First-order branching in diffusion-limited aggregation

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Diffusion-limited-aggregation (DLA) deposits with no branching ("zeroth-order branching DLA") resemble a "forest of whiskers;" the fractal dimension d_f of this forest is significantly less than that of normal DLA deposits. We study a generalization of this whisker model in which only first-order branching is allowed: attachment to the sides as well as the tips of the whiskers is permitted. Using various analysis methods, we find that d_f may be only very slightly smaller than d_f for conventional DLA (the full infinite-order branching case).

I. INTRODUCTION

Considerable interest has recently developed in the study of diffusion-limited aggregation (DLA).[1] One starts with a seed particle and releases random walkers in a d-dimensional space from a large hypersphere centered on the seed. Among the variants of pure DLA is the surface-deposition model. Here one begins with not a "zero-dimensional" seed particle, but with a one-dimensional line of seed particles. (In general, one can study hypersurfaces of seed particles provided the hypersurface has a dimension smaller than d.) Although surface deposition models are relevant to many physical systems, they have not been fully explored nor fully understood to date.[2,3] For this reason much recent interest has focused on a simpler variation of the surface-deposition model, a model in which growth is only allowed to occur on the main trunk of a cluster. This model, called DLA without branching, or *zeroth-order branching*, produces deposits that resemble a forest of whiskers.[4] For this model, Rossi[5] was able to derive a systematic renormalization-group treatment while Cates[6] solved the mean-field equations exactly. The fractal dimension of DLA without branching is significantly smaller than that of the random regular DLA deposition. In this paper, we study numerically a model intermediate between DLA without branching and regular DLA deposition.

II. FIRST-ORDER BRANCHING MODEL

We define the zeroth-order skeleton of the cluster as the set of sites which can be connected to the original base line by a straight line (i.e., the "trunks" of the trees). We define zeroth-order branching as the arrival of a random walker on the top of a tree, the random walker coming from above (see Fig. 1). The first-order branches are the set of points which do not belong to the zeroth-order skeleton, and can be connected to the original base line by a path on the cluster consisting of two straight lines.

We say that we have first-order branching when a random walker touches the trunk of a tree by the sides or when it hits a first-order branch by the end (see Fig. 1). Second- and higher-order branching is defined in a simi-

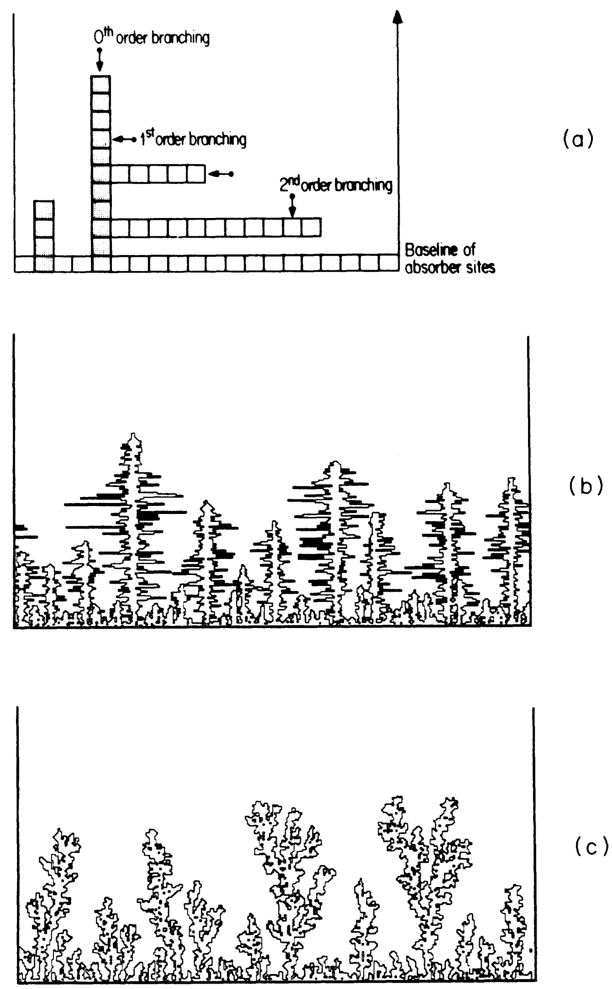

FIG. 1. (a) Definition of zeroth-order skeleton (in black). Also shown are the additional sites needed for first-order skeleton. The arrows indicate how different orders of branching are defined. (b) Typical simulation of the first-order branching case, with $L=1001$ and $N=40\,000$, compared with (c) the infinite-order branching case (DLA) with the same values of L and N.

