[14]V. G. Bhide and M. S. Hegde, Phys. Rev. B 5, 3488 (1972).
[15]E. Matthias, W. Schneider, and R. Steffen, Phys. Lett. 4, 41 (1963).
[16]L. Bertain, Ph.D. thesis (University of Nevada, June, 1970) (unpublished).
[17]G. Burns, J. Chem. Phys. 31, 1253 (1959).
[18]R. M. Sternheimer, Phys. Rev. A 6, 6 (1972).

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Double-Power Scaling Functions near Tricritical Points*

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We introduce invariants of the scaling equation about the tricritical point. Using these invariants, a modified version of the scaling hypothesis about the three critical lines meeting at the tricritical point is presented. From it we demonstrate that the thermodynamic equation of state near a tricritical point and near a critical line may be expressed as double-power scaling functions. These imply that experimental data should collapse from a *volume* onto a *line* (i.e., by two dimensions). This behavior is in contrast to ordinary "single-power" scaling functions, which predict data collapsing from a volume onto a surface or from a surface onto a line (i.e., by one dimension).

I. INTRODUCTION

There have been recent experimental measurements[1] near tricritical points[2] (TCPs) in metamagnets, NH_4Cl, and ^3He-^4He mixtures.[1] These data have been partially interpreted recently in terms of scaling arguments in which one makes not one but two scaling hypotheses.[3-5] Riedel and Wegner[6] were perhaps the first authors to note that in regions for which two scaling hypotheses are simultaneously valid, double-power-law behavior of certain functions results. In this work we present a *variation* of scaling for tricritical points, utilizing generalized homogeneous functions[7] (GHF's) of invariants of the scaling equation about the tricritical point. We obtain, in regions near a critical line and a tricritical point, double-power scaling functions which permit data to collapse from a volume onto a line, in contrast to the behavior of single-power scaling functions, which permit data to collapse by only one dimension (e.g., from a surface onto a line, or a volume onto a surface).

Before we can proceed to make the scaling hypothesis for a TCP, it will be necessary to determine the relevant directions for scaling.[3] The three thermodynamic fields (T, temperature; η, ordering field; and g, nonordering field) near a TCP are believed to constitute an affine space in which directions may be defined by parallelism only. A TCP is a point of intersection of three critical lines in this three-dimensional affine space (cf. Fig. 1). At each point P on a critical line, three different types of directions can be established. The first direction, $x_1(P)$, is a direction not locally parallel to the coexistence surface. The second, $x_2(P)$, is locally parallel to the coexistence surface but not parallel to the critical line. These are the "strong" and "weak" directions of Griffiths and Wheeler.[8] The third direction, $x_3(P)$, is locally parallel to the critical line.

As the point P moves toward the TCP, these directions attain limiting orientations. Since there are three critical lines terminating at a TCP, three "rival" sets of directions of this type exist at the TCP. It has been shown[4] that if scaling holds at a TCP, these three sets of directions are equivalent. Thus, we choose the relevant directions for scaling at a TCP as $\bar{x}_i \equiv \lim(P\to \text{TCP})x_i(P)$, where P is a point on the critical line L_1 (see Fig. 1).[9]

II. SCALING HYPOTHESIS FOR TCP

Having ascertained the relevant scaling directions $\bar{x}_i$ for TCP, we now introduce a scaling parameter λ (> 0) and make the homogeneity hypothesis[3-7] that the singular part of the Gibbs potential is asymptotically a GHF,

$$G(\lambda^{\bar{a}_1}\bar{x}_1,\ \lambda^{\bar{a}_2}\bar{x}_2,\ \lambda^{\bar{a}_3}\bar{x}_3) = \lambda G(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)\ , \tag{1}$$

where $\bar{a}_i$ are the scaling powers. Equation (1) is *equivalent* to the statement that

$$G = F_3\ (\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3) \tag{2}$$

is an invariant equation under the one-parameter (λ) group ($\mathcal{G}_3$) of transformations

$$G' = \lambda G,\quad \bar{x}_i' = \lambda^{\bar{a}_i}\bar{x}_i,\quad (i=1,\ 2,\ 3)\ . \tag{3}$$

In other words, under the transformations, Eq. (2) becomes $G' = F_3(\bar{x}_1',\ \bar{x}_2',\ \bar{x}_3')$.

