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"THERMODYNAMICAL FORMALISM" FOR AN INFINITE HIERARCHY OF FRACTAL RESISTOR NETWORKS

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A thermodynamical formalism for resistor networks is developed in order to extract full information on the multifractal scaling structure. We introduce a matrix representation and study the moments of the voltage distribution

$$\tilde{Z}(\beta) = \sum_{i=1}^{N} |V_i|^{\beta} \propto N^{-\tilde{F}(\beta)} ,$$

where N is the number of resistors. We find a generic phase transition at $\beta = \beta_c = -1$. Also, we develop a transfer matrix technique which determines all even positive moments. The thermodynamical formalism is applied to the Hilfer–Blumen hierarchy of generalized Sierpinski gasket fractal networks, and the crossover from fractal to lattice behavior is studied. At this crossover we find a sharp phase transition in the second moment ($\beta = 2$).

1. Introduction

Much attention has been focused recently on the scaling properties characterizing resistor networks [1–3]. Typically one determines how the moments of the voltage distribution $\tilde{Z}(\beta)$ scale with the size L or with the number of bonds N ($N \propto L^{d_f}$, where d_f is the fractal dimension). Here

$$\tilde{Z}(\beta) = \sum_{i=1}^{N} |V_i|^{\beta} \propto N^{-\tilde{F}(\beta)} , \tag{1}$$

where N is the number of resistors. This defines a function $\tilde{F}(\beta)$. When $\tilde{F}(\beta)$ is nonlinear, the scaling structure is denoted as multifractal since it is characterized not by one but by an infinite set of exponents. Notice that $\tilde{F}(0) = -1$ and that $\tilde{F}(2)$ yields the dc resistivity exponent $\tilde{\zeta}$, $\tilde{F}(2) = \tilde{\zeta}/d_f$.

To determine $\tilde{F}(\beta)$ one finds in practice the sum $\tilde{Z}(\beta)$, and estimates the scaling exponents from the logarithm of this sum relative to the logarithm of

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the size. However, it turns out that the negative moments (i.e. β negative) are very sensitive to boundary conditions, such as V, the applied voltage. For example, consider the Sierpinski gasket of fig. 1a. One corner is placed at potential $\phi = 1$, the other at potential $\phi = 0$, and the top is at potential $\phi = \phi_0 = 1 - V$. Also shown are the voltage drops across each of the three unit resistors. Fig. 1b shows this structure at the first level of construction. The

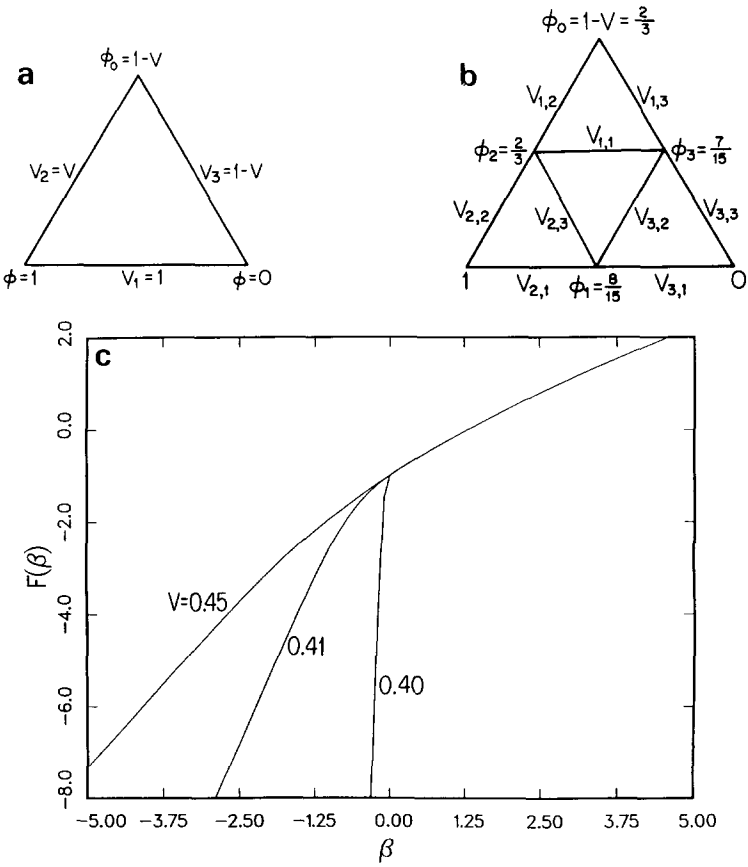

Fig. 1. (a) Notation for potentials ϕ and voltage drops V_j [$j = 1$–3] for the basic triangle from which are generated the hierarchy of Sierpinski gasket fractal resistor networks. Note that V is the only tunable parameter.

(b) Notation for potentials ϕ and voltage drops $V_{i,j}$ for the generator for the $b = 2$ member of the hierarchy of fractal resistor networks ($b = 2$ is the Sierpinski gasket). Note that if the parameter V is chosen to be $\frac{1}{3}$, then the voltage $V_{1,2}$ is strictly zero. Hence the negative moments of $\tilde{Z}(\beta)$ of eq. (1) are dominated by $V_{1,2}$ when V is close to $\frac{1}{3}$.

(c) $\tilde{F}(\beta)$ for values of V near 0.4 found from the second and third levels of construction ($N = 3^3$ and $N = 3^4$ bonds). To find $\tilde{F}(\beta)$, we (i) calculate all the voltage drops in the network, (ii) form the partition function $\tilde{Z}(\beta)$ of eq. (1), and (iii) extract $\tilde{F}(\beta)$ from knowledge of the $\tilde{Z}(\beta)$ for two different values of N.

potentials at the three new nodes are functions of V, found by straightforward application of Kirchoff's laws to be $\phi_1 = (3-V)/5$, $\phi_2 = 2(2-V)/5$, and $\phi_3 = (3-2V)/5$. Note that if $V = \frac{1}{3}$, then the voltage drop $\phi_0 - \phi_2 = 0$ so the negative moments in (1) diverge to infinity. Similarly, if V is very close to $\frac{1}{3}$, the negative moments are large and extremely sensitive to the actual choice of V. Thus the system is "sensitive to initial conditions."

At higher levels of construction of the gasket, there will occur *additional* choices of V for which $\tilde{F}(\beta)$ is dominated by a bond with zero voltage drop. For example, fig. 1c shows $\tilde{F}(\beta)$ for a range of values near $V = 0.4$ for the case of the second and third levels of construction. To find $\tilde{F}(\beta)$, we (i) calculate all the voltage drops in the network, (ii) form the partition function $\tilde{Z}(\beta)$ of (1), and (iii) extract $\tilde{F}(\beta)$ from knowledge of the $\tilde{Z}(\beta)$ for two different values of N. Thus fig. 1c is obtained for two levels, with $N = 3^3$ and $N = 3^4$ bonds.

The existence of such suddenly occurring close-to-zero voltage drops has nothing to do with the scaling properties. To repair the disadvantage of extreme small voltage drops we generalize to higher dimensions, embedding the set of voltage drops in a "voltage" space of dimension $d_V > 1$, i.e. combining the voltage drop into "voltage drop vectors" with d_V components. We show that by doing this, further information is gained on the voltage distribution. Moreover, we introduce a natural and powerful transfer matrix technique which determines the scaling exponents for all even positive moments $\tilde{Z}(2m)$. In this paper, we apply a generalized thermodynamical formalism to an infinite hierarchy of Sierpinski gasket fractals [4] that are strictly self-similiar. The generator is an equilateral triangle divided into a number $N_b = b(b+1)/2$ of smaller equilateral triangles, where the number of triangles along a side is denoted b, see fig. 2.

Recently, this hierarchy of Sierpinski gasket fractals has been studied in order to derive the transport properties at the crossover from fractal to

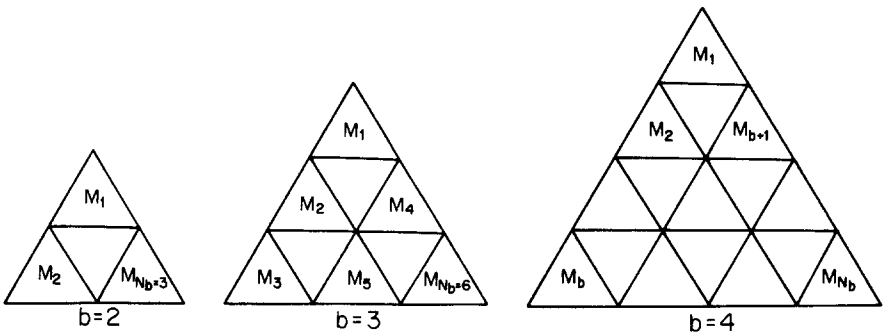

Fig. 2. Generators for small b members of the hierarchy of Sierpinski gasket fractal networks, showing also the notation for the matrices M_j used in the text.

nonfractal geometry. The spectral dimension d_s of the first 200 members of the family was calculated by Borjan et al. [5], who determined $d_s = d_s(b)$ to increase monotonically from

$$d_s(2) = \frac{2 \ln 3}{\ln 5} \simeq 1.365 \tag{2a}$$

to the value $d_s(200) \simeq 1.643$, but they were not able to find the asymptotic behavior of $d_s(b)$ as b approaches infinity. Milošević et al. [6] extended the calculations to include the first 650 members and found that $d_s(b)$ approaches the asymptotic value $d_s(\infty) = 2$ following the law

$$d_s = 2 - \frac{\ln(\ln b)}{\ln b} - \frac{c}{\ln b} - \mathcal{O}\left(\frac{\ln(\ln b)}{(\ln b)^2}\right). \tag{2b}$$

The main correction was recently found analytically by Dhar using Fourier analysis [7].

