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LETTER TO THE EDITOR

Continuum percolation in two dimensions: Monte Carlo tests of scaling and universality for non-interacting discs

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Abstract. Detailed results are reported for the connectivity properties of a system of discs of unit radius free to be situated *anywhere* within a square of area $2L^2$. Ordinary lattice percolation would correspond to the discs being situated on the vertices of a $\sqrt{2}L \times \sqrt{2}L$ lattice. Computer simulations are carried out for a sequence of increasing system sizes ranging from $L = 20$ to $L = 1000$; for each value of L a large number of realisations are generated for 25 values of the disc concentration x. We calculate a variety of estimates for the threshold parameter x_c, as well as the critical exponents β, γ, τ and ν. Our exponent estimates are in close agreement with accepted values for ordinary lattice percolation; therefore, this continuum system appears to be in the same 'universality class' as lattice percolation.

In recent years, lattice percolation has been used increasingly to model phenomena occurring in systems that are hardly lattice systems—such as polyfunctional condensation of monomers and cross-linking of polymers (see e.g., Conglio *et al* (1979) and references therein). However, virtually all of the calculations of sufficient accuracy to describe the quantities of interest are for lattice systems. In fact, the simple question 'Are the critical exponents for continuum percolation the same as for lattice percolation?' is yet to be answered. This is the question addressed in the present paper.

At first sight, one might imagine that continuum systems will conform to the behaviour predicted for lattice systems—for example, one might imagine that if we view a percolating lattice system from a sufficiently large distance, the lattice will blur and the lattice percolation clusters would be similar to continuum percolation clusters—and the fractal dimension $d_f = 2y_H - d$ of the system and the fractal dimension $d_f^+ = y_H$ of the incipient infinite cluster would be the same for both lattice and continuum systems (notation in Stanley (1977)). Intuitive arguments can be misleading, however, and in percolation one is often confronted with facts that may contradict initial intuitive guesses. Examples that come to mind include the facts that (a) exponents for percolation on a linear chain lattice with second-neighbour interactions are not equal to those for a chain with only nearest-neighbour interactions (Klein *et al* 1978), (b) the limit of infinitely long-range interactions appears not to be described by classical theory (Stauffer and Coniglio 1980, Stephen and Aharony 1981), and (c) the surface to volume ratio for large percolation clusters does not approach zero (Domb 1974, Stauffer 1975, Kunz and Souillard 1978, Coniglio and Russo 1979).

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In this work we consider the simplest prototype two-dimensional system: non-interacting discs. This system has been studied previously, but not with sufficient detail to permit accurate determination of critical exponents (see e.g. Ottavi and Gayda (1974), Pike and Seager (1974), Seager and Pike (1974), Fremlin (1976), Wintle and Puhach (1978)). First, we describe in some detail the system and the simulation method used, and then we present our new results. We conclude with a discussion and summary. We should note at the outset that after the completion of the present work, we received a preprint (Vicsek and Kertész 1981) discussing the same system from a different point of view, large-cell Monte Carlo position-space renormalisation group (Reynolds *et al* (1978, 1980) and references therein). Vicsek and Kertész estimate one critical exponent, ν, and conclude that within the accuracy of their method ν agrees with results from lattice studies. Also, Shalitin (1981) has very recently demonstrated that universality holds exactly for continuum percolation in one dimension.

We now describe our continuum system. It consists of N discs of unit radius, randomly distributed over an area $\sqrt{2}L \times \sqrt{2}L$. Two discs are said to be in the same cluster if their centres are separated by less than one diameter. To simulate the infinite system we chose $L = 1000$ and ran 25 realisations at each of 25 concentrations $x = N/2L^2$ in the range $0.65 < x < 0.83$. For the finite-size scaling tests we fixed the concentration at x_c (as determined from the $L = 1000$ simulation) and ran $\rho(L)$ realisations for six additional values of L, namely, $\rho(L = 20) = 8000$, $\rho(40) = 4000$, $\rho(80) = 2000$, $\rho(160) = 1000$, $\rho(320) = 500$, and $\rho(640) = 250$. We also examined these same system sizes at 25 concentrations away from x_c in the same range as given above for $L = 1000$. However only 20% as many realisations were run for these concentrations as were run at x_c.