The group $\mathcal{G}_3$ admits a basis set of three ($i=0$, 1, 2) functionally independent absolute invariants, $y_i(G', \bar{x}_1',\ \bar{x}_2',\ \bar{x}_3') = y_i(G, \bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)$, such that all

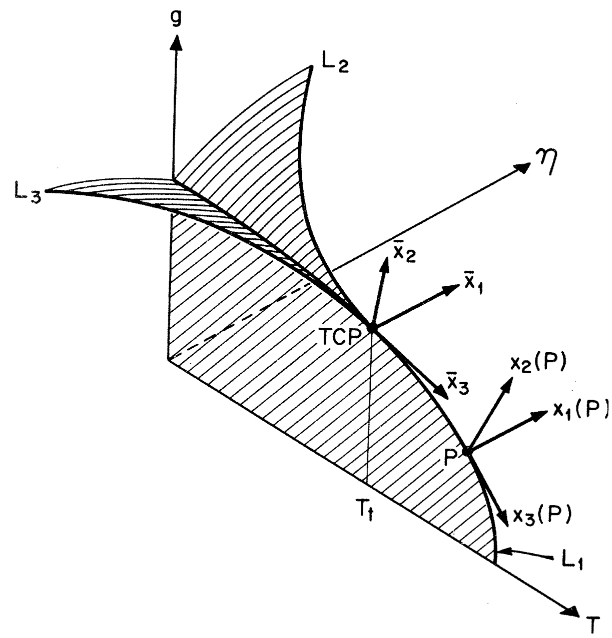

FIG. 1. Schematic phase diagram showing a TCP (at $T = T_t$). Shaded areas are coexistence surfaces. At a point P on L_1, a triad of directions $x_i(P)$ are shown. This triad becomes $\bar{x}_i$ at TCP.

other absolute invariants are expressible in terms of these. One such basis set is

$$y_0 \equiv G/\bar{x}_3^{1/\bar{a}_3}, \quad y_1 \equiv \bar{x}_1/\bar{x}_3^{\bar{a}_1/\bar{a}_3}, \quad y_2 \equiv \bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3} \quad . \tag{4}$$

The scaling hypothesis, Eq. (1), requires Eq. (2) to be expressible in terms of the basis set as a "single-power" scaling function,

$$y_0 = \bar{F}_2(y_1, y_2) , \tag{5}$$

which states that G (and other thermodynamic functions), when appropriately scaled, are functions of the invariants (y_1, y_2) alone. This result allows data near a TCP to collapse from a *volume* onto a *surface*.

We remark that, using Eq. (1), it is possible to determine all exponent relations and "single-power" scaling laws for a TCP.[3-5]

III. GEOMETRY OF SURFACES AND CURVES NEAR TCP

Since the quantities y_1 and y_2 defined in Eqs. (4) form a basis set of functionally independent absolute invariants of $\bar{x}_i$ under the group of transformations $\bar{x}_i' = \lambda^{\bar{a}_i}\bar{x}_i$, *points* in the invariant (y_1, y_2) plane give rise to invariant *curves* in the $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ space. We have seen that the scaling hypothesis requires scaled thermodynamic functions near a TCP to depend on y_1 and y_2 only. This implies that each of the three critical *lines* near the TCP can be expressed as a *point* $y_i = k_i$ in the (y_1, y_2) plane, where k_i are constants.

Usually, for systems exhibiting a TCP, one of the critical lines is a planar curve lying entirely in the (g, T) plane (e.g., L_1 of Fig. 1). Since $\bar{x}_1 = 0$, Eq. (4) implies that L_1 is given by $(y_1, y_2) = (0, -k)$ in the invariant plane.

