Three-Dimensional Toom Model: Connection to the Anisotropic Kardar-Parisi-Zhang Equation

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(Received 31 March 1992)

A three-dimensional Toom model is defined and the properties of the interface separating the two stable phases are investigated. Using symmetry arguments we show that in the zero-noise limit the model has only nonequilibrium fluctuations and that the scaling is described by the anisotropic Kardar-Parisi-Zhang equation. The scaling exponents are determined numerically and good agreement with the theoretical predictions is found.

PACS numbers: 68.35.Fx, 05.40.+j, 64.60.Ht

Fluctuations of equilibrium and nonequilibrium interfaces have attracted much attention in the recent past, in part because of their relevance to many fields, ranging from spin systems to growth models [1]. Derrida, Lebowitz, Speer, and Spohn (DLSS) [2] studied the novel behavior of an interface separating the two stable phases in the two-dimensional Toom model, which is a relatively simple probabilistic cellular automaton [3]. The model has attracted much attention since Toom proved its nonergodicity in the presence of small perturbations. The robust nonergodicity and its implications for generalized Ising models have also been studied [4]. DLSS raise the interesting possibility that the Toom model is "generic" for a variety of physical systems. Here we take a concrete step in this direction by (i) extending the Toom model to three dimensions, and (ii) mapping the interface of this probabilistic cellular automaton (with purely geometric rules) onto the anisotropic Kardar-Parisi-Zhang (AKPZ) equation [5]. We focus on the fluctuations of the two-dimensional interface separating the two stable phases existing in the three-dimensional system. Finally we report numerical measurements on the interface predicted by the AKPZ equation and determine the scaling exponents in the strong coupling regime.

The Toom model.— In the two-dimensional Toom model, spins with values $S=\pm 1$ are simultaneously updated at every time step as follows: $S(i,j)$ becomes equal to 1 with probability p, -1 with probability q, and, with probability $1-p-q$, becomes aligned with the majority of itself and a specified set $\{S\}$ of neighboring spins. In the simplest version, $\{S\}$ is defined as the northern and eastern neighbors (NE model).

Here we introduce the three-dimensional model [6], with the set $\{S\}$ being comprised by four of the neighboring spins $\{(i+1,j,k);(i,j+1,k);(i,j,k-1);(i+1,j+1,k-1)\}$ (see Fig. 1). When $p=q=0$, the Toom model is deterministic. For small enough noise (small p and q) the model for any dimension is nonergodic, with two stable phases formed by spins $+1$ and -1. The mechanism is best explained first for the case of the $d=2$ NE model. Here a 135° diagonal interface between up and down spins drifts southwestward with unit speed because sites just southwest of the interface, at each instant of time, have neighborhood majorities dominated by their north and east neighbors on the other side. For the three-dimensional model the $\{S\}$ vicinity will influence the value of the spin (i,j,k); thus the transition rule is characterized by a transport direction determined by a vector with direction $\vec{AB}\equiv(-1,-1,1)$ (see Fig. 1).

Symmetry properties of the interface.— The direction of the vector $\vec{AB}$ together with the boundary conditions determine the position of the interface separating the two phases. The symmetry properties of the transition rule imply a reflection symmetry with respect to the plane containing $\vec{AB}$ and perpendicular to the plane $k=0$. To obtain a flat well-defined interface, we choose boundary conditions as follows [7]: We divide the $k=0$ plane in two by the line $i-j=0$; for the spins with $i-j>0$ we fix $S=-1$, while for spins with $i-j<0$ we take $S=+1$. The volume in which the Toom rule operates is determined by the conditions $k>0$ and $i-j<0$.