The simulation algorithm is an extension to the continuum of the cluster multi-labelling technique of Hoshen and Kopelman (1976). An essential idea (Nakanishi 1980) is to overlay an 'imaginary covering mesh' onto the $\sqrt{2}L \times \sqrt{2}L$ area, each mesh cell being square and of area two (figure 1). For each cell we select a random number ξ_0 between 0 and 1. We then determine an integer D by requiring

$$\sum_{k=0}^{D+1} P(x, k) > \xi_0 > \sum_{k=0}^{D} P(x, k) \tag{1a}$$

where

$$P(x, k) = (\mathrm{e}^{-2x})(2x^k)/k! \tag{1b}$$

in the Poisson distribution. We then place D discs in the current cell, with their positions *within the cell* chosen randomly. Note that the statistics of the spatial distribution of discs thus obtained are identical to the statistics of a distribution of discs whose positions are randomly chosen to be *anywhere* in the $\sqrt{2}L \times \sqrt{2}L$ area, and hence the 'imaginary covering mesh' *does not distort the continuum nature of the simulation.*

We construct a cluster label tree by extending to the continuum the lattice algorithm of Hoshen and Kopelman (1976). The continuum algorithm begins by placing a pointer in the upper left corner 'imaginary mesh cell.' Since the diagonal of a mesh cell equals the disc diameter, two discs can overlap only if their centres lie within the fourth-neighbour shell of the cell in question. Hence the algorithm proceeds cell by cell, column by column, searching *previously visited* cells that are either first, second, third or fourth neighbours (these cells are shown shaded in the example of figure 1(*b*)). Let us colour green all discs in the 'current cell' (indicated by an X in figure 1(*b*)), and let us colour red all discs in the ten shaded mesh cells. For each green disc, the algorithm

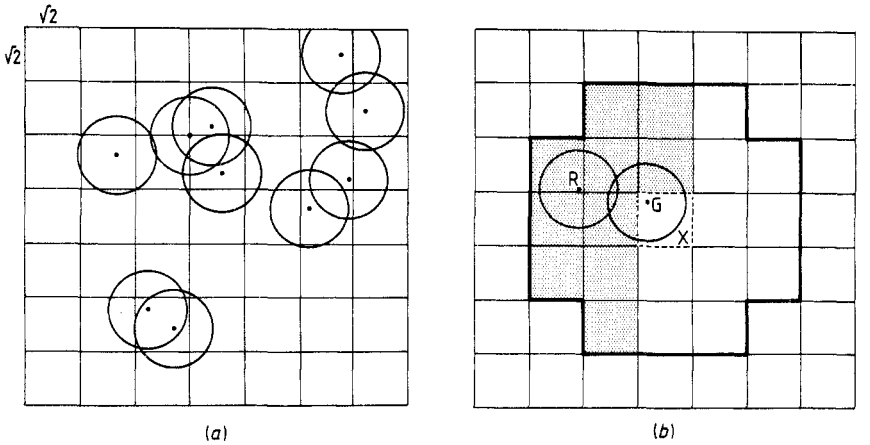

Figure 1. (*a*) Typical configuration of ten discs situated in a $\sqrt{2}L \times \sqrt{2}L$ area with $L = 7$ at a concentration below the percolation threshold x_c. The ten discs shown form three clusters, two four-disc clusters and one two-disc cluster. Note that the diagonal of the covering mesh equals the disc diameter, so that two discs can overlap if and only if their centres are either in the same cell or in two different mesh cells that are no further than fourth-nearest neighbours in separation. (*b*) Schematic illustration of the algorithm used. Two discs are shown in this example. Disc G, coloured 'green', is in the cell where the pointer is. Disc R, coloured 'red', is in a cell which is the fourth-nearest neighbour of cell x. The algorithm calculates the centre-to-centre distance d between disc G and disc R. Since $d < 2$, disc G and disc R belong to the same cluster.

calculates the centre-to-centre distances to all the red discs in order to ascertain which red discs overlap the green disc. This procedure is repeated for all the green discs (which automatically belong to the same cluster†), and then the pointer moves on to a new 'current cell.'