All these studies are based on the relation between the spectral dimension d_s and the dc resistivity exponent $\tilde{\zeta}$,

$$d_s = \frac{2 d_f}{d_f + \tilde{\zeta}}. \tag{3}$$

Due to the strict self-similarity for the Sierpinski gasket type fractals, both $\tilde{\zeta} = \tilde{\zeta}(b)$ and $d_f = d_f(b)$ are given from the generator, $\tilde{\zeta}(b)$ by its dc resistance R_b from one corner to the other,

$$\tilde{\zeta} = \frac{\ln(\frac{3}{2} R_b)}{\ln b}, \tag{4}$$

and $d_f(b)$ by its number of triangles N_b,

$$d_f = \frac{\ln N_b}{\ln b} = 2 - \mathcal{O}\left(\frac{1}{\ln b}\right). \tag{5}$$

Since the correction from $d_f(\infty) = 2$ for the fractal dimension $d_f(b)$ only is of order $1/\ln b$, both the first and second correction to the spectral dimension come from the asymptotic behavior of the dc resistance R_b, which accordingly is found to scale logarithmically [6],

$$R_b \simeq a \ln b, \tag{6}$$

with $a \simeq 1.0827$. Hence, in eq. (2b) $c = \ln(3a/2) \simeq 0.4850$. Milošević et al. [6] pointed out that this logarithmic behavior also is found in the case where all

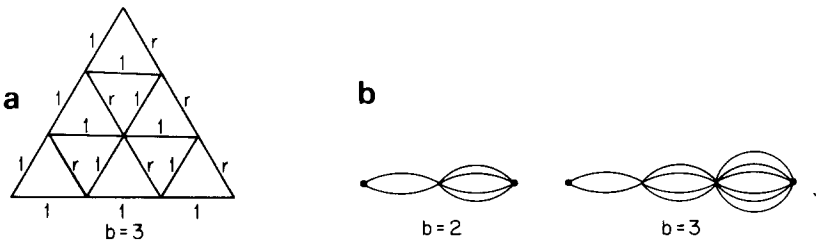

Fig. 3. (a) The case where the left-leaned resistors have resistance r $(0 < r < 1)$. (b) Reductions into unit resistor networks for the case $r = 0$.

resistors in the generator that are learned, for instance, to the left are put to zero, see fig. 3. Here we generalize the above considerations to all positive moments of the voltage distribution. In particular, we show that for the configuration with zero left-leaned resistors the functions $\tilde{F}(\beta)$ has a kink at $\beta = 2$ (a "phase transition"), and we argue that this is true also for the general hierarchy.

2. Thermodynamical formalism

To extract the information on scaling we shall introduce a matrix representation. Consider first the Sierpinski gasket ($b = 2$). The generator has $N_b = 3$ triangles, each of which has 3 voltage drops as labeled in fig. 1b. Since Kirchoff's laws are linear, the voltage drops in the generator can be obtained by a *linear* map (a matrix) from the voltage drops across the original triangle of fig. 1a. Adopting the sign convention shown in fig. 4, $V_3 = V_1 - V_2$, so we do not need 3×3 matrices but rather 2×2 matrices M_i $[i = 1, 2, 3]$ where M_i maps

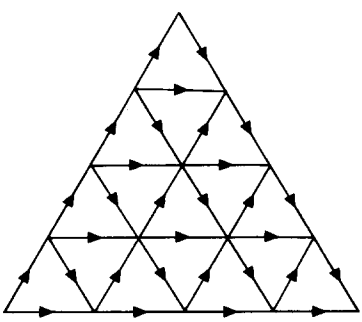

Fig. 4. Sign convention (indicated by arrows) adopted at all levels for the Sierpinski gasket fractals.

the voltage drops (V_1, V_2) to $(V_{i,1}, V_{i,2})$ (fig. 1). Moreover, it is sufficient to obtain M_1 since M_2 and M_3 are obtained by rotation. By inspection

$$M_1 = \frac{1}{5}\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}, \tag{7a}$$

$$M_2 = U^{-1} M_1 U = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{7b}$$

and

$$M_3 = U M_1 U^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}, \tag{7c}$$

where

$$U^3 = -I \tag{7d}$$

is the identity matrix, or

$$U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7e}$$

For $b > 2$, there are N_b matrices $M_1, \ldots, M_{N_b}$, which can be obtained either by a succession of star–triangle transformations [3, 5, 6] or by solving Laplace's equation (using, e.g., the standard relaxation method) on the generator for just two different initial conditions, for instance $V = 0$ and $V = 1$.

We emphasize that all moments (not only the dc resistance) are determined from the properties of the generator. According to the strict self-similarity the potentials at the triangle corners will not change by raising the level, i.e. replacing every resistor triangle with the resistor generator. Therefore, the voltage drops at higher levels are determined from a simple tree structure as is shown for the Sierpinski gasket in fig. 5a. The initial condition is given by a vector $(1, V)$ where the first coordinate is taken to be 1 by normalization.

In order to extract *all* scaling properties properly we first notice that the tree structures for the hierarchy of Sierpinski gasket fractals are very similar to the tree structures for simple multiscale Cantor sets (see fig. 5b), which is the one-dimensional analog and often have been used to illustrate multifractal scaling behavior [8]. In these Cantor sets a whole set of scaling exponents E_j are obtained, describing the decay of step sizes Δ, defined by

$$\Delta_j \equiv \prod_{i=1}^{n} m_{j_i} \propto N^{-E_j}, \tag{8}$$

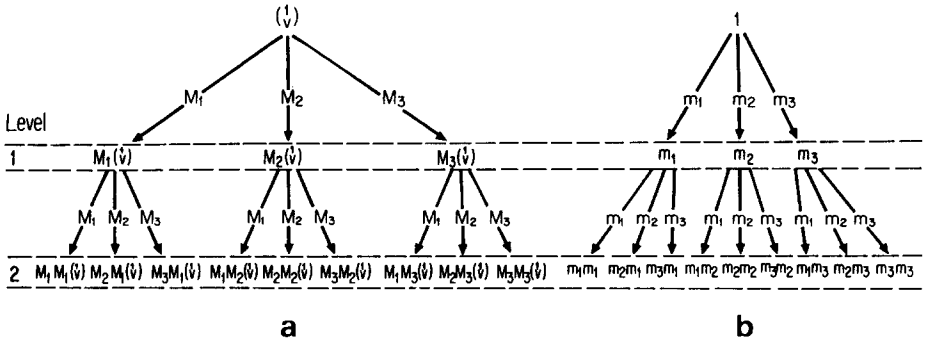

Fig. 5. (a) Tree structure from which all relevant thermodynamic information can be derived for the Sierpinski gasket ($b = 2$). M_j [$j = 1, 2, 3$] are matrices mapping the set of voltage drops at one level to the set of voltage drops at the next level. (b) Tree structure for a three-scale Cantor set. m_j [$j = 1, 2, 3$] are scalars that allow an easy determination of the number of intervals of equal size at a given level.