With these boundary conditions we shall obtain an in-

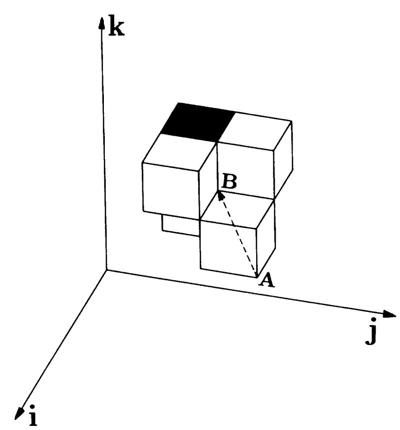

FIG. 1. The three-dimensional Toom rule used in this work. The black box $S(i,j,k)$ is updated with the majority of itself and the vicinity $\{S\}$ (shown by the white boxes, which correspond to the three nearest neighbors and one next-nearest neighbor in the eighth octant of a Cartesian coordinate system centered on the shaded box). The vector $\vec{AB}$ represents the transport direction (see text).

terface (Fig. 2) which is anchored to the line ($i-j=0$, $k=0$) and which separates two phases, with $+1$ spins under it and -1 above. In the zero-noise limit ($p\to 0$, $q\to 0$), we add a $+$ or $-$ spin to the interface and wait until the resulting perturbation has settled down—operationally, we apply the deterministic part of the Toom model until no further changes are observed.

To obtain some intuitive understanding of the $d=3$ case, we shall call *particles* those sites which have spin $+1$, while the -1 sites will be empty. Thus, converting a spin -1 to $+1$ corresponds to inserting a particle, while converting a spin from $+1$ to -1 corresponds to removing a particle. One can easily verify the following properties: (i) Starting initially only with the boundary conditions (particles only on the semiplane $k=0$, $i-j<0$), the interface will be stable if $p=0$, i.e., the noise does not generate particles on the interface; (ii) $q=0$, $p\neq 0$ will generate a stepped interface which forms an angle with the horizontal plane ($k=0$) equal to the angle formed by $\vec{AB}$ with the plane $k=0$. So for general p and q, the mean position of the interface will be an inclined plane between limit (i) and (ii), at an angle determined by the values of p and q.

Using the system of coordinates of Fig. 2, we note that the rule determining interface fluctuations (a) has *reflection symmetry* $x_\perp\to -x_\perp$, (b) has a *translational invariance in* h, $x_\perp$ and $x_\parallel$ [8], (c) *lacks reflection symmetry in* $x_\parallel$, and (d) *lacks reflection symmetry in* h. The (d) property needs some more comments. In the case of the two-dimensional NE model the 1D interface fluctuates under equilibrium conditions for $p=q$ because the deterministic part, defined by the choice $\{S\}$, has reflection symmetry in h. The nonequilibrium fluctuations appear only when this symmetry is broken by imposing $p\neq q$. In our case the deterministic part is characterized by the lack of the up-down symmetry in h, related to the fact that there is no plane containing the ($i-j=0$, $k=0$) line to serve as symmetry plane for the transition rule, so for any p and q values the interface will fluctuate under *nonequilibrium* conditions [9].

The KPZ approach is to consider the lowest-order Langevin equation compatible with the symmetry conditions of the transition rule. In the present case, the continuum equation compatible with the above symmetries [10,11] is the AKPZ equation [5]

$$\frac{\partial h}{\partial t}=\nu_\parallel\frac{\partial^2 h}{\partial x_\parallel^2}+\nu_\perp\frac{\partial^2 h}{\partial x_\perp^2}+a_1\frac{\partial h}{\partial x_\parallel}+\lambda_\parallel\left(\frac{\partial h}{\partial x_\parallel}\right)^2+\lambda_\perp\left(\frac{\partial h}{\partial x_\perp}\right)^2+\eta(x,t)\,. \quad (1)$$

Equation (1) is obtained by writing all possible terms and then eliminating those that do not satisfy the symmetry conditions (a)-(d).

The noise is assumed to be uncorrelated, with zero mean and two-point correlation function $\langle\eta(x,t)\eta(x',t')\rangle=D\delta(t-t')\delta(x-x')$. For an unpinned interface the a_1 term in (1) can be eliminated by the coordinate transformation $x_\parallel\to x_\parallel+a_1t$. For the pinned one this term cannot be eliminated, but has no effect on the value of the scaling exponents [2].