The algorithm was implemented as a Fortran program. The execution time was 310 μs per disc and the total CPU time expended was 80 hours on an IBM 370/168 computer.

We now give the results for β, γ and τ. To simulate the infinite system we consider the $L = 1000$ simulation. From the cluster size distribution for this case we determine the 'magnetisation' exponent β, the 'susceptibility' exponent γ, and their associated amplitudes. In figure 2 we show log–log plots of the functions $P(x, L = 1000)$ and $S(x, L = 1000)$ against $|x - x_c|$. Here $P(x, L)$ is the probability of any disc chosen at random belonging to the infinite cluster (here taken to be the largest cluster), $S(x, L) = \Sigma_s' s^2 n(s, x)/\Sigma_s' sn(s, x)$ is the second moment of the cluster number distribution $n(s, x)$, and x_c is the critical concentration. The prime on the summation denotes the omission of the largest cluster when $x > x_c$. The critical behaviour of these two functions in the vicinity of the percolation threshold is described by power law singularities

$$P(x, \infty) = B|x - x_c|^{\beta}, \tag{2a}$$

$$S(x, \infty) = C_+|x - x_c|^{-\gamma}. \tag{2b}$$

The exponents β and γ appear to depend on the value of x_c chosen in (2*a*) and (2*b*). To determine the true value of x_c we treat it as an adjustable parameter and determine

† This would not be true if the disc diameter were smaller than 2. Were the disc diameter larger than 2, then further-neighbour mesh cells would have to be searched. Thus the choice 2 is 'optimal.'

its value from the best correlation between the raw data and the functional forms in (2*a*) and (2*b*)†. In this way we find

$$x_c = 0.718 \pm 0.003 \tag{3}$$

which is in good agreement with previous estimates‡ $x_c = 0.715$ (Pike and Seager 1974), $x_c = 0.70 \pm 0.03$ (Fremlin 1976) and $x_c = 0.73 \pm 0.05$ (Haan and Zwanzig 1977), but not with the estimates $x_c = 0.742 \pm 0.010$ (Vicsek and Kertész 1981) and $x_c = 0.65 \pm 0.03$ (Ottavi and Gayda 1974). We also note the earlier work of Roberts (1967) who finds $x_c = 0.62$, and that of Domb (1972) who finds $x_c = 0.72$ from the study of percolation on regular lattices with long-range interactions. From the least-squares fit to the raw data of figure 2 we find

$$\beta = 0.14 \pm 0.02, \qquad B = 1.4 \pm 0.1, \tag{4a}$$

$$\gamma = 2.43 \pm 0.04, \qquad C_+ = 1.0 \pm 0.1. \tag{4b}$$

The 'error bars' have been propagated down from the errors in the individual data points, which in turn are estimated by $\sigma/\sqrt{\rho}$ where σ is one standard deviation and ρ is the number of realisations. Both exponents in (4) agree well with the estimates for random site lattice percolation; cf the series estimates $\beta = 0.138 \pm 0.007$ (Sykes *et al* 1976b) and $\gamma = 2.43 \pm 0.03$ (Sykes *et al* 1976a) and the Monte Carlo position-space renormalisation group estimates $\beta = 0.138^{+0.006}_{-0.005}$ (Reynolds *et al* 1978, 1980) and $\gamma = 2.432 \pm 0.035$. Our continuum values also are consistent with the values $\beta = \frac{5}{36} = 0.138\,88\ldots$ and $\gamma = \frac{43}{18} = 2.3888\ldots$ obtained from the den Nijs (1979) conjecture $y_T = \frac{3}{4}$ (see also Black and Emery (1981)) and the 'extended' conjecture $y_H = \frac{91}{48}$ (Pearson 1980, Nienhuis *et al* 1980).