each associated with a certain route $j = (j_1, \ldots, j_n)$ through the tree. In (8), $N = N(n) = [N(1)]^n$ is the number of step sizes at level n. The degeneracy of these exponents, i.e., the density $\mathcal{N}(E)\,\mathrm{d}E$ of exponents E between E and $E + \mathrm{d}E$, also scales according to an exponent $S(E)$,

$$\mathcal{N}(E) \propto N^{S(E)}\,, \tag{9}$$

which is determined by the binomial distribution [8]. Due to the formal similarity to thermodynamics, we speak of the *thermodynamical formalism*, and call E and $S(E)$ the energy and entropy, respectively. The point at the end of a route, i.e., a point of the Cantor set is called a state, and the free energy is defined as the Legendre transformed to the entropy,

$$F(\beta) = \beta E - S(E)\,, \tag{10a}$$

where

$$\beta = S'(E)\,, \tag{10b}$$

or equivalently, as the scaling exponent related to the partition function,

$$Z(\beta) = \sum_{j=1}^{N} \Delta_j^{\beta} \propto N^{-F(\beta)}\,. \tag{11}$$

Again $F(0) = -1$, normalizing the maximum entropy to $S_{\max} = 1$. From eqs. (10) and (11) we notice that the bounds on the energy $E_{\min} \leqslant E \leqslant E_{\max}$ are given by the slopes of the free energy in the infinite β limits ($\beta \to \pm\infty$) which

on the other hand are determined by the largest ($\lambda_{\max} = \max_j\{m_j\}$) and smallest ($\lambda_{\min} = \min_j\{m_j\}$) scaling factors,

$$E_{\min} = \frac{-\ln \lambda_{\max}}{\ln N(1)} \tag{12a}$$

and

$$E_{\max} = \frac{-\ln \lambda_{\min}}{\ln N(1)} \,. \tag{12b}$$

The thermodynamical formalism for $d_V = 1$ can be generalized for our $d_V = 2$ structures. However, due to the fact that matrices in general do not commute, the entropy term can not be straightforwardly calculated. The number of step sizes at the first level is $N(1) = N_b$ and the step sizes are taken to be the lengths Δ of the voltage drop vectors $V = (V_1, V_2)$ according to some natural norm d_k,

$$\Delta = d_k(V) = (|V_1|^k + |V_2|^k)^{1/k} \tag{13}$$

($1 \leqslant k \leqslant \infty$), where $k = 1$ is the (absolute) sum and $k = \infty$ is the (absolute) maximum norm. *Since the scaling properties are associated solely with the matrices, the thermodynamical functions do not depend on the norm chosen.* In appendix A, we show that this is indeed the case for all positive k. In particular, since $\tilde{F}(\beta)$ is determined by d_β, we conclude that

$$\tilde{F}(\beta) = F(\beta) \tag{14}$$

for all *positive* values of β (and obviously also for $\beta = 0$); and $F(\beta)$ can be determined using solely, say, the maximum norm d_∞.

In the infinite β limits the energies (i.e. the slopes of $F(\beta)$) are determined by the largest ($\lambda_{\max}$) and the smallest ($\lambda_{\min}$) eigenvalue,

$$E_{\min} = \frac{-\ln \lambda_{\max}}{\ln N_b} \leqslant E \leqslant E_{\max} = \frac{-\ln \lambda_{\min}}{\ln N_b} \,, \tag{15}$$

corresponding to the smallest and largest scaling factor in the one-dimensional case. For the $b = 2$ Sierpinski gasket the eigenvalues are $\lambda_{\min} = \frac{1}{5}$ and $\lambda_{\max} = \frac{3}{5}$ from (7). Hence from (15),

$$E_{\max} = \frac{\ln 5}{\ln 3} \simeq 1.465 \tag{16a}$$

and

$$E_{\min} = E_{\max} - 1 \simeq 0.465 \,, \tag{16b}$$

which therefore are the slopes of $F(\beta)$ obtained for large negative and positive moments.

3. Negative moments

Since $F(\beta)$ describes the scaling properties, it is independent of initial conditions. In contrast, we saw in the introduction that $\tilde{F}(\beta)$ for negative β can be very sensitive to initial conditions due to the occurrence of extremely small voltage drops. At a given level of construction, an entire set of initial conditions exists for which $\tilde{F}(\beta)$ changes drastically. However, as we will show below, even if these initial conditions are not considered, $\tilde{F}(\beta)$ differs from $F(\beta)$ due to a general mixing of entropy.

In fig. 6a the two free energies are shown for the Sierpinski gasket. The most striking difference is the finite entropy $\tilde{S}(E_{\max}) > S(E_{\max}) = 0$ obtained in the $\beta \to -\infty$ limit [$\tilde{S}(E)$ being the Legendre transformed of $\tilde{F}(\beta)$]. We determine $\tilde{S}(E_{\max})$ to be in accordance with the approximate value $\tilde{S}(E_{\max}) \simeq \ln 1.609/\ln 3 \simeq 0.433$ found by Roux and Mitescu [9].

To understand the origin of this finite entropy we need the result that any pseudonorm d_k for k negative gives the same free energy curve. This is shown in appendix A. For convenience we use the (absolute) minimum norm $d_{-\infty}$. Now, the entropy mixing is a result of the circumstance that although the length Δ of a voltage drop vector is associated with an energy E, the minimum norm will add it to a higher energy state if one of its coordinates are small compared to Δ. To calculate this mixing we assume that the phases θ of the voltage drop vectors have a normalized distribution $g_E(\theta)$, which is bounded and nonzero at almost all phases and energies in the thermodynamic limit. For the Sierpinski gasket this assumption seems to be fulfilled, as it appears from figs. 7a, b which show the density $g_E(\theta)\mathcal{N}(E)\mathrm{d}\theta\,\mathrm{d}E$ at levels 7 and 8. Fig. 7c shows the average distribution $g(\theta)$ obtained from figs. 7a, b. Indeed, $g(\theta)$ seems to converge as the level increases.

Based on the assumption above, at an energy $\hat{E}$ the distance between the vector tips is

$$\delta(\hat{E}) \propto N^{-[S(\hat{E})+\hat{E}]} . \tag{17}$$

The entropy contribution to a higher energy state given by a scaling $\Delta(E) \propto N^{-E}$, $E \geq \hat{E}$, is therefore

$$N^{\tilde{S}(E,\hat{E})} \propto \Delta(E)/\delta(\hat{E}) \propto N^{[S(\hat{E})+\hat{E}-E]} . \tag{18}$$

The total contribution from lower energy states is

$$N^{\tilde{S}(E)} \propto \sum_{\hat{E} \leq E} N^{\tilde{S}(E,\hat{E})} \propto \sum_{\hat{E} \leq E} N^{[S(\hat{E})+\hat{E}-E]} . \tag{19}$$

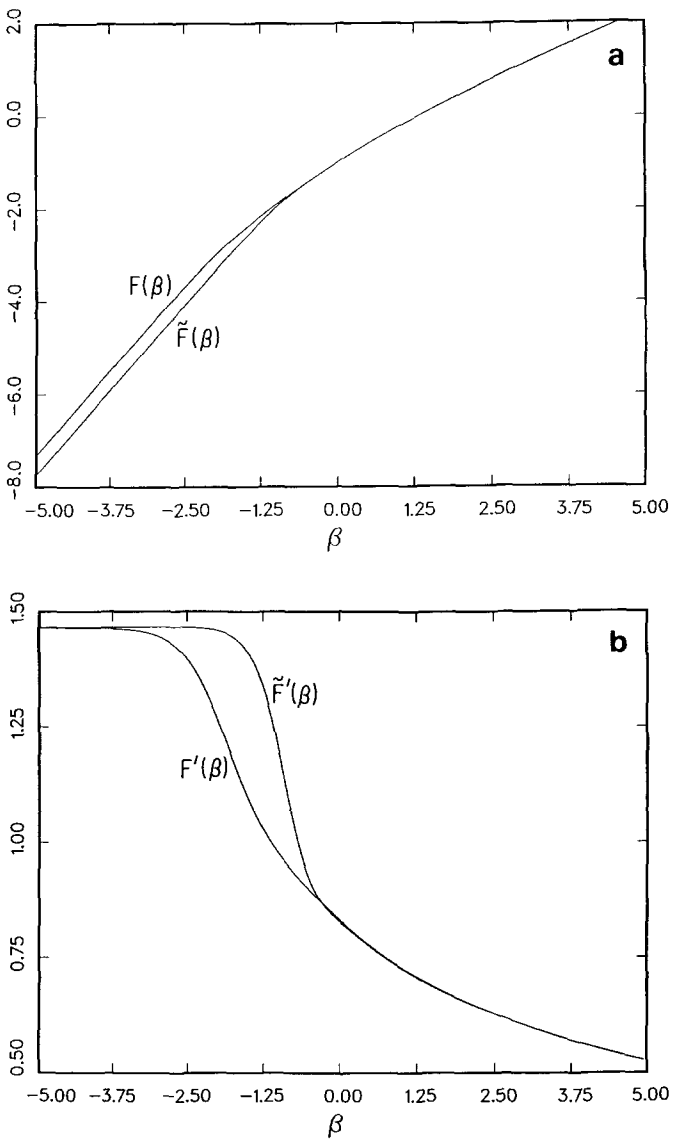

Fig. 6. (a) The two free energies $\tilde{F}(\beta)$ [from eq. (1)] and $F(\beta)$ [using d_∞ in eq. (11)] characterizing the Sierpinski gasket, determined from the 7th and 8th level of construction. The two thermodynamical functions separate significantly for $\beta < -1$. (b) Derivatives of $F(\beta)$ and $\tilde{F}(\beta)$. A substantial decrease in slope is observed for $\tilde{F}(\beta)$ at $\beta = -1$.