lar way. Infinite-order branching is a conventional DLA deposit. We use the DBM boundary condition.[6] If the movement of the random walker corresponds to zeroth- or first-order branching, then when a random walker first touches a *cluster* site, the previously visited *perimeter* site becomes part of the cluster. However, if the movement of the random walker corresponds to higher-order branching, the random walker is reflected back to its original position and another step is taken.

III. NUMERICAL SIMULATIONS AND RESULTS

Periodic boundary conditions are assumed on the sides of the box. Walkers are launched from a point located just one lattice site above the height of the cluster but of random abscissa. Walks are performed in the continuum until the walker comes to a distance R smaller than some number R_{min} from the cluster; a walk is then performed on a lattice (square lattice).[7,8] If the random walker goes beyond a certain distance R_{max} from the cluster, then the walker is reset on the continuum. If the random walker goes beyond some very large distance from the base line, it is killed (typically we chose this distance to be 1000 times the height of the tallest tree). A typical width is $L=1001$ lattice units, and typically we deposit $N=10^4-10^5$ particles.

We are interested in the density profile $\rho(y)$, the number of cluster sites between height y and $y+1$. For one sample, the density profile looks very "noisy," so we average on many samples. After an anomalous region very close to the surface, we find a power-law decay of the density

$$\rho(y) \sim y^{-\alpha} , \tag{1}$$

followed by a plateau corresponding to the density of one finger, and in the end a falloff. When N, the number of particles deposited, is not at least two orders of magnitude larger than the width L, the average density profile has the shape shown in Fig. 2(a) ($N=56\,000$, $L=1001$, 87 samples). As N increases, eventually, one finger outgrows the others so that the density profile will have a plateau as shown in Fig. 2(b) ($N=13\,250$, $L=121$ 100 samples).

The exponent α is simply related to the fractal dimension d_f of the forest by

$$\alpha = d - d_f . \tag{2}$$

We have determined the exponent α in a situation where $L=1001$, $N=56\,000$; the average has been taken over 120 samples. We got the result $\alpha=0.34\pm0.04$. This gives $d_f=1.66\pm0.04$, very close to the value for infinite ordinary branching DLA deposit.

However, we know that the lattice has a dramatic effect on regular DLA. The very small difference in our numerical simulations between first-order and infinite-order DLA deposition could be due to the fact that the fractal dimension of first-order DLA converges quicker to its lattice value than the one of ordinary DLA does.[4] Simulations done at widths L of 121 and 301 give approximately the same values of d_f. For $L=301$ and for $L=121$, we got $\alpha=0.34\pm0.04$ (see Fig. 3).

To see if d_f for first-order branching is significantly less than d_f for infinite-order branching (ordinary DLA), we calculated $\rho(y)$ for ordinary DLA. From Fig. 4, we see that $d_f^{(\infty)}=1.72\pm0.015$, 3–4% larger than d_f for first-order branching but only just barely outside the error bars $d_f^{(1)}=1.66\pm0.05$.

