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# Symmetry of the continuum percolation threshold in systems of two different size objects

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## Abstract

We study the continuum percolation in systems composed of overlapping objects of two different sizes. We show that when treated as a function of the volumetric fraction  $f$  as opposed to the concentration  $x$ , the percolation threshold exhibits the symmetry  $\eta_c(f, r) = \eta_c(1 - f, r)$  where  $r$  is the ratio of the volumes of the objects. Knowledge of this symmetry has the following benefits: (i) the position of the maximum of the percolation threshold is then known to be at exactly  $f = 1/2$  for any  $r$  and (ii) full knowledge of the percolation threshold is obtained by performing simulations only for  $f \in [0, \frac{1}{2}]$  or  $f \in [\frac{1}{2}, 1]$ , whichever is computationally easier.

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## 1. Introduction

For almost 50 years [1], percolation theory has been used to model static and dynamic properties of porous media and other disordered physical systems [2–4]. When the size and shape of objects in the percolation system are an important aspect of the system being modeled, continuum percolation—as opposed to simple site or bond

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percolation—must be used [2–4]. The size and shape of the objects are important in real world systems such as gas and oil reservoirs. A first step in taking into account the various size objects found in these reservoirs is to model the systems using objects of two different sizes.

Recently, the critical density of continuum percolation systems composed of overlapping disks (2D) and spheres (3D) of two different sizes was determined to high accuracy [5,6]. When plotted as a function of concentration of the two-sized objects, the critical density was found to be asymmetric. The focus of this paper is to show analytically and with Monte Carlo simulations that when plotted against the relative volumetric fraction, the critical density is a symmetric function.

For systems of volume  $V$  containing  $N_A$  objects of volume  $v_A$  and  $N_B$  objects of volume  $v_B$ , the system is characterized by the quantities

$$\eta \equiv \frac{N_A v_A + N_B v_B}{V} \quad (1)$$

or the volume fraction [6–8]

$$\phi = 1 - e^{-\eta} . \quad (2)$$

The percolation threshold  $\eta_c$  is a function of the relative concentration of the objects

$$x \equiv \frac{N_A}{N} \quad (3)$$

and the ratio of object sizes

$$r \equiv \frac{v_B}{v_A} . \quad (4)$$

That is,

$$\eta_c = \eta_c(x, r) . \quad (5)$$

## 2. Percolation threshold symmetry

Conventionally  $\eta_c$  is chosen to be a function of  $x$  and  $r$ . It is useful, however, to consider  $\eta_c$  as a function of the volumetric fraction  $f$

$$f \equiv f_A \equiv \frac{N_A v_A}{N_A v_A + N_B v_B} = \frac{1}{[1 + r(1/x - 1)]} . \quad (6)$$

In this section, we show that

$$\eta_c(f, r) = \eta_c(1 - f, r) . \quad (7)$$

For given values of the volumes  $v_A$  and  $v_B$ , let the values of  $N_A$  and  $N_B$  at the percolation threshold  $\eta_c$  be

$$N_A = \bar{N}_A , \quad (8)$$

$$N_B = \bar{N}_B . \quad (9)$$

We then note that there is a second set of values of  $N_A$  and  $N_B$  with the same percolation threshold

$$N_A = \bar{N}'_A = \frac{v_B}{v_A} \bar{N}_B, \tag{10}$$

$$N_B = \bar{N}'_B = \frac{v_A}{v_B} \bar{N}_A. \tag{11}$$

Since

$$\eta'_c \equiv \frac{\bar{N}'_A v_A + \bar{N}'_B v_B}{V} = \frac{\bar{N}_A v_A + \bar{N}_B v_B}{V} \equiv \eta_c. \tag{12}$$