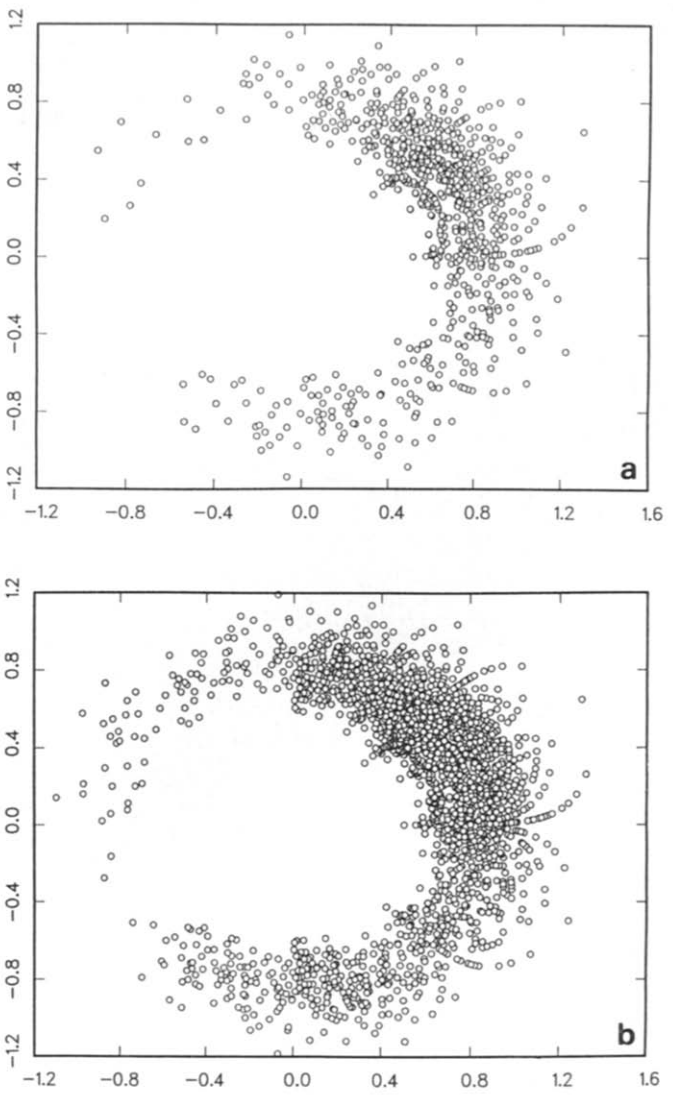

Fig. 7. Phase distribution of the voltage vectors for the Sierpinski gasket for $V = 0.5$ at (a) level 6 (3^6 points) and (b) level 7 (3^7 points). The radial axis is logarithmic, i.e., an energy axis. Only the vector tips are shown, which falls between $E_{\min} \simeq 0.465$ and $E_{\max} \simeq 1.465$ [from (16)]. (c) Normalized average distribution $g(\theta)$ of phases of the voltage drop vectors for level 6 (□) and level 7 (△). The vector indicates the center of gravity. (d) Entropy contribution $\tilde{S}(E, \hat{E})$ to higher energy states, which leads to the development of a phase transition. For $\hat{E} = E' < E_{-1}$ the dominant contribution to $\tilde{S}(E')$ comes from E'; for $\hat{E} = E'' > E_{-1}$ the dominant contribution comes from E_{-1}.

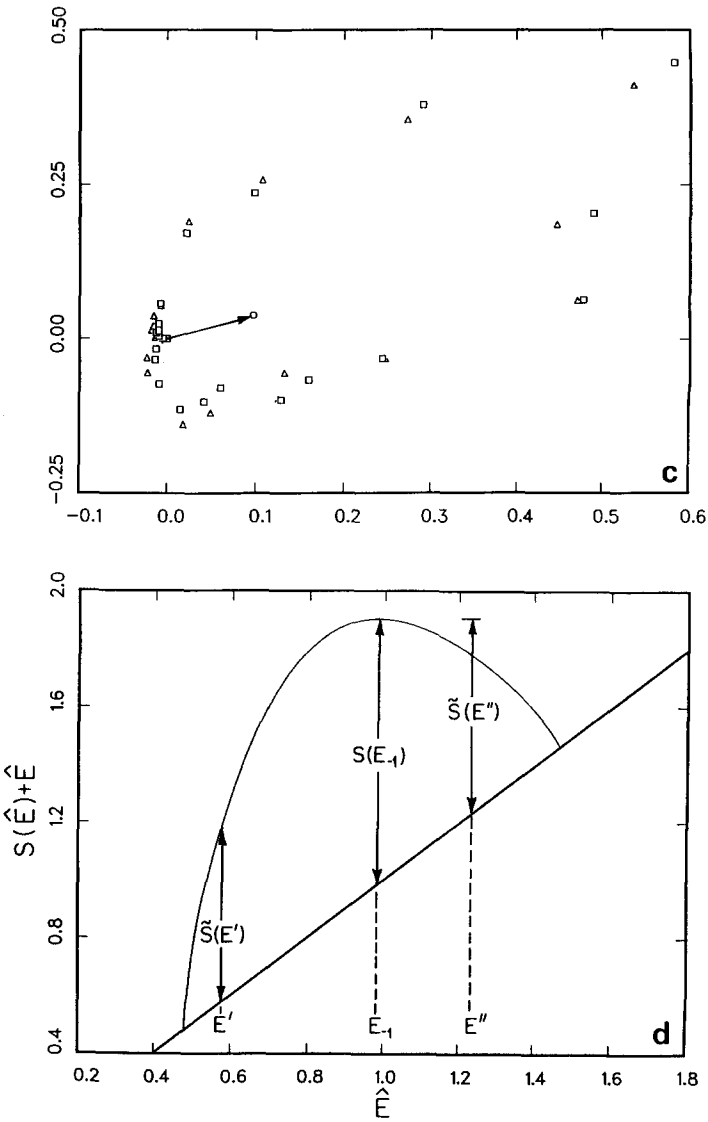

Fig. 7 (cont.).

Now, the maximal value of $\tilde{S}(E, \hat{E})$ is given by the value of $\hat{E}$ for which $S(\hat{E}) + \hat{E}$ is maximal which happens when $S'(\hat{E}) = -1$. Denoting this value of $\hat{E}$ by E_{-1} yields (see fig. 7d)

$$\tilde{S}(E) = \begin{cases} |F(-1)| - E\,, & \text{if } E \geqslant E_{-1}\,, \\ S(E)\,, & \text{if } E \leqslant E_{-1}\,, \end{cases} \tag{20}$$

where eqs. (10a, b) have been used. For the Sierpinski gasket we obtain $|F(-1)| \simeq 1.900$, giving $\tilde{S}(E_{\max}) \simeq 0.435$ in extremely good agreement with the value found directly.

We want to add two comments to eq. (20). First, we stress that $\tilde{F}(\beta)$ is determined solely by use of the minimum norm, and seems to become identical to $F(\beta)$ (determined solely by use of the maximum norm) at $\beta = -1$ giving occasion to a *phase transition*. We point out, however, that a real bump in the free energy is hard to achieve due to finite level effects (even in one dimensional systems). However, as is shown in fig. 6b, the derivative of $\tilde{F}(\beta)$ changes drastically from its maximal value $E_{\max}$ at $\beta = -1$. A second point is that the bounded phase distribution assumption has no natural break-down at $E = E_{\max}$, but first at $E = |F(-1)|$ where $\tilde{S}(E)$ becomes zero. Nonetheless, level to level correlations become important when $E = E_{\max}$ since even a very small voltage drop at most can scale from level to level by a factor of $\lambda_{\min}$.

Finally, we point out that in general an *order parameter* $0 \leq r_E \leq 1$ is defined by $r_E \boldsymbol{e}_E = (\Sigma_{j=1}^{N} \Delta_j^{\beta-1} \boldsymbol{V}_j)/Z(\beta)$, where E is related to β through eqs. (10a, b) as the derivative of the free energy $F(\beta)$. Hence, a uniform (totally disordered) distribution $g_E(\theta)$ has $r_E = 0$, and a peak (totally ordered) distribution has $r_E = 1$. Also, $r_E \boldsymbol{e}_E/2\pi$ denotes the "center of gravity" of the distribution curve $g_E(\theta)$. In particular, since almost all points have energies corresponding to $\beta = 0$, the vector $r_E \boldsymbol{e}_E/2\pi$ obtained for $\beta = 0$ represents the center of gravity of the distribution curve $g(\theta)$ in the thermodynamical limit $N \to \infty$ (see fig. 7c).