Near L_1, it is expected that the symmetry property of the critical line will also influence the asymptotic form of the thermodynamic functions. The region of influence is bounded by some "crossover" curve, $f_x(y_1, y_2) = 0$ [Fig. 2(a)], or

$$f_x(x_1/x_3^{\bar{a}_1/\bar{a}_3}, \bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3}) = 0, \tag{6}$$

which is a conical surface surrounding L_1 in the $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ space [Fig. 2(b)]. Scaling cannot tell us the *actual* shape of the curve in the (y_1, y_2) plane,[10] but it does limit the shape of the conical "crossover" surface in the $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ space, since all points in the (y_1, y_2) plane give rise to curves approaching the TCP along the $\bar{x}_3$ axis (corresponding to the minimum $\bar{a}_i$).[11]

IV. DOUBLE-POWER SCALING FUNCTIONS FOR L_1

We now proceed to deduce the restriction on the asymptotic form of the thermodynamic functions near a TCP adjacent to the critical line L_1.[12] Along L_1, the conventional scaling hypothesis is

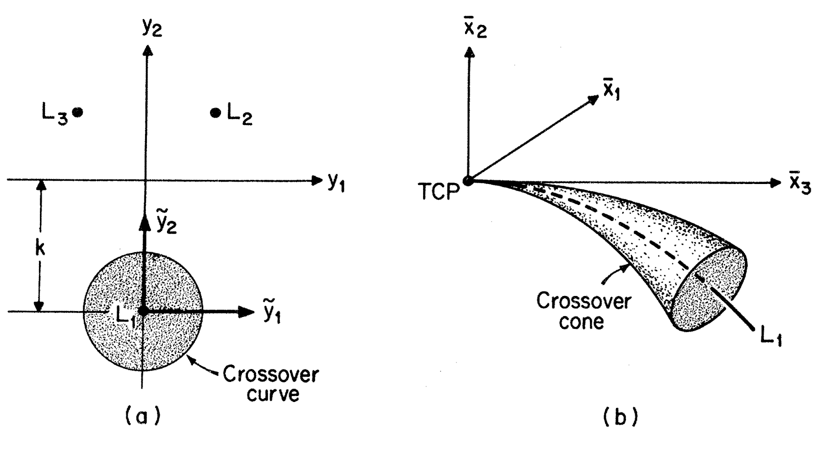

FIG. 2. (a) Invariant (y_1, y_2) plane. The strong and weak directions for L_1 are y_1 and y_2, and the crossover curve is shown (Ref. 8). (b) Principal points of interest of (a) in the $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ space.

normally stated in terms of a GHF equation

$$G(\mu^{a_1} x_1,\ \mu^{a_2} x_2;\ x_3) = \mu G(x_1,\ x_2;\ x_3), \tag{7}$$

where $\mu(>0)$ is an arbitrary parameter, a_1 and a_2 are the scaling powers for L_1, and x_3 is an "inactive" variable which does not scale. Near the TCP, by Eq. (5) the value of $y_0 \equiv G/\bar{x}_3^{1/\bar{a}_3}$ changes only if the values y_1 and/or y_2 change. It is much better therefore to make a scaling hypothesis about L_1 near the TCP using y_1 and y_2.[13]

Since the coexistence surface bounded by L_1 maps into the vertical axis of the $(y_1,\ y_2)$ plane, the direction y_1 is strong and y_2 is weak. Thus, we deduce that the proper scaling variables for L_1 (near the TCP) are

$$\tilde{y}_1 \equiv y_1,\quad \tilde{y}_2 \equiv y_2 + k; \tag{8}$$

these vanish at the line $(y_1,\ y_2) = (0,\ -k)$.

We now hypothesize that along L_1 near the TCP, $\tilde{y}_0 \equiv y_0$ is a GHF of $(\tilde{y}_1,\ \tilde{y}_2)$:

$$\tilde{y}_0(\mu^{a_1}\tilde{y}_1,\ \mu^{a_2}\tilde{y}_2) = \mu\tilde{y}_0(\tilde{y}_1,\ \tilde{y}_2). \tag{9}$$

In other words, instead of (7) we postulate that

$$\tilde{y}_0 = F_2(\tilde{y}_1,\ \tilde{y}_2) \tag{10}$$

is an invariant equation under the group $(\mathcal{G}_2)$ of transformations $(\tilde{y}_0' = \mu\tilde{y}_0, \tilde{y}_1' = \mu^{a_1}\tilde{y}_1$, and $\tilde{y}_2' = \mu^{a_2}\tilde{y}_2)$.

