PROPOSAL FOR THE CLASSIFICATION OF CRITICAL POINTS BY ORDER☆

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Received 20 March 1973

Critical points of complex thermodynamic systems are classified by their order $\mathcal{O}$, defined here for the first time. A convenient notation is developed for such points.

The geometric approach to phase transitions in multicomponent systems [1], which was used by Griffiths to propose the existence of tricritical points [2], naturally leads one to pose the following question: "In more complex systems, with more than three field or field-like†[1] variables, what sort of more complex critical points and critical phenomena can be expected?" In this work we propose the classification of critical points by their *order*, denoted by an index $\mathcal{O}$, which we discuss below. For a particular space†[2] of critical points (denoted CRS in ref. [1]) the value of $\mathcal{O}$ depends only upon the topological properties of that CRS in an appropriately general space of field variables.

A tricritical point was defined [2] as a point where three lines of ordinary critical points meet. Thus, the dimensionality of a space of tricritical points is at least one less than the dimensionality of spaces of ordinary critical points which intersect there. Spaces of *ordinary* critical points are known to be of dimension d two less than the total number of field variables n; hence a space of *tricritical* points is of dimension $d \leqslant n-3$. If $n = 4$ we may have lines of tricritical points, for $n = 5$, surfaces, and so on.

A point of intersection of lines of tricritical points will be as different from a tricritical point as a tricritical point is from a simple critical point. This can be clearly seen by considering the topology at such a point. At a line of tricritical points, three surfaces of critical points meet, whereas at the special point several (more than three) distinct surfaces of critical points converge on the special point, each surface being bounded by two of the lines of tricritical points.

Such a point, being qualitatively different, must be distinguished from ordinary critical points and from tricritical points. To accomplish this, we propose a definite number $\mathcal{O}$, called the order, associated with each different kind of critical point and defined as follows. One can define ordinary critical points to be of order $\mathcal{O} = 2$, and a special point where lines of order $\mathcal{O}$ intersect to be a critical point of order $\mathcal{O} + 1$ ($\mathcal{O} \geqslant 2$). Thus a tricritical point is of order $\mathcal{O} = 3$, while a point of intersection of lines of tricritical points is of order $\mathcal{O} = 4$.

In general, the dimensionality of spaces of critical points can be larger than zero or one, but the dimensionality, d, of a space of critical points of order $\mathcal{O}$ is always less than the dimensionality of the spaces of critical points of order $\mathcal{O} - 1$ which intersect at it. Griffiths and Wheeler [1] reasoned that the dimensionality of a space of ordinary critical points is $(n-2)$; the value of d for arbitrary $\mathcal{O}$ therefore satisfies

$$d \leqslant n - \mathcal{O} \tag{1}$$

where n is the total number of field (and field-like) variables available.

A specific example which demonstrates the impor-

†[1] The term field-like variables refers to quantities, such as parameters in a Hamiltonian, which do not suffer a discontinuity across a coexistence surface. Hereafter the term field-variable will mean both sorts of variable. See also ref. [1].

†[2] In this context the word *space* refers to any line, surface, or higher dimensional entity of critical points. They are all smooth subspaces of the total space of field and field-like variables.

☆Work supported by the NSF, ONR, AFOSR, NASA, and the MIT Laboratory for Nuclear Science. This work forms a portion of a Ph.D. thesis submitted to the Physics Department of MIT by A. Hankey.

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tance of distinguishing the *order* of a critical point from the *number* of critical lines meeting there is the tetracritical point [3] of Nagle and Bonner. This is a point of order $\mathcal{O} = 3$ with field variables (H, H_{st}, T), where H_{st} is the staggered magnetic field. When a field-like variable (the ratio of short-range to long-range interaction strengths) is also included, this becomes *part of a line of tricritical points*. Eq. (1) becomes an equality since $n = 4$ and $\mathcal{O} = 3$. The tetracritical point is simply a point on a smooth line of tricritical points†3 — it is of order $\mathcal{O} = 3$ and not $\mathcal{O} = 4$. This reasoning corroborates the fact, noted by Nagle and Bonner, that the *exponents at the tetracritical point are the same as at the tricritical points*.

To be concise we now introduce a notation for critical points of arbitrary order. This notation is an extension of that of ref. [1]. As proposed there, the letters CRS denote a space of critical points, but now the order $\mathcal{O}$ will be given by a preceding superscript, and the dimensionality d by a subscript (${}^{\mathcal{O}}\mathrm{CRS}_d$). Thus a tricritical point, which is a point (space of dimension zero) of order 3 is denoted by a ${}^3\mathrm{CRS}_0$. A line of ordinary critical points is a ${}^2\mathrm{CRS}_1$: three ${}^2\mathrm{CRS}_1$ meet at a ${}^3\mathrm{CRS}_0$.

Four field or field-like variables are needed to achieve a critical point of order 4. Examples of systems exhibiting a critical point of order four are the following: (i) a three-dimensional Ising model with variable interaction RJ between planes; here the critical point of order four occurs in the limit $R \to 0$ on the temperature axis [i.e., at the point $R = H = H_{st} = 0$, and $T = T_c(R = 0)$], (ii) a variation of the Nagle-Bonner model with the long-range interaction acting separately on odd and even sublattices, and (iii) a one-dimensional model containing staggered magnetic fields of different wavelength. Both models (ii) and (iii) are exactly soluble [5]. In addition to these model systems, several experimental systems may have critical points of order greater than three. For example, the order of the "triple point" of the spin-flop transition is at least three, but whether it is truly three or four has yet to be determined.

In conclusion, it should be noted that there are also non-geometrical ways of defining order which are connected with the scaling hypothesis. The order of a space of critical points turns out to be equal to the number of relevant scaling variables about each point of that space. In most of the examples known to the authors, the two definitions coincide; in particular, this is true for the examples mentioned above, in which eq. (1) is satisfied as an equality. The definition of order in terms of the number of relevant scaling variables may be more appropriate in very complex systems such as fluid mixtures with more than three components.

†3 A detailed analysis may be found in ref. [4]; the tetracritical point arises because we have chosen a section of the four-dimensional space that is *tangent* to the line of tricritical points, rather than a section which *intersects* it.

References

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