4. Transfer matrices

The scaling exponents for even positive moments can be calculated from the matrix representation by use of a transfer matrix technique. The idea is to use that the sum $V_1^{2k} + V_2^{2k}$ is a product of sum of squares,

$$V_1^{2k} + V_2^{2k} = \prod_{p=1}^{k} (V_1^2 - \epsilon_p V_2^2) , \tag{21}$$

where ϵ_p are the roots of $y^k + 1 = 0$. Every term on the right hand side of eq. (21) can be written

$$(V_1\ V_2) \Lambda_{p,0}^{(k)} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} , \tag{22a}$$

where

$$\Lambda_{p,0}^{(k)} = \begin{pmatrix} 1 & 0 \\ 0 & -\epsilon_p \end{pmatrix} . \tag{22b}$$

Now, to calculate a moment $Z_n(2m)$ at level n we use a $2k$-norm where k divides m. Defining in general $\Lambda_{p,0}^{(m)} = \Lambda_{(p \bmod k),0}^{(k)}$, $1 \leqslant (p \bmod k) \leqslant k$, yields

$$Z_n(2m) = \prod_{p=1}^{m} \left[(1 \;\; V) \Lambda_{p,n}^{(m)} \begin{pmatrix} 1 \\ V \end{pmatrix} \right], \tag{23}$$

where the matrices $\Lambda_{1,n}^{(m)}, \ldots, \Lambda_{m,n}^{(m)}$ are defined recursively by

$$\prod_{p=1}^{m} \left[(V_1 \;\; V_2) \Lambda_{p,n}^{(m)} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \right] = \sum_{j=1}^{N_b} \prod_{p=1}^{m} \left[(V_1 \;\; V_2) \{ M_j^{\dagger} \Lambda_{p,n-1}^{(m)} M_j \} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \right], \tag{24}$$

$M_j^{\dagger}$ denoting the adjoint of M_j. The derivation of eq. (23) is carried out in appendix B.

In order to reduce the complexity we point out that eq. (24) is an identity for polynomials of degree $2m$, and therefore actually consists of exactly $2m+1$ equations, one for each coefficient. This means that we are free to choose, for instance, all $\Lambda_{p,n}^{(m)}$ to have zero off-diagonal elements, except for $\Lambda_{1,n}^{(m)}$ having one nonzero off-diagonal element. Hence

$$\Lambda_{p,n}^{(m)} = \begin{pmatrix} \lambda_{p,0,n}^{(m)} & \lambda_{p,1,n}^{(m)} \\ 0 & \lambda_{p,2,n}^{(m)} \end{pmatrix}, \tag{25a}$$

where

$$\lambda_{p,1,n}^{(m)} = 0 \qquad [p \geqslant 2]. \tag{25b}$$

Thus, $\lambda_{p,l,n}^{(m)}$ is the coefficient of $V_1^{2-l} V_2^{l}$ in the pth term on the left side of eq. (24). Denoting the coefficient to $V_1^{2m-q} V_2^{q}$ in eq. (24) by $c_{q,n}^{(m)}$, the *transfer matrix* $T^{(m)} = \{a_{q,\hat{q}}^{(m)}\}$ is defined by

$$c_{q,n}^{(m)} = \sum_{\hat{q}=0}^{2m} a_{q,\hat{q}}^{(m)} c_{\hat{q},n-1}^{(m)}. \tag{26}$$

In appendix B we show that $T^{(m)}$ is well defined, and given by

$$a_{q,\hat{q}}^{(m)} = \sum_{j=1}^{N_b} \sum_{i_1 + \cdots + i_m = q} \prod_{p=1}^{m} b_{i_p,l_p}^{(j)}. \tag{27}$$

Here $\{l_1, \ldots, l_m\}$ is an arbitrary representative of l indices with sum $\hat{q}$ and with $l_p \neq 1$ for $p \geqslant 2$, and

$$B_j = \begin{pmatrix} b_{0,0}^{(j)} & b_{0,1}^{(j)} & b_{0,2}^{(j)} \\ b_{1,0}^{(j)} & b_{1,1}^{(j)} & b_{1,2}^{(j)} \\ b_{2,0}^{(j)} & b_{2,1}^{(j)} & b_{2,2}^{(j)} \end{pmatrix}$$

$$= \begin{pmatrix} m_{0,0}^{(j)2} & m_{0,0}^{(j)} m_{1,0}^{(j)} & m_{1,0}^{(j)2} \\ 2m_{0,0}^{(j)} m_{0,1}^{(j)} & m_{0,0}^{(j)} m_{1,1}^{(j)} + m_{0,1}^{(j)} m_{1,0}^{(j)} & 2m_{1,0}^{(j)} m_{1,1}^{(j)} \\ m_{0,1}^{(j)2} & m_{0,1}^{(j)} m_{1,1}^{(j)} & m_{1,1}^{(j)2} \end{pmatrix}, \tag{28a}$$

where

$$M_j = \begin{pmatrix} m_{0,0}^{(j)} & m_{0,1}^{(j)} \\ m_{1,0}^{(j)} & m_{1,1}^{(j)} \end{pmatrix}. \tag{28b}$$

Note that B_j depends solely on M_j.

The transfer matrix $T^{(m)}$ allows us to rewrite eq. (23),

$$Z_n(2m) = (1 \;\; V \ldots V^{2m}) \begin{pmatrix} c_{0,n}^{(m)} \\ \vdots \\ c_{2m,n}^{(m)} \end{pmatrix} = (1 \;\; V \ldots V^{2m}) T^{(m)n} \begin{pmatrix} c_{0,0}^{(m)} \\ \vdots \\ c_{2m,0}^{(m)} \end{pmatrix}. \tag{29}$$

The free energy $F(2m)$ is given by the largest eigenvalues $\lambda^{(m)}$ of $T^{(m)}$,

$$F(2m) = -\frac{\ln \lambda^{(m)}}{\ln N_b}. \tag{30}$$

Previous studies [1, 9] have concerned the $2m$-norm, however our analysis works for any $2k$-norm, where k is a divisor of m, since only the initial matrices $\Lambda_{p,0}^{(m)}$ are changed. For the usual length norm ($k = 1$) all initial matrices equal the identity matrix I, and $c_{2q,0}^{(m)} = \binom{m}{q}$ ($c_{2q+1,0}^{(m)}$ is always zero). Before we show some examples of the transfer matrix technique in practice, we comment on the odd moments. Using, say, the 2-norm, we see that for the first moment the products in eq. (24) are replaced by square roots; to get rid of them we must accept mixing of the $(n-1)$ terms with the n term. *However, the main reason that general methods do not work is the occurrence of sign shifts.* To this end we stress that a sign convention which puts all voltage drops positive on the generator creates negative voltage drops on higher levels. If all voltage drops were positive, we could write for the first moment

$$Z(1) = \sum_{i=1}^{N_b^n} (V_{i,1} + V_{i,2}) = \sum_{i=1}^{N_b^n} (1\ 1)\begin{pmatrix} V_{i,1} \\ V_{i,2} \end{pmatrix}$$

$$= \sum_{i=1}^{N_b^{n-1}} \sum_{j=1}^{N_b} (1\ 1) M_j \begin{pmatrix} V_{i,1} \\ V_{i,2} \end{pmatrix}$$

$$= \sum_{i=1}^{N_b^{n-1}} (1\ 1)\left\{\sum_{j=1}^{N_b} M_j\right\}\begin{pmatrix} V_{i,1} \\ V_{i,2} \end{pmatrix}. \tag{31}$$

Hence, an appropriate transfer matrix would be the sum of the matrices. For the Sierpinski gasket this sum is a diagonal matrix with eigenvalue $\lambda = \frac{6}{5}$, which gives $F(1) = -\ln \frac{6}{5} / \ln 3 \simeq -0.166$. This is in disagreement with the actual value $F(1) \simeq -0.224$, which emphasizes that the occurrence of sign shifts has importance.

To show how the transfer matrix analysis above works, we go through the two cases $\beta = 2$ and $\beta = 4$ for the Sierpinski gasket. We mention, however, that $F(2)$ can be found more easily for the hierarchy of Sierpinski gasket fractals knowing that the maximal voltage drop, $V_{\max}$ actually is found at one (or both) of the lower corners, where also practically all current J go through, i.e. $V_{\max} \propto J$. Since $V_{\max} \propto N^{-E_{\min}}$ and $J \propto Z(2)$ we get

$$F(2) = E_{\min}\,. \tag{32}$$

For the Sierpinski gasket we obtain from eq. (16b) the result $F(2) = (\ln 5/\ln 3) - 1$. The direct relation, eq. (3), to the spectral dimension, $d_s = 2/(F(2)+1)$, reproduces the result (2a).

In order to calculate $F(2)$ using the transfer matrix method and to calculate $F(2m)$ in general, we must first find the B_j. From eqs. (7) and (28) we obtain

$$B_1 = \frac{1}{25}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -6 \\ 0 & 0 & 9 \end{pmatrix}, \tag{33a}$$

$$B_2 = \frac{1}{25}\begin{pmatrix} 4 & 2 & 1 \\ 4 & 5 & 4 \\ 1 & 2 & 4 \end{pmatrix} \tag{33b}$$

and

$$B_3 = \frac{1}{25}\begin{pmatrix} 9 & 0 & 0 \\ -6 & 3 & 0 \\ 1 & -1 & 1 \end{pmatrix}. \tag{33c}$$

From eq. (27), $T^{(1)}$ is found to be the sum of these matrices,

$$T^{(1)} = \sum_{j=1}^{3} B_j = \frac{1}{25}\begin{pmatrix} 14 & 1 & 2 \\ -2 & 11 & -2 \\ 2 & 1 & 14 \end{pmatrix}. \tag{34}$$

In general, inversion symmetry splits the eigenvalue problem, adding in the characteristic determinant $\det(T-\mu I)$ for $q \neq m$ the qth row to the $(2m-q)$th row and then subtract the $(2m-q)$ column from the qth. Hence the eigenvalue problem for $T^{(1)}$ splits into the eigenvalue $\mu_1 = \frac{12}{25}$ and the eigenvalues of

$$\hat{T}^{(1)} = \frac{1}{25}\begin{pmatrix} 11 & -2 \\ 2 & 16 \end{pmatrix} \tag{35}$$

which are $\mu_2 = \frac{12}{25}$ and $\mu_3 = \frac{3}{5}$. The largest, eigenvalue $\lambda^{(1)} = \mu_3 = \frac{3}{5}$, determines the behavior of $Z(2)$ in the thermodynamic limit and gives the predicted free energy by eq. (30).