If we denote the value of  $f$  corresponding to the solution specified by Eqs. (8) and (9) as  $\bar{f}$ , we find that the value of  $\bar{f}'$  corresponding to the solutions specified by Eqs. (10) and (11) satisfies

$$\bar{f}' = 1 - \bar{f}, \tag{13}$$

from which it follows

$$\eta_c(f, r) = \eta_c(1 - f, r). \tag{14}$$

This symmetry can be expressed in an alternative form; by simply exchanging the labels  $A$  and  $B$ , we have from Eqs. (1), (4) and (6) that

$$\eta_c(f, r) = \eta_c(1 - f, 1/r), \tag{15}$$

and using Eq. (7), we find the symmetry

$$\eta_c(f, r) = \eta_c(f, 1/r). \tag{16}$$

### 3. Simulations

Here we demonstrate the symmetry represented by Eq. (7). In Fig. 1, we plot  $\phi_c$  versus both  $x$  and  $f$  for a two-dimensional system of disks with various values of the ratio  $r$  [5]. While the plots versus  $x$  are highly asymmetric, the plots versus  $f$  exhibit the expected symmetry. As an example, consider the sharpest asymmetric curve ( $\lambda=0.1$ ) in which the points are plotted with asterisks and the corresponding symmetric curve with corresponding points represented by open circles. We see that for the largest concentrations of  $x$ , the points in the plot versus  $f$  are compressed into the region near  $f = 1$ .

In Fig. 2, we plot  $\phi_c$  versus both  $x$  and  $f$  for a three-dimensional system of spheres with volume ratio  $r = 0.125$ . Again we see the asymmetry in the plot versus  $x$  as opposed to the symmetry of the plot versus  $f$  [6].

Since  $f$  has a range of  $[0, 1]$  and  $\eta$  function is symmetric with respect to  $f$ , the critical density attains its maximum value at  $f^* = 0.5$ . Solving Eq. (6) for  $x$ , when  $f^* = 0.5$  we find the corresponding value of  $x$  at which the critical density attains its maximum

$$x^* = \frac{r}{1+r} = \frac{1}{1+\lambda^{-d}}, \tag{17}$$

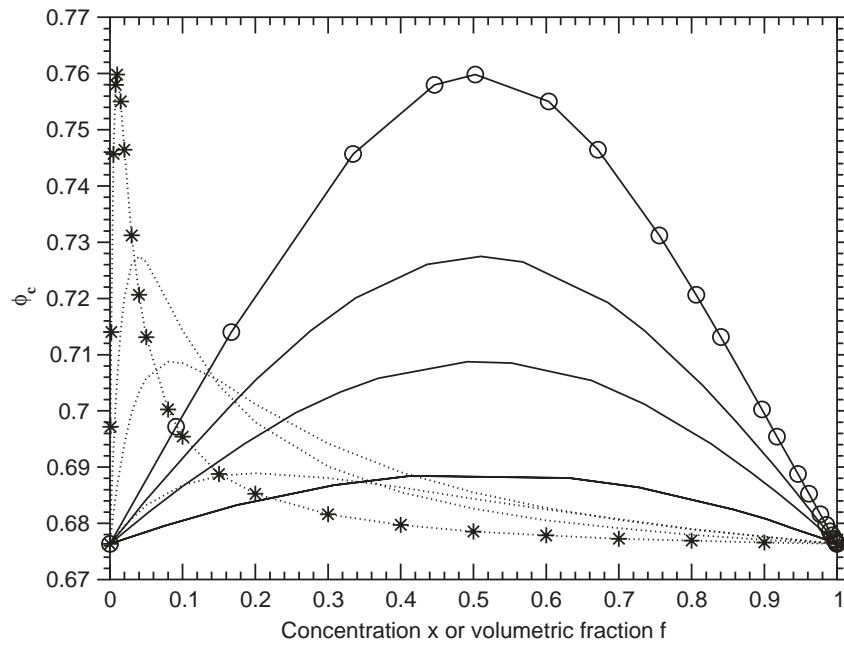


Fig. 1. Results from Ref. [5] for the percolation threshold for 2D disks versus both concentration  $x$  of larger disks (dotted line) and versus volumetric fraction  $f$  (solid line) for values of the ratio  $\lambda$  of the radii (from top to bottom) = 0.1, 0.2, 0.3 and 0.5.