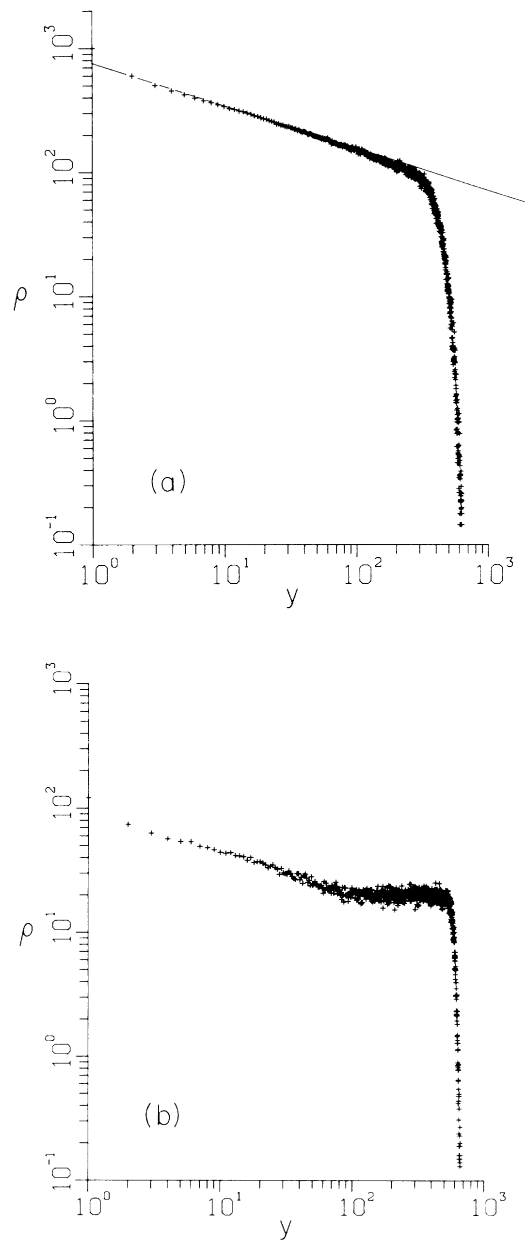

FIG. 2. (a) Log-log plot of the density ρ vs y, the height from the base line, after averaging over 187 samples of mass $N=56\,000$, for a strip of width $L=1001$ lattice units. The solid line is the linear fit in the scaling region, yielding $d_f=1.66\pm0.04$. (b) Log-log plot of the density ρ vs y, after averaging over 100 samples of mass $N=13\,250$ for a strip of width $L=121$. A plateau corresponding to one single finger outgrowing the others can clearly be seen.

We also studied other quantities such as the rms thickness defined as

$$T \equiv \frac{1}{N} \sum_{i=1}^{N} y_i^2 , \tag{3}$$

where the sum extends over all the cluster sites i and y_i denotes the height of site i from the base line. It has been shown[3] that the scaling of T can be related to the fractal dimension,

