

Generalized Scaling Hypothesis in Multicomponent Systems. II. Scaling Hypothesis at Critical Points of Arbitrary Order*

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We make the scaling hypothesis at a space of critical points of arbitrary order, using the geometric classification of critical points by order proposed in a preceding paper, part I. It follows from the definition of order that there are regions in which several scaling hypotheses are simultaneously valid, one at each space of critical points of order $\Theta \geq 2$ which is connected to the point of highest order. The successive hypotheses are conveniently framed in terms of invariants of the groups of transformations about the points of higher order. This procedure facilitates the formation of multiple-power scaling functions. We suggest that the regions of validity of later hypotheses are bounded by hypersurfaces which scale and which are therefore most easily visualized in the spaces of invariants. A general critical point of order 4 is treated as a preliminary example. We then apply the hypothesis in detail to the critical point of order 4 in a specific example: the Ising model with variable interplanar interaction. This is done in two ways: the first shows specifically the relationship to the case of a metamagnet and to scaling with a parameter for change of lattice dimension; the second explicitly demonstrates that the connectivity between the various surfaces of ordinary ($\Theta=2$) critical points is the same as for the edges of a tetrahedron. As a consequence of this scaling hypothesis, we can make predictions about the shapes of the coexistence and critical surfaces meeting at the critical point of order 4.

I. INTRODUCTION

In a preceding paper¹ (hereafter referred to as I) we introduced the geometric concept of order for a space of critical points (CRS) and also a convenient notation ${}^{\Theta}R_d$ and ${}^{\Theta}X_d$ for critical and coexistence spaces in complex phase diagrams. Well-known examples of tricritical points, and other critical points of order 3 were discussed, and critical spaces of order 4 were shown to exist in certain model systems involving four field or fieldlike variables. The purpose of this paper is to show how to formulate a scaling hypothesis at a space of critical points of arbitrary order; to derive multiple-power scaling functions at such points; and to explicitly analyze the critical point of order 4 in the three-dimensional Ising model with variable interplanar interaction in terms of invariants.²

To formulate a scaling hypothesis at a critical point of arbitrary order, it is first necessary to select special directions at the point analogous to the strong and weak directions described in I. As discussed in Sec. IV of I, the problem of finding the principal directions of scaling at a critical point of arbitrary order is exactly the same as that for a tricritical point (or 3R_0). Once the principal directions at a space of points of order Θ have been found, we can take the limiting orientation of the principal directions at a point of order $\Theta + 1$, as it is approached along a line of points of order Θ . The problem of uniqueness of this coordinate system is the same for a point of order $\Theta = 3$ and its

solution may be obtained by the same principles as treated in Sec. IV of I.

It is therefore possible to induce a set of coordinate systems for a thermodynamic system showing critical spaces of order 4 or more, starting at a ${}^2R_{n-2}$ and proceeding by induction up to the ${}^{\Theta}R_{n-\Theta}$. As each higher order is reached, another principal direction is defined; at an ${}^{\Theta}R_{n-\Theta}$, Θ directions out of the ${}^{\Theta}R_{n-\Theta}$ have been defined. They will be denoted by $\{x_{\Theta 1}, x_{\Theta 2}, \dots, x_{\Theta \Theta}\}$. The geometric possibilities are too complex (even for $\Theta = 4$) to warrant the case by case analysis done earlier for a 3R_0 ; however, it is clear that the consistency arguments demonstrated in I will apply equally well for any order Θ .

The outline of this paper is as follows. In Sec. II, we form the scaling hypothesis and show how to make it at each ${}^{\Theta}R_d$ of successively smaller order. We thereby obtain a total group of scaling transformations which is the direct product of several subgroups.

In Sec. III we treat a general 4R_0 , considering a particular 3R_1 and 2R_2 . We schematically plot the crossover surfaces and show how they may be expected to be expressible as functional relations between invariants of the groups of scaling transformations.