For $\beta = 4$, i.e. for $m = 2$, the transfer matrix has five rows and columns. From eqs. (27) and (33) we find

$$T^{(2)} = \frac{1}{625}\begin{pmatrix} 98 & 7 & 5 & 1 & 2 \\ -76 & 58 & 14 & 22 & -4 \\ 78 & 3 & 51 & 3 & 78 \\ -4 & 22 & 14 & 58 & -76 \\ 2 & 1 & 5 & 7 & 98 \end{pmatrix}. \tag{36}$$

Utilizing the inversion symmetry, the eigenvalues are identified as the eigenvalues to the matrices

$$\hat{T}_1^{(2)} = \frac{1}{625}\begin{pmatrix} 96 & 6 \\ -72 & 36 \end{pmatrix} \tag{37a}$$

and

$$\hat{T}_2^{(2)} = \frac{1}{625}\begin{pmatrix} 51 & 3 & 78 \\ 28 & 80 & -80 \\ 10 & 8 & 100 \end{pmatrix}, \tag{37b}$$

which are $\mu_1 = 6(11-\sqrt{13})/625$ and $\mu_2 = 6(11+\sqrt{13})/625$ for $\hat{T}_1^{(2)}$, and $\mu_3 = \mu_1$, $\mu_4 = \mu_2$, and $\mu_5 = 99/625$ for $\hat{T}_2^{(2)}$. The largest, $\lambda^{(2)} = \mu_5 = 99/625$, gives the free energy,

$$F(4) = -\frac{\ln \lambda^{(2)}}{\ln 3} = \frac{\ln(625/99)}{\ln 3} \simeq 1.677\,, \tag{38}$$

in accordance with previous results [2, 9] and our numerical results (see fig. 1c or fig. 6a).

5. Crossover from fractal to lattice

We now turn our attention to the large b limit. Since the number of triangles increases quadratically in b, we expect that already at an early stage the free energy will be well approximated from the voltage distribution obtained at the first level (the generator). From earlier studies on the dc resistance [6], we know how $F(2)$ behaves. According to eq. (32) exactly the same behavior is obtained for the slope in the large positive β limit. Then knowing that $F(2)$ and $E_{\min}$ converge to zero (in the same way as d_s converges to two, see eq. (2b)), $F(\beta)$ must converge to zero for *all* $\beta \geq 2$. However, the convergence is slow as is observed in fig. 8 where we show the free energy for $b = 20, 40, 60, 80$ for which eq. (6) is known to be very accurate. Motivated by the identical scaling behavior for $\beta = 2$ (and $\beta = \infty$), we compare the moments obtained for the hierarchy of Sierpinski gasket fractals in the large b limit with the moments obtained in the case where all left-leaned resistors are taken to be zero. In the latter situation the problem reduces (taking $V = 1$) to a study of a multiscale Cantor set. For every $i = 1, \ldots, b$ there are i scaling factors λ_i with value

$$\lambda_i = \frac{i^{-1}}{\sum_{p=1}^{b} p^{-1}} \,. \tag{39}$$

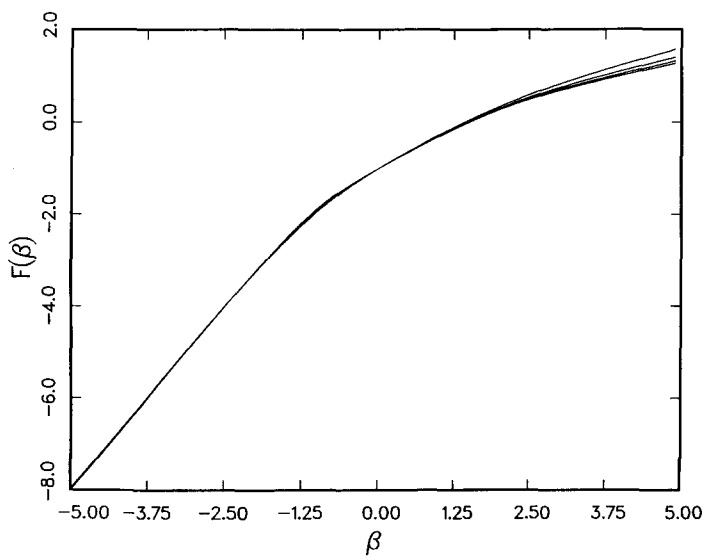

Fig. 8. Free energy $F(\beta)$ obtained from the voltage distribution on the generator for the hierarchy of Sierpinski gasket fractal networks. The top curve has $b = 20$, and remaining curves are for $b = 40, 60, 80$ (all using the norm d_∞). As b increases, $E_{\min}$ decreases, and $E_{\max}$ increases.

(On the generator, see fig. 3b, $V_i = \lambda_i(\frac{1}{1})$.) Therefore, $Z_n(\beta) \propto \lambda(\beta)^n$ [8], where

$$\lambda(\beta) = \sum_{p=1}^{b} p\lambda_p^{\beta} = \frac{\Sigma_{p=1}^{b} p^{1-\beta}}{(\Sigma_{p=1}^{b} p^{-1})^{\beta}} . \tag{40}$$

We notice that in the large b limit, eq. (40) yields

$$\lambda(\beta) \simeq \begin{cases} \dfrac{1}{2-\beta}\, b^{2-\beta}(\ln b)^{-\beta} , & \text{if } \beta < 2 , \\ (\ln b)^{-1} , & \text{if } \beta = 2 , \\ \zeta(\beta - 1)(\ln b)^{-\beta} , & \text{if } \beta > 2 . \end{cases} \tag{41}$$

In eq. (41) $\zeta(x)$ is the Riemann zeta function. Hence, as $b \to \infty$ the free energy displays a phase transition at $\beta = 2$,

$$F(\beta) = -\frac{\ln \lambda(\beta)}{\ln N_b} = \begin{cases} \frac{1}{2}\beta - 1 , & \text{if } \beta < 2 , \\ 0 , & \text{if } \beta \geqslant 2 , \end{cases} \tag{42}$$

which expresses the fact that the dominating voltage drops change abruptly at $\beta = 2$ from being the minimal nonzero voltage drops to being the maximal voltage drops.

Regarding the first level approximation we find the partition function for the "zero-resistor" gaskets at the first level to be

$$Z_1(\beta) = 2^{\beta/k}\lambda(\beta) . \tag{43}$$

Here, k refers to the norm used, so eq. (43) shows that the first level approximation is limited to $k \geqslant \beta$, which include the β norm as well as the maximum norm ($k = \infty$). In any case, to avoid uninteresting prefactors in the comparison, we study the b dependence at fixed β values. To be more precise we have for several β values ($0 \leqslant \beta \leqslant 5$) plotted $\ln[(\Sigma_{p=1}^{b} p^{1-\beta})/Z_1(\beta)]$ versus $\ln(\ln b)$ for the hierarchy of Sierpinski gasket fractal networks (see fig. 9a). For every β value, a slope $s(\beta)$ is determined from a least square fit. Fig. 9b shows $s(\beta)$, where the diagonal ($s(\beta) = \beta$) constitutes the "zero-resistor" gasket behavior. It is noticed that the disagreement is largest at $\beta = 2$. Nonetheless, the infinite b behavior is actually known to be identical for $\beta = 2$. For the b values used here Milošević et al. [6] showed that eq. (6) is valid to a high accuracy. In contrast, the "zero-resistor" gasket dc resistance,