We also studied the critical behaviour of $S(x, L = 1000)$ for $x > x_c$. Our results

$$\gamma' = 2.31 \pm 0.19, \qquad C_- = 0.02 \pm 0.01, \tag{5}$$

are rather uncertain due to the fact that $S(x, L)$ decreases rapidly as x increases above x_c. The amplitude ratio $C_+/C_- = 50 \pm 26$ is smaller than for lattice percolation (see e.g. Nakanishi and Stanley (1980) and references therein).

Finally, we estimate the exponent $\tau = 2 + 1/\delta$, defined through $n(s, x_c) \sim s^{-\tau}$. We find $\tau = 2.0 \pm 0.1$, consistent with lattice percolation estimates (cf, e.g., Stauffer 1979, Pike and Stanley 1981).

We now give the results for ν. To calculate the connectivity length exponent ν, we turn to the finite-size scaling approach (Fisher 1971, Levinshtein *et al* 1975, Sur *et al* 1976). First we consider the finite-size scaling behaviour of the functions $S(x, L)$ and $P(x, L)$ for $x = x_c$,

$$P(x_c, L) \sim L^{-\beta/\nu}, \tag{6a}$$

$$S(x_c, \mathrm{L}) \sim \mathrm{L}^{\gamma/\nu}. \tag{6b}$$

† For each trial value of x_c we pick a succession of data 'windows' of various widths and centres. These data windows are then least-squares fitted to a straight line characterised by a linear correlation coefficient r and a parameter χ^2. We next calculate $P_1(r, N)$, the probability that a completely random distribution of N points in the xy plane has a linear correlation coefficient greater than or equal to r. We also calculate the analogously defined function $P_2(\chi^2, N)$. We choose the window that minimises P_1 and maximises P_2.

‡ Note that some authors report the critical area fraction ϕ_c, which is easily related to x_c. For discs of radius r and concentration of disc centres x, the Poisson distribution (1*b*) gives $\exp(-ax)$ for the probability of any point chosen at random *not* lying within a circle of radius r, with area $a = \pi r^2$. Hence the area fraction ϕ is simply given by $\phi = 1 - \exp(-\pi r^2 x)$. For $x_c = 0.718 \pm 0.003$, $\phi_c = 0.676 \pm 0.002$.

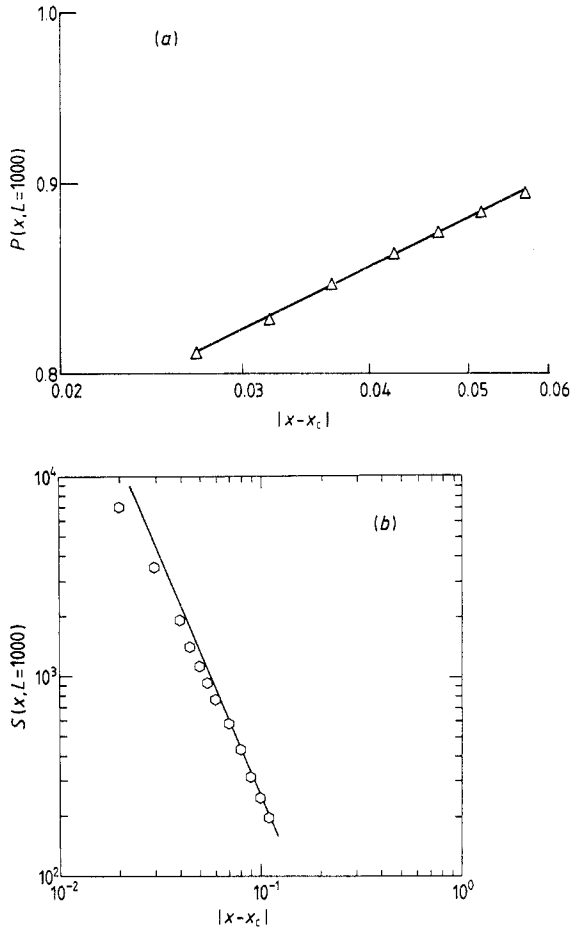

Figure 2. Double logarithmic plot showing the dependence on $|x - x_c|$ of (*a*) $P(x, L = 1000)$ and (*b*) $S(x, L = 1000)$. The linear fit in (*b*) includes only the bottom five data points. The deviation of the remaining points from the line is a finite-size effect.