By the same reasoning used for the derivation of Eq. (5), we see that (9) requires that (10) may be written in the form

$$z_0 = F_1(z_1), \tag{11}$$

where $z_0 \equiv \tilde{y}_0/\tilde{y}_2^{1/a_2}$ and $z_1 \equiv \tilde{y}_1/\tilde{y}_2^{a_1/a_2}$ form a set of functionally independent absolute invariants of $\mathcal{G}_2$ and therefore of the variables $G,\ \bar{x}_1,\ \bar{x}_2,\ \bar{x}_3$ under the direct-product group $\mathcal{G}_2 \otimes \mathcal{G}_3$.

Reexpressing Eq. (11) in terms of the original variables G, x_1, x_2, and x_3 we obtain the "double-power" form[14]

$$\frac{G}{\bar{x}_3^{1/\bar{a}_3}(\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3}+k)^{1/a_2}} = F_1\left(\frac{\bar{x}_1}{\bar{x}_3^{\bar{a}_1/\bar{a}_3}(\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3}+k)^{a_1/a_2}}\right). \tag{12}$$

Equation (12) predicts that near the TCP *and* L_1, data will collapse from a *volume* onto a *line*. Clearly, this happens only within the crossover cone of Eq. (6) [cf. Fig. 2(b)].

In the plane $\bar{x}_1 = 0$ [i.e., the $(g,\ T)$ plane], Eq. (12) requires[15]

$$G \sim \bar{x}_3^{1/\bar{a}_3}(\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3}+k)^{1/a_2}, \tag{13}$$

and the conical surface of Eq. (6) becomes two crossover lines (cf. Fig. 3) $\bar{x}_2 = C_i\bar{x}_3^{\bar{a}_2/\bar{a}_3}$ or $y_2 = C_i$, where $i = 1,\ 2$. The crossover exponent $\varphi \equiv \bar{a}_3/\bar{a}_2$, which determines the shape of the crossover lines, can be obtained directly from the shape of L_1.[3]

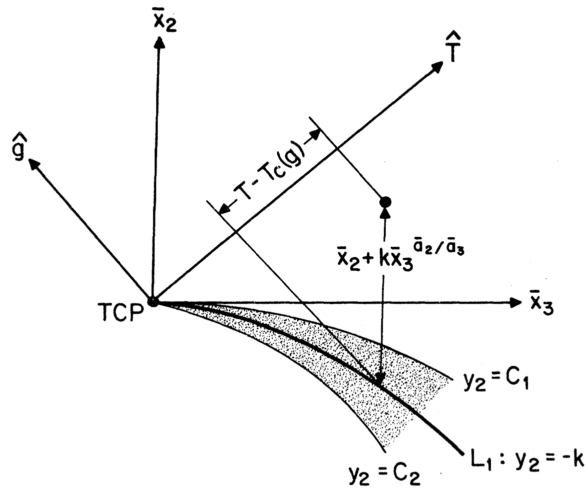

FIG. 3. Figure 2(b) sliced in the $\bar{x}_1 = 0$ plane. The crossover lines are labeled $y_2 = C_1,\ C_2$. The projection of $\bar{x}_2 + k\bar{x}_3^{\bar{a}_2/\bar{a}_3}$ along the T axis is $T - T_c(g)$, and $(\hat{T},\ \hat{g}) \equiv (T - T_t,\ g - g_t)$.

V. EXTENSIONS AND CONCLUDING REMARKS

The entire discussion in this paper may be extended to the scaling of any thermodynamic function f. For example, for the staggered susceptibility $\chi_{st} \equiv \partial^2 G/\partial\eta^2 \propto \partial^2 G/\partial\bar{x}_1^2$ (or $\partial^2 G/\partial x_1^2$) and the direct susceptibility $\chi \equiv \partial^2 G/\partial g^2 \propto \partial^2 G/\partial\bar{x}_2^2$ (or $\partial^2 G/\partial x_2^2$) of a metamagnet, the expression analogous to Eq. (13) is

$$f_i = \bar{x}_3^{(1-2\bar{a}_i)/\bar{a}_3}(\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3}+k)^{(1-2a_i)/a_2}, \tag{14}$$

where $f_1 \equiv \chi_{st}$ and $f_2 \equiv \chi$. We note that Eq. (13) has the appropriate divergence properties at the critical line *and* at the TCP.