Wolf showed [5] using a dynamic renormalization-group analysis that if the two λ terms have opposite signs, the nonlinear terms are not relevant, and the interface fluctuations are determined only by the diffusion terms $\nu_\parallel\partial^2h/\partial x_\parallel^2$ and $\nu_\perp\partial^2h/\partial x_\perp^2$. This is the case for the vicinal surfaces in which context this equation was originally introduced. The Toom interface is described by the strong coupling fixed point becoming relevant if the nonlinear terms have the same sign. Rescaling (1) according to $x_\perp\to bx_\perp$, $x_\parallel\to b^\chi x_\parallel$, $h\to b^\alpha h$, $t\to b^z t$, from the nonrenormalizability of the λ's one gets the scaling relations [5]

$$z+\alpha=2\,, \quad (2a)$$

$$\chi=1\,. \quad (2b)$$

The dynamic renormalization-group analysis fails to give the exact values of the exponents for the strong coupling regime.

We succeeded [12] in measuring z and α for the two-dimensional interface of the three-dimensional Toom model in the zero-noise limit. Given the interface generated by the model (see Fig. 3), the fluctuations in the $x_\parallel$ direction are extracted from the scaling relation $\langle[h(x_\perp,x_\parallel)-\bar{h}(x_\perp,x_\parallel)]^2\rangle_{x_\perp}\sim x_\parallel^{2\beta}$, where $\beta=\alpha/z$, and $\bar{h}$ is the time-averaged value of $h(x_\perp,x_\parallel)$. In this direction, due to the unidirectional flow of the information, there

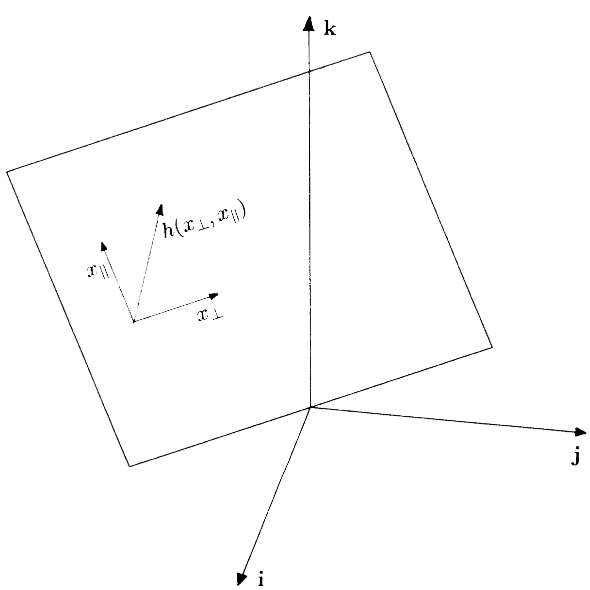

FIG. 2. The height function $h(x_\perp,x_\parallel)$ is defined as a deviation from the flat steady-state interface profile (the inclined plane). The system of coordinates of the interface is defined by $x_\perp$ and $x_\parallel$. The system of coordinates of the Toom rule is given by the i,j,k axes.

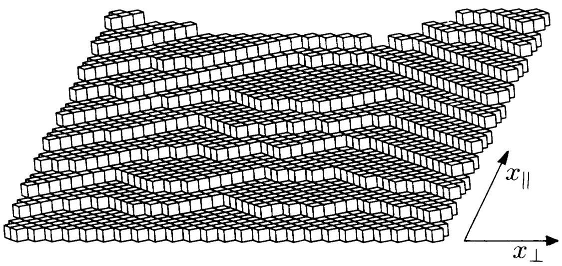

FIG. 3. An example of the two-dimensional Toom interface for a 25×25 system with $p \neq 0$ and $q \neq 0$.

are no finite-size effects. This flow and the pinning at $x_\parallel = 0$ are believed to be responsible for the appearance of the β exponent in this scaling relation. In the $x_\perp$ direction we imposed periodic boundary conditions and the scaling was determined using the height-height correlation function $\langle [h(x_\perp + l, x_\parallel) - h(x_\perp, x_\parallel)]^2 \rangle \sim l^{2\alpha}$ with $x_\parallel$ held fixed. The applicability of this relation is a result of the reflection symmetry in $x_\perp$, as discussed above.