In figure 3 we show the dependence on lg L of $\lg[P(x_c, L)]^5$ and lg $S(x_c, L)$ respectively. From the slopes of the straight lines we estimate

$$\beta/\nu = 0.127 \pm 0.025, \tag{7a}$$

$$\gamma/\nu = 1.75 \pm 0.15. \tag{7b}$$

Using the results from (4) for β and γ, we find two independent estimates for ν,

$$\nu_1 = 1.10 \pm 0.27, \tag{8a}$$

$$\nu_2 = 1.39 \pm 0.12. \tag{8b}$$

There is an alternative method of estimating ν that does not depend on our prior estimates for β and γ (Roussenq *et al* 1976, Blumberg *et al* 1980). We define $\pi(x, L)$ as the fraction of realisations of a $\sqrt{2}L \times \sqrt{2}L$ system which span at a disc concentration x. In figure 4(*a*) we plot $\pi(x, L)$ for $L = 20, 40, 80, 160$ and 320. With increasing cell size, $\pi(x, L)$ approaches a step function with $\pi(x, L) = 0$ for $x < x_c$ and $\pi(x, L) = 1$ for $x > x_c$.

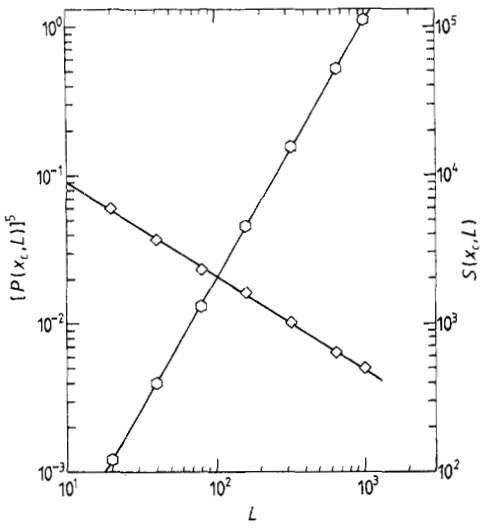

Figure 3. Double logarithmic plot showing the dependence on L of $S(x_c, L)$ and $[P(x_c, L)]^5$. Finite-size scaling, equation (6), predicts that the data should be linear with slopes γ/ν and $-5\beta/\nu$ respectively. ⊙ $S(x_c, L)$; ◇ $[P(x_c, L)]^5$.

For each value of L, we fit the curve $\pi(x, L)$ with a fourth-order polynomial which we denote $\tilde{\pi}(x, L)$. We define $x_f(L)$ and $x_{1-f}(L)$ as the disc concentrations at which the fractions f and $1-f$ of all realisations span (i.e. $\tilde{\pi}(x_f, L) = g$ and $\tilde{\pi}(x_{1-f}, L) = 1-f$). We expect from finite-size scaling theory

$$x_f(\infty) - x_f(L) \sim L^{-1/\nu}, \tag{9a}$$

$$x_{1-f}(L) - x_{1-f}(\infty) \sim L^{-1/\nu'}. \tag{9b}$$

Assuming $\nu' = \nu$ and $x_{1-f}(\infty) = x_f(\infty) = x_c$, we have

$$x_{1-f}(L) - x_f(L) \sim L^{-1/\nu}. \tag{10}$$

Equation (10) implies that the double logarithmic plots of figure 4(b) should be straight lines with slope $-1/\nu$, and least-squares fits to the curves shown ($f = 0.1$, 0.2, 0.3 and 0.4) give four estimates ν_f that increase monotonically with f. Repeating this analysis for a large number of values of f in the range $0.10 \leq f \leq 0.46$ leads to a smooth curve of ν_f against f (figure 4(c)) that may be readily extrapolated to $f = \frac{1}{2}$. From these considerations we estimate