$$R_b^{(0)} = \sum_{p=1}^{b} p^{-1} \simeq \ln b + c , \tag{44}$$

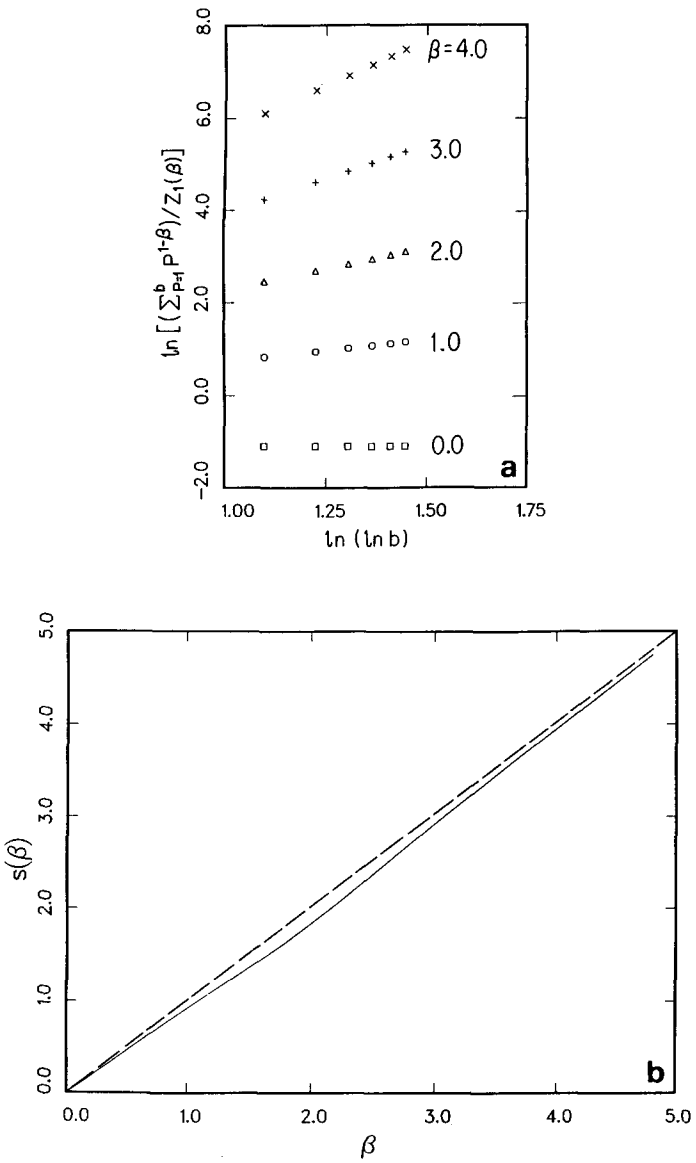

Fig. 9. (a) $\ln[(\Sigma_{p=1}^{b} p^{1-\beta})/Z_1(\beta)]$ vs. $\ln(\ln b)$ for the hierarchy of Sierpinski gasket fractal networks with $b = 20, 30, 40, 50, 60, 70$, where $Z_1(\beta)$ denotes the partition function for the generator (level 1). (b) Slopes $s(\beta)$ obtained in the way illustrated by (a), using least square fits. The diagonal, representing the "zero-resistor" gasket behavior, is shown as a dashed line.

has the correction $c = 0.5772\ldots$ (Euler's constant), which even for large b values has a clear effect on the second moment decreasing the value of $s(2)$ from its value at infinite b.

From the considerations above we conjecture that $F(\beta)$ is given by eq. (42) in the infinite b limit. Nevertheless, to better understand this conjecture, let us set all left-leaned resistors equal to a certain value r (fig. 3a). Hence, $r = 1$ corresponds to the normal hierarchy of Sierpinski gasket fractal networks while $r = 0$ corresponds to the "zero-resistor" gaskets. For $r \neq 0, 1$ the strict self-similar tree structure is no more preserved for a finite level gasket. The matrices depend on r, and this dependence changes from level to level. These changes happen, however, in a controllable way, which by renormalization can be put into a function $f_b(r)$; this is illustrated for the Sierpinski gasket in fig. 10a. Thus the thermodynamical behavior is governed by the fixed points of f_b,

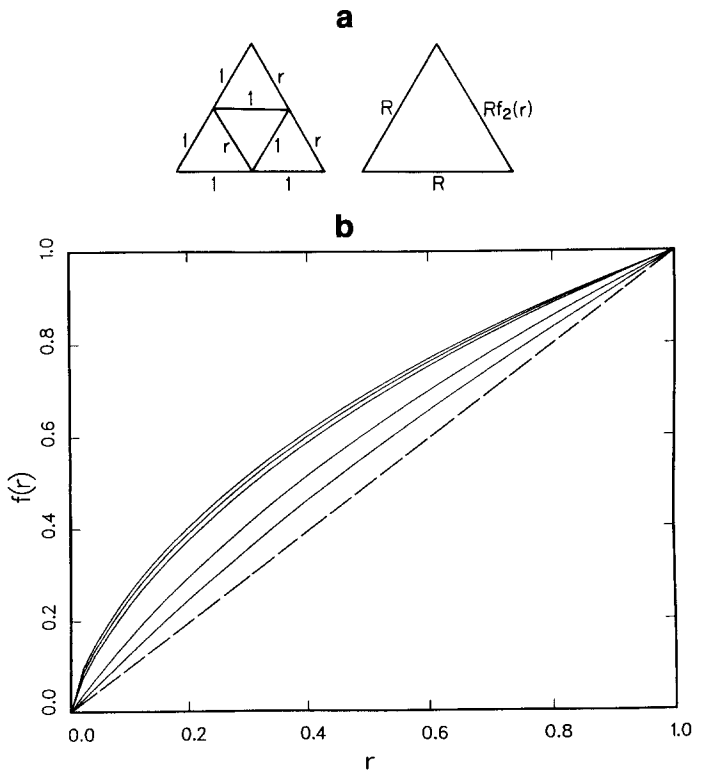

Fig. 10. (a) Renormalization function $f_2(r)$ for the Sierpinski gasket network, defined as shown. In general, R determines the dc resistivity exponent $\tilde{\zeta}$. (b) Renormalization functions $f_b(r)$ for various b values. It is observed that $f_b(r)$ only has two fixed points, the unstable fixed point $r = 0$, and the stable fixed point $r = 1$. From right to left: $b = 2, 4, 30, 70, 200$. The diagonal is shown as a dashed line.

which $r = 0$ and $r = 1$ are known to be among. For the Sierpinski gasket we find

$$f_2(r) = \frac{2r(3r+2)}{r^2+6r+3}, \tag{45}$$

which is shown in fig. 10b. This function has only the two mentioned fixed points $r = 0$ and $r = 1$. Furthermore, while $r = 1$ is a stable fixed point, $r = 0$ is not. *This means that for all $r \neq 0$ the thermodynamical functions are identical to the normal $r = 1$ functions.*

To see the behavior for larger values of b we have numerically determined $f_b(r)$, see fig. 10b. We find that $r = 1$ always is a stable fixed point, $r = 0$ always is unstable, and no other fixed points exist. In the limit $b \to \infty$ fig. 10b suggests that $f_\infty(r) = 1$ for all values of $r > 0$. In this limit the thermodynamics is given by the first level; therefore, taking the limit $r \to 0$ leads to the conclusion that the thermodynamical functions must be identical for all r. Finally, regarding the negative moments, the same thermodynamic analysis for all r values implies that the minimal voltage drop for $r > 0$ decays exponentially in b, i.e., $E_{\max}$ approaches infinity when b does. We have, however, not been able to show this.

6. Discussion and conclusions

In this paper, we have presented a matrix formalism for fractal resistor networks which allows more information on the voltage distribution than previously achieved. To characterize the voltage distribution the thermodynamical formalism is applied, which separates the set of voltage drops into subsets, each characterized by its own exponents. However, in order to obtain maximal information the thermodynamical formalism has to be applied in a sufficiently high dimension d_V, which means that close voltage drops are connected in an appropriate way to form a vector. The projection onto the one-dimensional system of voltage drops gives rise to a phase transition at $\beta = -1$, corresponding to the first negative moment. While the projection above this value of β, and in particular for all positive moments, produces the correct thermodynamics, no information aside from the minimal scaling factor $\lambda_{\min}$ can be extracted at β values below -1. Furthermore, numerically even $\lambda_{\min}$ can be difficult to determine.

We have here treated the Sierpinski gasket hierarchy of fractal networks, where $d_V = 2$. The voltage distributions can be extracted from a tree structure based on a finite number of matrices. For the Sierpinski gasket in particular, we have determined the appropriate matrices, which immediately give the

extreme scaling factors. We have also shown that the finite entropy obtained for the one-dimensional distribution at large *negative* moments is due to the phase transition occurring at $\beta = -1$.

The matrix representation also gives occasion for a natural definition of transfer matrices $T^{(m)}$ for the even positive $2m$ moments. For the hierarchy of Sierpinski gasket fractal networks we have deduced a finite expression for $T^{(m)}$ which is a $2m + 1$ square matrix, and illustrated the technique on the normal Sierpinski gasket for $m = 1$ and $m = 2$.