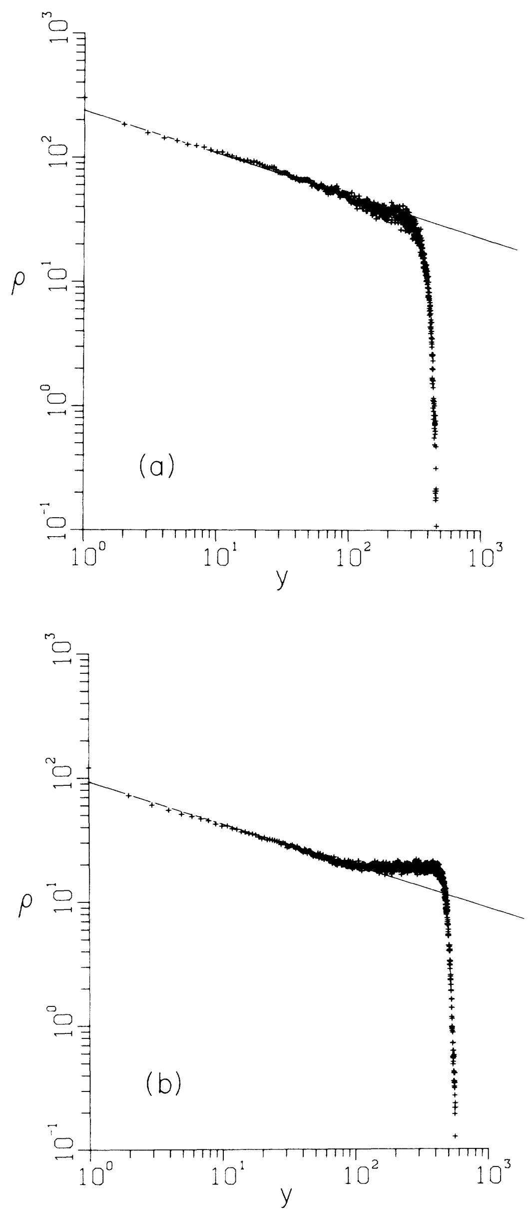

FIG. 3. Log-log plot of the density ρ vs y after averaging over 120 samples of mass $N = 17\,801$ for a strip of width $L = 301$. The linear fit to experimental points in the scaling region gives $d_f = 1.66 \pm 0.04$. (b) Log-log plot of ρ vs y after averaging over 300 samples of mass $N = 10\,000$ for a strip of width $L = 121$. The linear fit gives $d_f = 1.66 \pm 0.03$.

$$T \sim N^{\epsilon} , \tag{4}$$

with

$$\frac{1}{\epsilon} = d_f - (d-1) . \tag{5}$$

We measured the rms thickness for a width of $L = 1001$ as a function of N up to $N = 56\,000$, and the results were averaged on 20 samples [see Fig. 5(a)]. We tried to fit our data with the form

$$\langle T \rangle \sim A N^{\epsilon} (1 + B N^{-\Delta}) \tag{6}$$

to include corrections to scaling, where $\langle T \rangle$ denotes the average of T over 20 realizations. A plot of $d \log_{10} \langle T \rangle / d \log_{10} N$ versus $N^{-\Delta}$, with $\Delta = 0.6$ was found to be approximately a straight line. From the ordinate at the origin we obtained $\epsilon = 1.59 \pm 0.04$ [see Fig. 5(b)]. This in turn gives $d_f = 1.63 \pm 0.02$ by relation (5).

We have also looked at the distribution of cluster sizes. Let us denote by $n_s(N)$ the number of clusters of size s; the scaling form

$$n_s(N) = s^{-\tau} f(s^{\sigma}/N) \tag{7}$$

has been proposed in Ref. 3. Here σ and τ both depend *only* on the fractal dimension, since

$$\sigma = 2 - \tau \tag{8}$$

and

$$\tau = 1 + (d-1)/d_f . \tag{9}$$

To determine σ and τ, we looked at the moments of

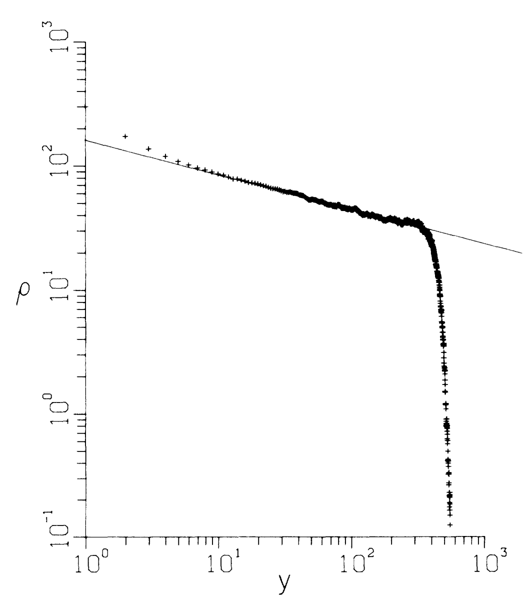

FIG. 4. Log-log plot of density ρ vs height y for ordinary DLA deposition on a strip of width $L = 1001$, after averaging over 200 samples. We find $d_f^{(\infty)} = 1.720 \pm 0.015$.