$$\nu_3 = 1.35 \pm 0.02. \tag{11}$$

To determine a final estimate of ν we perform a weighted average of ν_1, ν_2 and ν_3 where the weight of each value is the reciprocal of its uncertainty. From (8a), (8b) and (11) we thereby find

$$\nu = 1.343 \pm 0.019, \tag{12}$$

comparable to the den Nijs (1979) conjecture $\nu = \frac{4}{3}$, with a variety of estimates for the lattice percolation system in the range 1.32–1.36, based on Monte Carlo renormalisation group (Reynolds *et al* 1978, 1980, Eschbach *et al* 1981, Blöte *et al* 1981),

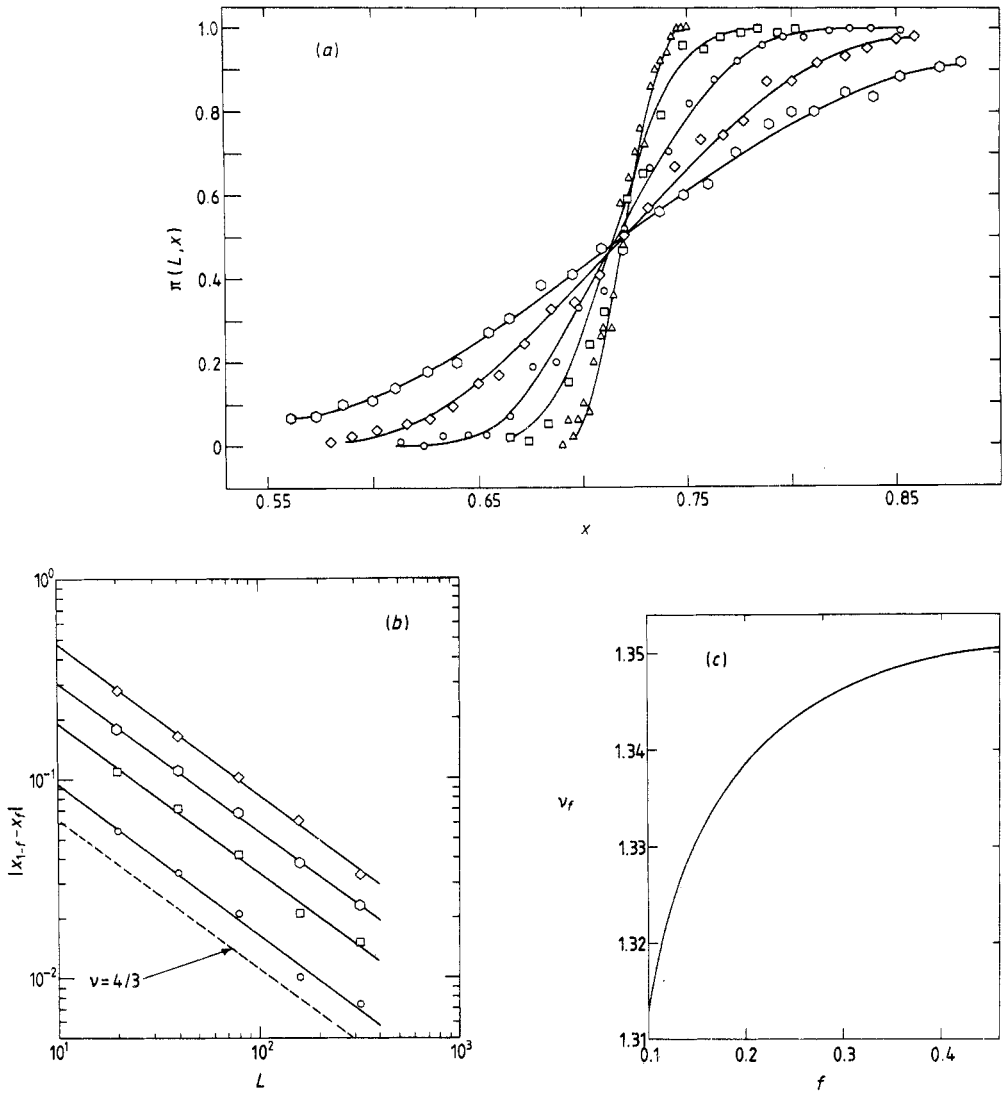

Figure 4. (*a*) Dependence on x of $\pi(x, L)$ for $L = 20$, 40, 80, 160 and 320. ⊙ $L = 20$; ◇ $L = 40$; ○ $L = 80$; □ $L = 160$; △ $L = 320$. (*b*) Double-logarithmic plot showing dependence on L of $|x_f(L) - x_{1-f}(L)|$ for $f = 0.1$, 0.2, 0.3 and 0.4. Finite-size scaling (equation (10)) predicts that the data are linear with slope $-1/\nu$. ◇ $f = 0.1$; ○ $f = 0.2$; □ $f = 0.3$; ○ $f = 0.4$. (*c*) Dependence on f of the estimate for ν obtained by the procedure illustrated in part (*b*). As $f \to 0.5$, these estimates should converge to the true value of ν. In light of the smoothness of the curve in figure 4(*c*), one may question the apparently large uncertainty in ν_3 of equation (11). Note that ν_f is determined from the polynomial fits $\tilde{\pi}(x, L)$; thus the statistical uncertainty of the raw data has been smoothed out and replaced by uncertainties in the fitting parameters.

series (Dunn *et al* 1975), finite-size scaling (Reynolds *et al* 1978, 1980) and with the value 1.33 ± 0.07 (Vicsek and Kertész 1981) for the continuum percolation system.

In summary, then, we have performed the first Monte Carlo computer simulations of sufficient accuracy to obtain estimates of the continuum percolation exponents (cf