Finally, we make a few remarks about the exponents and directions of approach toward the TCP and L_1. Using the experimentally accessible function χ_{st} as an example, we note that if we approach the TCP along a curve $\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3} = \text{const}$, the scaling exponent is $-\bar{\gamma}_{st}\varphi^{-1} = (1 - 2\bar{a}_1)/\bar{a}_3$. If we approach the critical line L_1 along a line $\bar{x}_3 = \text{const}$, the scaling exponent is $-\gamma_{st} = (1 - 2a_1)/a_2$ as expected. On the other hand, if the TCP is approached along a path outside the crossover lines, χ_{st} scales with an exponent $-\bar{\gamma}_{st} = (1 - 2\bar{a}_1)/\bar{a}_2$. Similar remarks may be made with respect to the three-dimensional "double-power" scaling functions of Eq. (12).

Equation (14) may be cast in "mixed-exponent" forms; e.g., $\chi_{st} \sim C[T - T_c(g)]^{-\gamma_{st}}$, in which $\bar{x}_2 + k\bar{x}_3^{\bar{a}_2/\bar{a}_3}$ has been replaced by its projection along the T axis (Fig. 3), and $C \sim \bar{x}_3^{(\gamma_{st}-\bar{\gamma}_{st})/\varphi}$ is the asymptotic amplitude. Depending on the relative magnitudes of γ_{st} and $\bar{\gamma}_{st}$, the asymptotic amplitude may diverge, vanish, or stay constant as the TCP is approached within the crossover cone.

The ideas of this work provide the basis of a formulation of the scaling hypothesis near critical points that are more complex than tricritical points. For these points, the direct product of more than two groups of scaling transformations arises naturally. A detailed account of this extension will be published elsewhere.

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[1](a) D. P. Landau, B. E. Keen, B. Schneider, and W. P. Wolf, Phys. Rev. B **3**, 2310 (1971); (b) C. W. Garland and B. B. Weiner, Phys. Rev. B **3**, 1634 (1971); (c) E. H. Graf, D. M. Lee, and J. D. Reppy, Phys. Rev. Lett. **19**, 417 (1967); (d) R. J. Birgeneau (private communication). For additional references to the vast experimental literature, see *Cooperative Phenomena near Phase Transitions,* edited by H. E. Stanley (MIT Press, Cambridge, Mass., 1973).

[2]R. B. Griffiths, Phys. Rev. Lett. **24**, 715 (1970).

[3]E. K. Riedel, Phys. Rev. Lett. **28**, 675 (1972).

[4]A. Hankey, H. E. Stanley, and T. S. Chang, Phys. Rev. Lett. **29**, 278 (1972).

[5]R. B. Griffiths, Phys. Rev. B **7**, 545 (1973).

[6]E. K. Riedel and F. Wegner, (a) Z. Phys. **225**, 195 (1969); (b) Phys. Rev. Lett. **24**, 730 (1970).

[7]A. Hankey and H. E. Stanley, Phys. Rev. B **6**, 3515 (1972). See also H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford U.P., London and New York, 1971), Chap. 11. (1972). See also, H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford U. P., London, 1971), Chap. 11.

[8]R. B. Griffiths and J. C. Wheeler, Phys. Rev. A **2**, 1047 (1970).

[9]In general, the $\overline{x}_j$ may be nonorthogonal, but in this work we need consider only rectilinear orthogonal coordinates.

[10]Although the crossover curve may be of any form, it is shown for convenience as a circle in Fig. 2(a).

[11]In this paper, we assume $a_1 \neq a_2 \neq a_3$. (See Ref. 4 for discussion of other cases.)

[12]The scaling ideas for $L_{2,3}$ are similar. However, it will be necessary to rotate the invariant axis to determine the strong and weak directions.

[13]This assumption is equivalent to the scaling assertion of Ref. 6, that the extended homogeneity relation (1) must encompass the relation (7) asymptotically close to the critical line.

[14]Equation (12) is similar to that found in Ref. 6(b) for dynamic scaling in anisotropic magnetic systems. We emphasize that scaled equations may change their values or functional forms when the scaling variables change signs.

[15]Equation (13) is essentially equivalent to Eq. (8) of Ref. 3, provided the variable μ_1 is interpreted as the critical line scaling field.