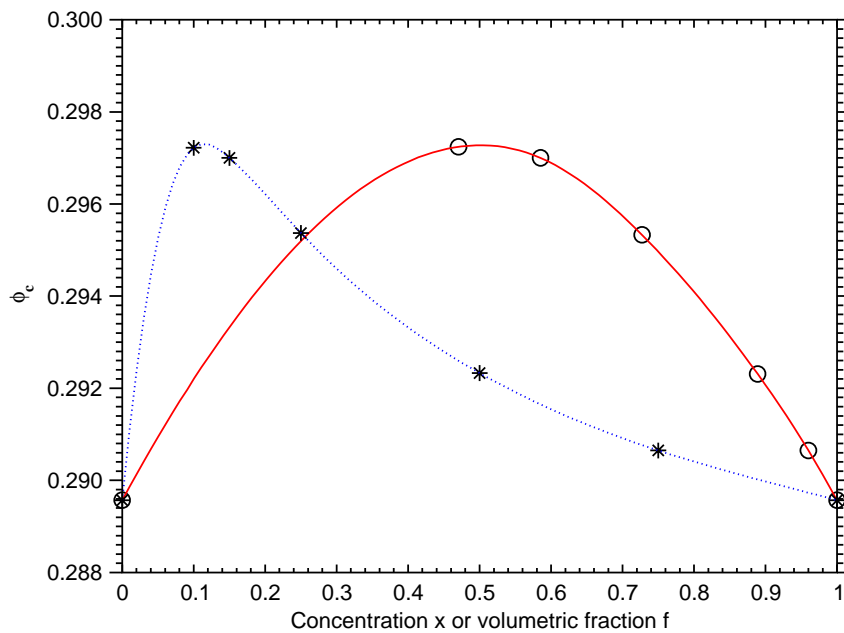


Fig. 2. Results from Ref. [6] for the percolation threshold for 3D spheres versus both concentration  $x$  of larger spheres (dotted line) and versus  $f$  (solid line) for the volume ratio  $r$  of the spheres = 0.125.

where  $\lambda$  is the ratio between the diameter of disks or spheres, and  $d$  is the dimension of objects.

In Ref. [5], it was noted that for small  $\lambda$ , the percolation threshold was maximized near  $x = 1 - \lambda^{-2}$ . Expanding in a Taylor series, this is a first approximation to our result in Eq. (17).

#### 4. Discussion and conclusions

We have shown that when considered as a function of the variable  $f$ , the percolation threshold exhibits the symmetry

$$\eta_c(f, r) = \eta_c(1 - f, r). \quad (18)$$

For  $f = 0$  and 1, this is just a statement of the fact that for systems with objects of only one size, the percolation threshold is independent of the size of the object.

Awareness of this symmetry has the following benefits:

(i) The position of the maximum of the percolation threshold is now known to be at exactly  $f = 1/2$  for any  $r$ .

(ii) Full knowledge of the percolation threshold is obtained by performing simulations only for  $f \in [0, \frac{1}{2}]$  or  $f \in [\frac{1}{2}, 1]$  whichever is computationally easier.

Subjects for further study are the effects of shape and the distribution of orientations on the critical threshold for systems composed of two different types of objects.

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#### References

- [1] S.R. Broadbent, J.M. Hammersley, Proc. Cambridge Philos. Soc. 53 (1957) 629.
- [2] D. Stauffer, A. Aharony, Introduction to Percolation Theory, Revised 2nd Edition, Taylor and Francis, London, 1994.
- [3] M. Sahimi, Applications of Percolation Theory, Taylor and Francis, London, 1994.
- [4] A. Bunde, S. Havlin, Fractal and Disordered Systems, 2nd Edition, Springer, Berlin, 1996;  
D. Ben-Avraham, S. Havlin, Diffusion and Reactions in Fractals and Disordered Systems, Cambridge University Press, Cambridge, 2000.
- [5] J. Quintanilla, Phys. Rev. E 63 (2001) 061 108.
- [6] R. Consiglio, D.R. Baker, G. Paul, H.E. Stanley, Phys. A 319 (2003) 49–55.
- [7] S. Chandrasekhar, Rev. Mod. Phys. 15 (1943) 1.
- [8] C.D. Lorentz, R.M. Ziff, J. Chem. Phys. 114 (2001) 3659.