We used systems of size $L \times L$, with $L = 128$ and 256; the best fit gives $\beta = 0.21 \pm 0.03$, and $\alpha = 0.43 \pm 0.04$. The existence of nonzero scaling exponents indicates the relevance of the nonlinear terms in (1), since the equilibrium fluctuations would lead to zero α and β. Moreover, the numbers differ only slightly from those of the (2+1)-dimensional isotropic KPZ equation (IKPZ) [13] as expected, since the strong coupling phase of the AKPZ and IKPZ equations are controlled by the same fixed point [expressed by Eq. (2b) also].

Finally, we comment on the zero-noise limit and its relation to the continuum approach. The zero-noise approximation is usual in the context of sandpile models, where—in contrast to the present situation—it indicates the end of the applicability of the continuum equations [14]. This limit gives rise to the appearance of a self-organized critical state, whose exponents differ from those predicted by the continuum theory [11]. The breakdown of the continuum approach can be attributed to the appearance of nonlocal events—avalanches—which in turn for finite noise have finite size due to the collisions between simultaneously appearing avalanches.

We now argue that in the Toom model the existence of the step structure is the mechanism which generates a characteristic length scale in the avalanches, thus making the dynamics local and justifying the continuum approach. As noted above, the values of p and q determine the angle formed by the interface with the $k=0$ plane, and thus determine, from simple geometric considerations, a mean distance between the steps. If we place a particle on a given step, it will generate a number of particles in the *same* layer, this number being determined by the position of the following step forming the next layer behind the new particle. Since the mean distance between the steps has a characteristic length scale, the number of generated particles (i.e., the size of the avalanches) will be bounded as well. These arguments apply also to the NE model.

One of us (A.-L.B.) would like to thank Péter Gács and Terence Hwa for many very helpful discussions. The authors are grateful to H. Larralde for his comments on the manuscript. M.A. acknowledges support of Intevep S.A., while A.-L.B. and H.E.S. thank the Hungary-U.S. exchange program of the Hungarian Academy of Sciences and the NSF for support.

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[6] However, this is not the only possible model for three dimensions; further details about other possible choices will be presented in an extended version of this paper. [M. Araujo, A.-L. Barabási, and H. E. Stanley (to be published).]

[7] Although there is a relative freedom in the choice of the boundary conditions, it is not completely arbitrary. Changing the boundary conditions may result in unwanted symmetry breaking, or even in an interface composed of two intersecting semiplanes [6].

[8] The existence of an equilibrium profile pinned at $x_\parallel = 0$ suggests that one is not free to translate the interface arbitrarily in the $x_\parallel$ direction. However, changing the boundary conditions to introduce pinning at a nonzero $x_\parallel$ generates an interface identical to the original one, without changing the *local* dynamics. In this sense one can assume a translational invariance in the $x_\parallel$ direction.

[9] For a definite proof one has to know the inclination

dependence of the growth velocity (without boundary conditions) [see J. Krug and H. Spohn, Phys. Rev. Lett. **64**, 2332 (1990)]. It cannot be excluded that for a particular inclination the coefficients of the nonlinear terms accidentally vanish (or change sign), although for the *pinned* interface for the relevant inclinations this has not been observed.

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[14] Here we refer to the difference between the limit of zero deposition rate (zero-noise approximation) and finite deposition rate in the running sandpiles [11]. Although the KPZ equation works also for small noise, large nonlocal events may lead to unexpected behavior.