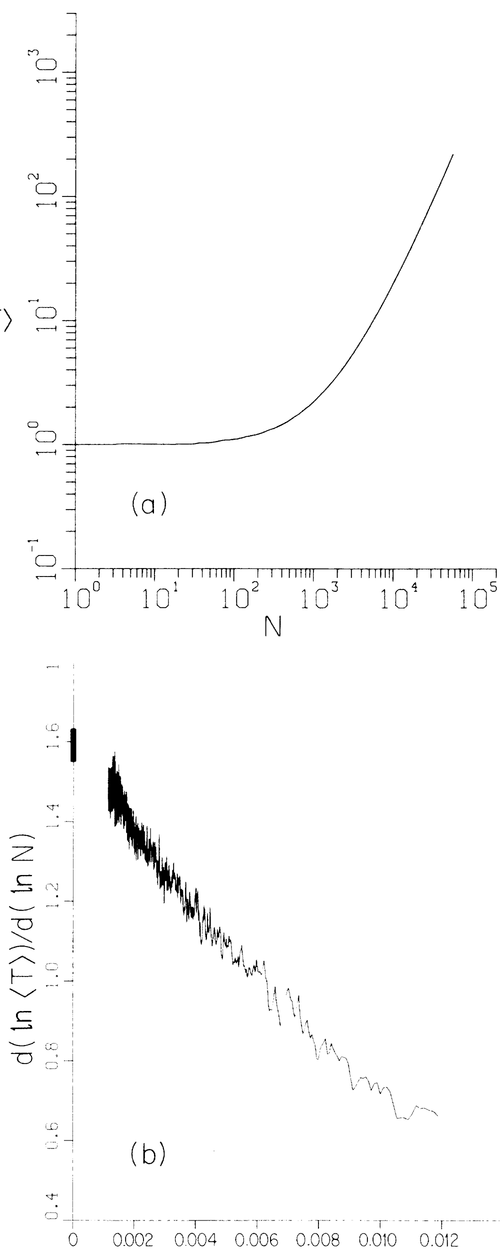

FIG. 5. (a) Plot of $\log_{10}\langle T \rangle$ vs $\log_{10}N$ for $N = 56\,000$ and $L = 1001$, averaging on 20 samples. For $N < L$, $\langle T \rangle$ is independent of N, while for $N \gg L$, $\langle T \rangle \sim N^{\epsilon}$. For fixed L, as $N \to \infty$, since one tree outgrows the others, we study how the rms thickness of this tree scales with its mass. Thus, for $N \to \infty$, we measure the fractal dimension of a single tree. Before one single tree emerges from the forest, the power-law decay of the density ρ gives the fractal dimension d_f of the forest. For example, the fractal dimension d_f^* of a single tree in "DLA without branching" *is* unity, while the fractal dimension d_f of the forest is greater than 1 (see Refs. 4 and 5). (b) Plot of $d\log_{10}\langle T \rangle / d\log_{10}N$ vs $N^{-\Delta}$ with $\Delta = 0.6$, for a strip of width $L = 1001$, where $\langle T \rangle$ is the thickness averaged on 20 samples. The solid line is a fit to experimental points in the range $9900 \le N \le 56\,000$. The extrapolation to the limit of infinite mass gives $\epsilon = 1.59 \pm 0.04$, corresponding to $d_f = 1.63 \pm 0.02$.

the cluster size distribution

$$M_p \equiv \sum_{s \ge 1} s^p n_s(N) \ . \tag{10}$$

Equation (8) arises from the normalization condition, $\sum_s s n_s = N$. In principle, the study of the behavior of one moment M_p with $p \neq 1$ should be sufficient to determine τ, since

$$M_p \sim N^{(p+1-\tau)/\sigma} \sim N^{(p+1-\tau)/(2-\tau)} \ . \tag{11}$$

Thus we get independent estimates of τ by looking at different moments M_p.

Figure 6(a) shows a log-log plot of $\langle M_2 \rangle$ versus N, where $\langle M_2 \rangle$ is the value of M_2 averaged on 180 samples. We also plotted $d\log_{10}\langle M_2 \rangle / dN$ versus $1/N$, since an extrapolation of $d\log_{10}\langle M_2 \rangle / dN$ to the limit $N \to \infty$ should give the exponent of $\langle M_2 \rangle$, $(3-\tau)/(2-\tau)$ [see Fig. 6(b)]. There are, however, three regimes.

Regime I: $N \le 12\,000$. In this regime, the mean thickness is still not appreciably greater than 1. We do not expect to attain the scaling law (11).

Regime II: $12\,000 \le N \le 30\,000$. Here, in addition to small trees, there are also many large trees competing. The scaling law (11) is expected to hold.