In Sec. IV we treat in detail the 4R_0 in the three-dimensional Ising model with variable interplanar interaction.² This is done in two ways. Firstly we scale with respect to the interaction parameter and thus reduce the model to the familiar metamagnet. Secondly we scale with respect to the variable

TABLE I. Steps in the formation of a scaling hypothesis in terms of invariants.

Step 1.	Hypothesize that the Gibbs potential is given by an invariant equation under the direct-product group $\bar{\mathcal{G}}_\beta \equiv \bar{\mathcal{G}}_\alpha \otimes \mathcal{G}_\beta$ ($\beta < \alpha$) in terms of the principal directions (or variables) of scaling in the invariant space of the group $\bar{\mathcal{G}}_\alpha$ which is a direct product of a subset of the groups \mathcal{G}_γ ($\theta \geq \gamma \geq \alpha$).
Step 2.	Form absolute invariants of the group $\bar{\mathcal{G}}_\beta$ from the principal variables of scaling in the invariant space of $\bar{\mathcal{G}}_\alpha$ by scaling each variable with respect to one of the variables.
Step 3.	Find the new principal directions of scaling in the invariant space of $\bar{\mathcal{G}}_\beta$ by forming linear combinations of a basis set of absolute invariants of the group $\bar{\mathcal{G}}_\beta$.

Quantity	Notation
Variables transformed by a group $\bar{\mathcal{G}}_\alpha$	$x_{\alpha i}$
Invariants of the group $\bar{\mathcal{G}}_\alpha$	$y_{\alpha i}$
Quantities parametrizing a critical space of order β ($< \alpha$)	$k_{\alpha i}$
Rotation matrix to obtain principal variables of scaling at critical space of order β ($< \alpha$)	${}^\alpha R_{ij}$

Equation connecting the above quantities: $x_{\beta i} = {}^\alpha R_{ij}(y_{\alpha i} - k_{\alpha i})$

$\tau \equiv T - T_c$ ($R = 0$) which is tangent to the lines of tri-critical points, and thus demonstrate that the connectivity between the various 2R_2 is the same as for the edges of a tetrahedron, a result that has been previously obtained by quite different arguments.

II. FORMATION OF SCALING HYPOTHESIS

The selection of principal directions proceeds step-by-step starting at a critical point of lowest order—a ${}^2R_{n-2}$. In contrast the scaling hypothesis starts at the critical point of highest order, and states the scaling hypothesis for critical spaces of lower order in terms of “invariants” of groups of transformations about spaces of higher order. The complete system of scaling equations that are valid at a particular point can be set up in the following series of steps.

Step 0. Determine principal directions of scaling $x_{01}, x_{02}, \dots, x_{0\theta}$ (as outlined above), by choosing an ordered set of critical subspaces ${}^iR_{n-i}$ of all orders $i < \theta$ such that ${}^0R_{n-\theta} \subset {}^iR_{n-i} \subset {}^jR_{n-j}$ for all $j < i < \theta$.

Step 1. Make the hypothesis that at the critical space of order θ the Gibbs potential G is given by an equation $G = \mathcal{F}(x_{01}, x_{02}, \dots, x_{0\theta}; x_{\theta, \theta+1}, \dots, x_{\theta n})$ which is invariant under the one-parameter continuous group of transformations \mathcal{G}_θ :

$$\mathcal{G}_\theta: x'_{0i} = \lambda^\theta x_{0i} \quad (i = 0, 1, \dots, n). \quad (2.1)$$

where $x_{00} \equiv G$, $a_{00} = 1$, and $a_{0i} = 0$ for $i > 0$.

Step 2. Re-express the variables $x_{01}, x_{02}, \dots, x_{\theta n}$ as absolute invariants $y_{\theta i}$ of the group \mathcal{G}_θ by scaling with respect to a suitable variable.³ In this section we assume it to be x_{00} :

$$y_{\theta 0} \equiv x_{00}/x_{00}^{1/a_{00}}, \quad (2.2a)$$

$$y_{\theta i} \equiv x_{0i}/x_{00}^{a_{0i}/a_{00}}, \quad i = 1, 2, \dots, \theta - 1 \quad (2.2b)$$

$$y_{\theta, j-1} \equiv x_{0j}, \quad j = \theta + 1, \theta + 2, \dots, n. \quad (2.2c)$$

But theorem 2 of I assures us that $y_{\theta 0}$ can be expressed as a function⁴ of the invariants $(y_{\theta 1}, y_{\theta 2}, \dots, y_{\theta, n-1})$.