Also, we have considered the fractal to lattice crossover as $b \to \infty$. To this end we have discussed the resistor networks obtained when all left-leaned resistors have resistance r. To determine the thermodynamics for $r \neq 0, 1$ a renormalization function has been introduced, which renormalizes the value of r from one level of construction to the preceding one. We have shown, analytically for the Sierpinski gasket, and numerically otherwise, that this function has only two fixed points: $r = 0$ which is unstable, and $r = 1$ which is stable. We conclude that the thermodynamics for all $r \neq 0$ gaskets coincide with that found for $r = 1$. In the lattice limit the renormalization function approaches unity for all positive r, and within our numerical accuracy we find that $r = 1$ and $r = 0$ reveal the same thermodynamics. At this point the free energy (which for $r = 0$ can be calculated analytically) has a first order phase transition at the second moment, where the energy characterizing the set of dominating voltage drops abruptly changes. We here point out that a rigorous proof of the conjecture that all "r-resistor" gaskets have the same thermodynamics in the lattice limit might be obtained through Fourier analysis [7]. From such an analysis analytical expressions for the extreme scaling factors should also emerge.

It should be interesting to apply the formalism presented here to the classic random resistor network problem and to relate the percolation problem to the study of products of random matrices. In particular, a sharp knee at $\beta = -0.5$ has been observed in the free energy for a square-lattice random resistor network [2]. To this end we point out that the phase transition obtained at $\beta = -1$ moves toward $\beta = 0$ when the number of vectors along coordinate axes are more dense than the bounded phase distribution assumption implies (in contrast hereto, the position of the phase transition does not directly depend on d_V). As a related subject where new information could be gained, we mention that growth phenomena such as diffusion-limited aggregation can be regarded as a problem for a superconducting cluster inside a normal resistor network [10]. In this analogy the voltage drops correspond to the gradients of the associated field, and matrices can be introduced through renormalization. Also here the behavior of negative moments has been hard to analyze.

Finally, we emphasize that our new framework for studying resistor net-

works has great similarity with the framework on which strange attractors (or repellors) of nonlinear dynamical systems are studied. The appropriate tree structure is there naturally defined by the dynamics, and accordingly the attractor must be embedded in a sufficiently high dimension to reproduce the correct thermodynamics.

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Appendix A

Independence of norm on thermodynamics

We want here to show that the free energies

$$F_k(\beta) \equiv -\lim_{N\to\infty} \ln \frac{\Sigma_{j=1}^{N} d_k(V_j)^{\beta}}{\ln N} , \tag{A.1}$$

only depend on the sign of k,

$$F_k(\beta) = \begin{cases} F_{-\infty}(\beta) , & \text{if } k<0 , \\ F_{\infty}(\beta) = F(\beta) , & \text{if } k>0 . \end{cases} \tag{A.2}$$

Here, d_k is defined by (13). For $0<k<1$ and k negative, d_k is strictly speaking not a norm since Minkowski's inequality $[d_k(\boldsymbol{U}+\boldsymbol{V}) \leq d_k(\boldsymbol{U}) + d_k(\boldsymbol{V})]$ does not hold.

The proof of (A.2) is based on the following inequalities valid for a, b positive and α between 0 and 1,

$$(a+b)^{\alpha} \leq a^{\alpha} + b^{\alpha} \leq 2(a+b)^{\alpha} . \tag{A.3}$$

Putting $a = |V_1|^k$, $b = |V_2|^k$, and $\alpha = l/k$ yields

$$d_k(\boldsymbol{V})^l \leq d_l(\boldsymbol{V})^l \leq 2 d_k(\boldsymbol{V})^l \qquad [0<l/k\leq 1] , \tag{A.4}$$

with $V=(V_1, V_2)$. For the partition functions the inequalities (A.4) give for β/l positive

$$2^{-\beta/l}\sum_{j=1}^{N} d_l(V_j)^\beta \leqslant \sum_{j=1}^{N} d_k(V_j)^\beta \leqslant \sum_{j=1}^{N} d_l(V_j)^\beta \leqslant 2^{\beta/l}\sum_{j=1}^{N} d_k(V_j)^\beta\,, \tag{A.5}$$

with $0<l/k\leqslant 1$. For β/l negative, the inequalities are reversed. Since prefactors do not contribute to the free energy [see (A.1)], it follows that $F_k(\beta)=F_l(\beta)$ for any l/k between 0 and 1. This concludes the proof of eq. (A.2).

Appendix B

Evaluation of the transfer matrix

Based on the definition of $\Lambda_{p,n}^{(m)}$ in section 4, we here deduce eq. (23). Moreover, we prove that the transfer matrix $T^{(m)}$ is well defined by eq. (26) and derive the expression, eq. (27), for its elements.

The partition function $Z_n(2m)$ at level n is by definition

$$Z_n(2m)=\sum_{i=1}^{N_b^n}\Delta_i^{2m}=\sum_{i=1}^{N_b^n}[d_{2k}(V_i)]^{2m}=\sum_{i=1}^{N_b^n}(V_{i,1}^{2k}+V_{i,2}^{2k})^{m/k}\,. \tag{B.1}$$

When k divides m, eqs. (21) and (22) give

$$Z_n(2m)=\sum_{i=1}^{N_b^n}\prod_{p=1}^{m}\left[(V_{i,1}\ V_{i,2})\Lambda_{p,0}^{(m)}\begin{pmatrix}V_{i,1}\\ V_{i,2}\end{pmatrix}\right]. \tag{B.2}$$

Now, uniting at level n the terms originating from the same step at level $n-1$ yields

$$\begin{aligned}Z_n(2m)&=\sum_{i=1}^{N_b^{n-1}}\sum_{j=1}^{N_b}\prod_{p=1}^{m}\left[(V_{i,1}\ V_{i,2})\{M_j^{\dagger}\Lambda_{p,0}^{(m)}M_j\}\begin{pmatrix}V_{i,1}\\ V_{i,2}\end{pmatrix}\right]\\ &=\sum_{i=1}^{N_b^{n-1}}\prod_{p=1}^{m}\left[(V_{i,1}\ V_{i,2})\Lambda_{p,1}^{(m)}\begin{pmatrix}V_{i,1}\\ V_{i,2}\end{pmatrix}\right],\end{aligned} \tag{B.3}$$

where the last equation comes from the definition, eq. (24). Eq. (23) follows, repeating this procedure n times.

Inserting eq. (25) into eq. (24), we find the coefficients on the left side of eq. (24),

$$c_{q,n}^{(m)} = \sum_{l_1+\cdots+l_m=q} \prod_{p=1}^{m} \lambda_{p,l_p,n}^{(m)} . \tag{B.4}$$

To evaluate the right side of eq. (24) we first determine the coefficient $\hat{\lambda}_{p,i,n-1}^{(j,m)}$ to $V_1^{2-i}V_2^i$ in the term given by j and p on the right side of eq. (24). This is a linear function of the $\lambda_{p,l,n-1}^{(m)}$,

$$\hat{\lambda}_{p,i,n-1}^{(j,m)} = \sum_{l=0}^{2} b_{i,l}^{(j)} \lambda_{p,l,n-1}^{(m)} , \tag{B.5}$$

with $\{b_{i,l}^{(j)}\}$ given by eq. (28). From the right side of eq. (24) we now get

$$\begin{aligned} c_{q,n}^{(m)} &= \sum_{j=1}^{N_b} \sum_{i_1+\cdots+i_m=q} \prod_{p=1}^{m} \hat{\lambda}_{p,i_p,n-1}^{(j,m)} \\ &= \sum_{j=1}^{N_b} \sum_{i_1+\cdots+i_m=q} \prod_{p=1}^{m} \left(\sum_{l=0}^{2} b_{i_p,l}^{(j)} \lambda_{p,l,n-1}^{(m)} \right) \\ &= \sum_{j=1}^{N_b} \sum_{i_1+\cdots+i_m=q} \sum_{\hat{q}=0}^{2m} \sum_{l_1+\cdots+l_m=\hat{q}} \left[\left(\prod_{p=1}^{m} b_{i_p,l_p}^{(j)} \right) \left(\prod_{p=1}^{m} \lambda_{p,l_p,n-1}^{(m)} \right) \right] \\ &= \sum_{\hat{q}=0}^{2m} \sum_{l_1+\cdots+l_m=\hat{q}} \left[\left(\sum_{j=1}^{N_b} \sum_{i_1+\cdots+i_m=q} \prod_{p=1}^{m} b_{i_p,l_p}^{(j)} \right) \left(\prod_{p=1}^{m} \lambda_{p,l_p,n-1}^{(m)} \right) \right] . \end{aligned} \tag{B.6}$$

Moving the sum over the i indices with sum q inside, the prefactor to the nonzero products of the $\lambda_{p,l_p,n-1}^{(m)}$ only depends on the l indices through $\hat{q}$. Thus, eq. (B.6) can be written

$$c_{q,n}^{(m)} = \sum_{\hat{q}=0}^{2m} \left[\left(\sum_{j=1}^{N_b} \sum_{i_1+\cdots+i_m=q} \prod_{p=1}^{m} b_{i_p,l_p}^{(j)} \right) \left(\sum_{l_1+\cdots+l_m=\hat{q}} \prod_{p=1}^{m} \lambda_{p,l_p,n-1}^{(m)} \right) \right] , \tag{B.7}$$

which by eq. (B.4) is equivalent to eqs. (26) and (27).

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