Regime III: $N \ge 30\,000$. There begins to be one large tree whose size is no longer negligible with respect to the total width L. This tree will eventually capture almost all the random walkers. In the extreme regime $N \gg L$, which is not attained in our simulations, the sum $\sum_s s^2 n_s(N)$ will be dominated by the largest cluster whose size is approximately equal to N. Thus $\langle M_2 \rangle \sim N^2$. We find that for $N \ge 30\,000$, approximately, (9) ceases to be valid in our system, owing to the fact that $L = 1001$ only. We fit the data of Fig. 6(b) in the range $12\,000 \le N \le 29\,800$ to a straight line. Though the range is quite small, the ordinate at the origin gives $(3-\tau)/(2-\tau) = 3.56 \pm 0.1$. From this follows $\tau = 1.61 \pm 0.02$ and $d_f = 1.64 \pm 0.04$.

We can do the same analysis for $\langle M_3 \rangle$ (see Fig. 7). We fit the data to a straight line in the range $12\,100 \le N \le 30\,600$ and obtained for the exponent of $\langle M_3 \rangle$, $(4-\tau)/(2-\tau) = 6.1 \pm 0.2$. This gives $\tau = 1.61 \pm 0.02$ and $d_f = 1.65 \pm 0.05$.

IV. CONCLUSION

We have shown that a modification of DLA without branching by allowing only first-order branching gives an apparent fractal dimension very close to that of a conventional "infinite-order" DLA deposit. The reason might be that in two dimensions, trees with *only* first-order branches screen each other very much like DLA trees with infinite-order branching. The situation should be quite different in three dimensions where a random walker can more easily find his way around the first-order branches.