Step 3. Near the critical space of order θ , the equations of the critical spaces of order $\theta - 1$ are given by points $(k_{\theta 1}, k_{\theta 2}, \dots, k_{\theta, n-1})$ in the space of invariants $(y_{\theta 1}, y_{\theta 2}, \dots, y_{\theta, n-1})$. The principal directions of scaling at the ${}^{\theta-1}R_d$ are determined by the geometry as linear combinations of the variables $(y_{\theta 1}, y_{\theta 2}, \dots, y_{\theta, \theta-1})$:

$$x_{\theta-1, i} = \sum_{j=1}^{\theta-1} {}^\theta R_{ij}(y_{\theta j} - k_{\theta j}), \quad i = 1, 2, \dots, \theta - 1. \quad (2.3)$$

We also define

$$\begin{aligned} x_{\theta-1, 0} &\equiv y_{\theta 0}, \\ x_{\theta-1, i} &\equiv y_{\theta i}, \quad i = \theta, \theta + 1, \dots, n - 1. \end{aligned} \quad (2.3a)$$

Step 3+1. Now make the scaling hypothesis at the ${}^{\theta-1}R_d$ in exactly the same fashion as in step 1, introducing a group of transformations $\mathcal{G}_{\theta-1}$ which transforms the $x_{\theta-1, i}$ as in Eq. (2.1). One can then proceed to steps 2 and 3 as before. In general a repetition of a cycle of steps 1, 2, and 3 is needed for the scaling hypothesis at each order of CRS. The steps are summarized in Table I.

When a ${}^2R_{n-2}$ is reached, and the last of the $(\theta - 1)$ scaling hypotheses has been made, an invariance is achieved under an $(\theta - 1)$ parameter continuous group of transformations:

$$\mathcal{G}_{\text{tot}} = \mathcal{G}_\theta \otimes \mathcal{G}_{\theta-1} \otimes \dots \otimes \mathcal{G}_2. \quad (2.4)$$

The validity of such a large group of transformations will be limited to a fairly small region close to all the appropriate ${}^iR_{n-i}$.

However, we have not exhausted *all* the possibilities near a ${}^\theta R_{n-\theta}$ because in general it is possible to pass directly from the ${}^\theta R_{n-\theta}$ to an ${}^{\theta'} R_{n-\theta'}$ with $(\theta - \theta') \neq 1$ and to remain outside the range of validity of the groups $\mathcal{G}_{\theta-1}, \dots, \mathcal{G}_{\theta'+1}$. This indicates that there may be regions where any particular subgroup (and only that subgroup) of the total group \mathcal{G}_{tot} is valid.

III. EXAMPLE: A GENERAL 4R_0

To give a little feeling for what is involved when $\theta > 3$, we offer in Fig. 1 a schematic representation

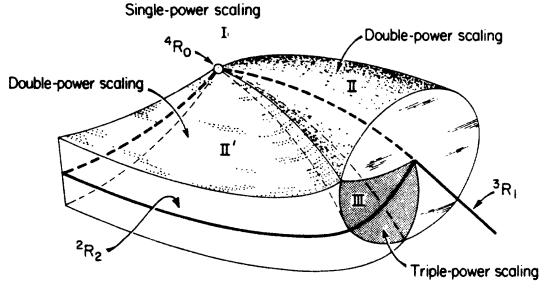


FIG. 1. Part of the region around a 4R_0 on a three-dimensional subspace of the total space. Only one 3R_1 and one 2R_2 are depicted. Boundaries of regions where various scaling laws are valid are indicated by surfaces.

of a three-dimensional projection of the four-dimensional field space in the vicinity of a 4R_0 . There are several different regions where different groups of transformations are valid.

