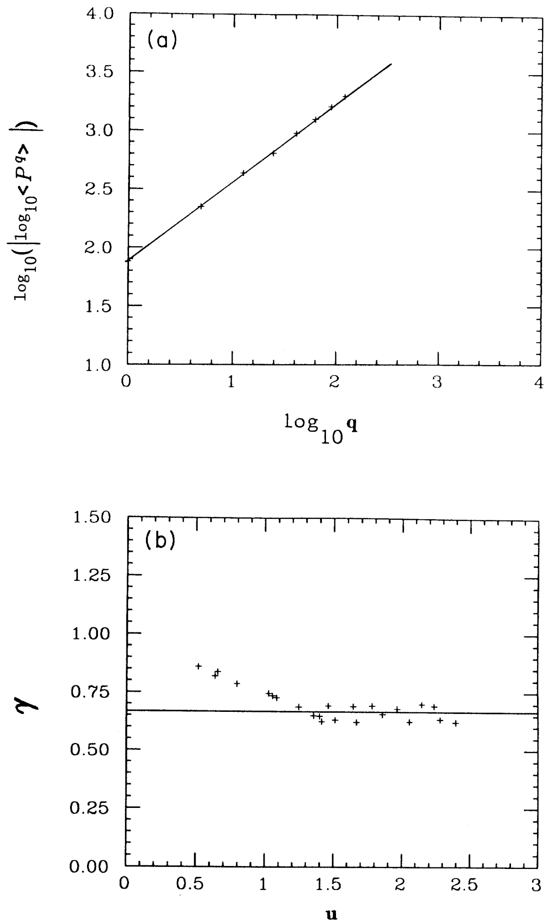

FIG. 2. (a) Plot of the logarithm of the absolute value of the logarithm of the qth moment of $P(x,t)$ vs the logarithm of q for ($q = 1$–8), $x = 45$ and $t = 96$ (or $u \equiv x/t^{3/4} = 1.47$). (b) Exponent γ in Eq. (6b) as a function of the nondimensional variable $u \equiv x/t^{3/4}$.

To derive the histogram $n(\log_{10} P)$, we express the qth moment of the probability density as

$$\langle P^q \rangle = \int_0^{\infty} P^q n(\log_{10} P) d(\log_{10} P) , \tag{7}$$

where now P denotes $P(x,t)$. By comparing (3) and (7) and using

$$\psi(t|R) dR = n(\log_{10} P)(d \log_{10} P) , \tag{8}$$

we find the form of $n(\log_{10} P)$

$$n(\log_{10} P) = A \frac{1}{(\log_{10} P)^3} e^{-B/(\log_{10} P)^2} , \tag{9}$$

where $A \equiv a_0 x^4/t^3 = a_0 u^4$ and $B \equiv b_0 x^4/t^3 = b_0 u^4$ and

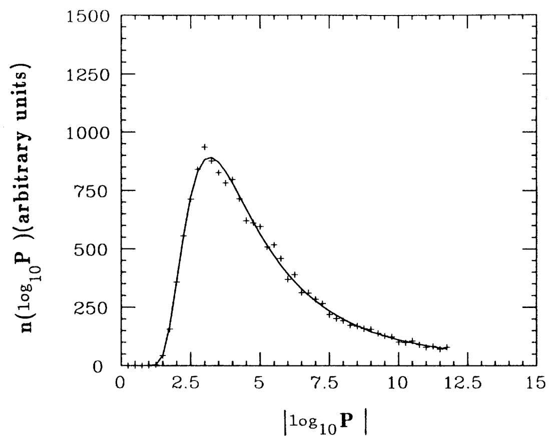

FIG. 3. Histogram $n(\log_{10}P)$ for $u \equiv x/t^{3/4}=1.47$. The continuous line corresponds to the prediction of Eq. (9) and the + symbols to the numerical calculations.

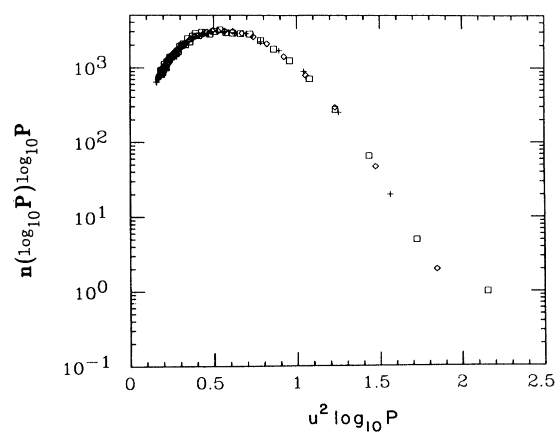

FIG. 4. Rescaled histogram $\log_{10}Pn(\log_{10}P)$ vs $u^2\log_{10}P$ for +, $u \equiv x/t^{3/4}=1.25$; ◇, $u=1.36$; and □, $u=1.47$.

a_0, b_0 are constants. There will be correction terms to (9), but Fig. 3 demonstrates that the functional form of (9) is sufficient to fit the numerical data.

The expression for $n(\log_{10}P)$ of Eq. (9) can be rescaled to make manifest its scaling properties. Figure 4 plots $\log_{10}Pn(\log_{10}P)$ as a function of $u^2\log_{10}P$ for three values of the scaled variable u. The data collapse supports the analytic expression (9).

The relative fluctuations $\delta P(x,t)$ of the probability density for a fixed x and t can be evaluated using the above results:

$$\delta P(x,t) \equiv \frac{\langle P^2(x,t)\rangle - \langle P(x,t)\rangle^2}{\langle P(x,t)\rangle^2} . \tag{10a}$$

from Eq. (6)

$$\delta P(x,t) = \langle P(x,t)\rangle^{2^{2/3}-2} - 1 \sim (e^{-u^{4/3}})^k - 1 , \tag{10b}$$

where $k \equiv 2^{2/3}-2$. Since $k<0$, $\delta P(x,t) \sim \exp(|k|x^{4/3}/t)$ grows stronger than exponentially with x.

In summary, we have shown that the average values of the tracer concentration by themselves are not sufficient to describe the dynamic behavior of the tracer concentration in stratified random media. To obtain a full description of the spatial distribution, one needs to calculate all the moments of the tracer concentration P. The broadening of the histogram of P should occur in experimental systems, and may explain previous difficulties encountered in obtaining numerical estimates for the average concentration [5].

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[12] This expression corresponding to the case $q=1$ in Eq. (3) is only valid asymptotically when $u \equiv x/t^{3/4} \gg 1$. For a detailed derivation of this relation see Ref. [9]. See also Ref. [5] for a discussion of the form of $P(x,t)$, Eq. (4).
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