table 1). We can thereby (i) test the accuracy of the scaling hypothesis for continuum percolation, and (ii) test whether continuum percolation belongs to the same universality class as the traditional lattice percolation.

Table 1. Comparison of the results of the present work with previous work on continuum percolation and lattice percolation.

	Method	x_c	β	γ	ν	τ
Present work	a	0.718 ± 0.003	0.14 ± 0.02	2.43 ± 0.04	1.343 ± 0.019	2.0 ± 0.1
Vicsek and Kertész (1981)	b	0.742 ± 0.010	—	—	1.33 ± 0.07	—
Haan and Zwanzig (1977)	c	0.73 ± 0.05	—	2.45 ± 0.10	—	—
Fremlin (1976)	a	0.70 ± 0.03	—	—	—	—
Pike and Seager (1974)	a	0.715	—	—	—	—
Ottavi and Gayda (1974)	a	0.65 ± 0.03	—	—	—	—
Domb (1972)	c	0.72	—	—	—	—
Roberts (1967)	c	0.62	—	—	—	—
Lattice	d	—	0.139	2.389	1.333	2.055

(a) Computer simulation.
(b) Monte Carlo position-space renormalisation group.
(c) Series expansions.
(d) The numbers shown are based on the extended den Nijs conjecture, with which most numerical estimates are consistent.

(i) *Scaling.* We have calculated four critical point exponents β, γ, τ and ν. Using our values we find $(2\beta + \gamma)/2\nu = 1.01 \pm 0.04$, which agrees with the prediction of scaling that the right-hand side be exactly unity. Also, we found agreement with the predictions of finite-size scaling theory for the dependence of percolation functions on L.

(ii) *Universality.* Our estimates for β, γ, τ and ν are all within the accepted range of estimates for the corresponding exponents for lattice percolation.

The essential idea of overlaying an 'imaginary covering mesh' is due to H Nakanishi, who initiated this project by writing a computer program for non-interacting squares. We also thank him, S Redner, G Shlifer and A Gonzalez for helpful discussions throughout this research. D Stauffer, F Family, P J Reynolds, D Shalitin, R Pike, H Gould, T Vicsek and J Kertész kindly offered constructive criticism on the manuscript.

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