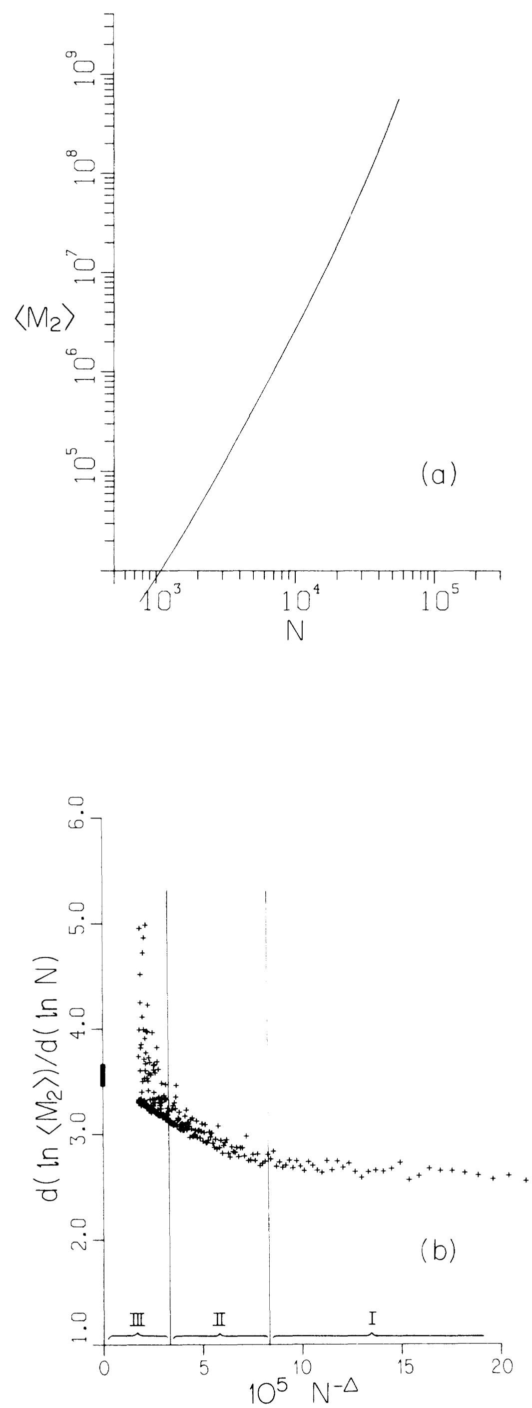

FIG. 6. (a) Log-log plot of the second moment of the distribution of cluster sizes $\langle M_2 \rangle$ vs mass N, after averaging on 180 samples of width $L=1001$. (b) Plot of $d\log_{10}\langle M_2 \rangle / d\log_{10}N$ vs $1/N^{\Delta}$ in the range $4887 \leq N \leq 56\,000$, with $\Delta=1$. Three different regions are observed: Region I, $N \leq 12\,000$; Region II, $12\,000 \leq N \leq 30\,000$; Region III, $N \geq 30\,000$. We considered the points in the range $12\,000 \leq N \leq 29\,800$ and made a fit to a straight line. The ordinate at the origin can be estimated to be 3.56 ± 0.1. This in turn yields $\tau = 1.61 \pm 0.02$, corresponding to $d_f = 1.64 \pm 0.04$.

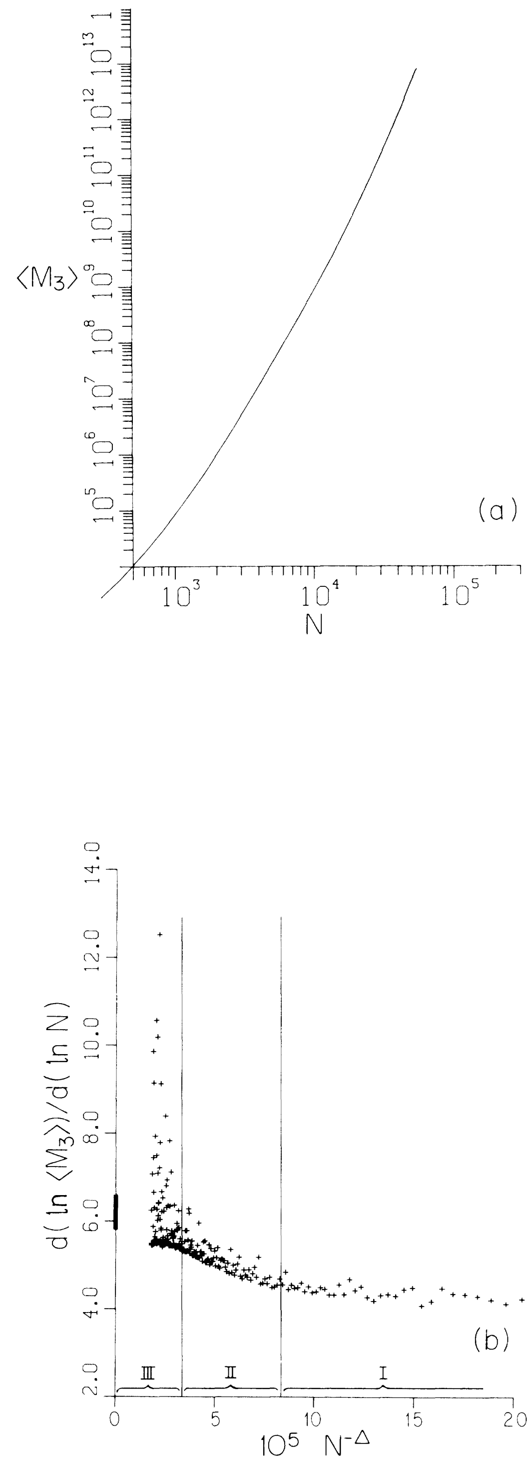

FIG. 7. (a) Log-log plot of the third moment of the distribution of cluster sizes $\langle M_3 \rangle$ vs mass N after averaging on 180 samples of width $L=1001$. (b) Plot of $d\log_{10}\langle M_3 \rangle / d\log_{10}N$ vs $1/N^{\Delta}$ in the range $4887 \leq N \leq 56\,000$, with $\Delta=1$. For the points in the range $12\,100 \leq N \leq 30\,600$ we made a fit to a straight line. The ordinate at the origin can be estimated to be 6.1 ± 0.2 giving $\tau = 1.61 \pm 0.02$, corresponding to $d_f = 1.65 \pm 0.05$.

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