Region I. Only the group \mathcal{G}_4 is valid and the singular part of Gibbs potential G is an invariant equation under \mathcal{G}_4 .

Region II. The singular part of the Gibbs potential is an invariant equation under the two-parameter group $\mathcal{G}_4 \otimes \mathcal{G}_3$ since this region is close to both the 4R_0 and to a critical line 3R_1 . Region II is bounded by a crossover surface defined by an equation connecting the invariants of the group \mathcal{G}_4^2 :

$$f^{II}(y_{41}, y_{42}, y_{43}) = 0 \quad (3.1)$$

Region II'. The singular part of the Gibbs potential is an invariant equation under the two-parameter continuous group of transformations $\mathcal{G}_4 \otimes \mathcal{G}_2$. Region II' is bounded by another crossover surface defined by a different relation between the invariants of \mathcal{G}_4^2 :

$$f^{II'}(y_{41}, y_{42}, y_{43}) = 0 \quad (3.2)$$

Region III. The singular part of the Gibbs potential is an invariant equation under the three-parameter continuous group of transformations $\mathcal{G}_4 \otimes \mathcal{G}_3 \otimes \mathcal{G}_2$. The boundary surface of this region is partly the surface defined by f^{II} in Eq. (3.1) and also a surface defined by an equation connecting the invariants under the group $\mathcal{G}_4^2 \otimes \mathcal{G}_3^2$:

$$f^{III}(y_{31}, y_{32}) = 0 \quad (3.3)$$

IV. ANALYSIS OF ISING MODEL WITH VARIABLE INTERPLANAR INTERACTION

A. The Scaling Hypothesis at the 4R_0

In Sec. II of I, it was demonstrated that the 3d Ising model with variable interplanar interaction [Hamiltonian given by Eq. (2.2) of I] provides an excellent example of a critical point of order 4. This aspect of the model has also been treated by Harbus *et al.*,² who showed that com-

bining the scaling hypothesis for change of lattice dimension when $\mathcal{R} > 0$, with that for $\mathcal{R} < 0$ leads to a general hypothesis for the critical point of order 4 of the form

$$G(\lambda^{a_H} H_2, \lambda^{a_{\mathcal{R}}} \mathcal{R}, \lambda^{a_H} H, \lambda^{a_{\tau}} \tau) = \lambda G(H_2, \mathcal{R}, H, \tau), \quad (4.1)$$

i. e., $G = F(H_2, \mathcal{R}, H, \tau)$ is an invariant equation under the group of scaling transformations \mathcal{G}_4 :

$$\mathcal{G}_4 \left\{ \begin{array}{l} G' = \lambda G, \quad H_2' = \lambda^{a_H} H_2, \quad \mathcal{R}' = \lambda^{a_{\mathcal{R}}} \mathcal{R}, \\ H' = \lambda^{a_H} H, \quad \tau' = \lambda^{a_{\tau}} \tau. \end{array} \right. \quad (4.2)$$

where H and H_2 have equal scaling power and $a_{\tau} < a_{\mathcal{R}} < a_H$ for reasons given in Ref. 2. Equation (4.1) is precisely of the form predicted for a critical point of order 4 with the principal directions of scaling given by

$$x_{41} \equiv H_2, \quad x_{42} \equiv \mathcal{R}, \quad x_{43} \equiv H, \quad x_{44} \equiv \tau. \quad (4.3)$$

We list the directions x_{4i} in the order shown because when referring to the critical surface 2R_2 of Fig. 2 of I, H_2 is a direction pointing out of the coexistence volume bounded by the 2R_2 , \mathcal{R} is a direction in the coexistence volume but pointing out of the 2R_2 , H is a direction parallel to the 2R_2 at the Néel point but not parallel to the tricritical lines 3R_1 , and τ , the "tangent variable," is parallel to the 3R_1 .

Since the field space is now four dimensional, it is now peculiarly advantageous to use the method of invariants outlined in Secs. V and VI of I and Sec. II of this paper. The space of invariants of \mathcal{G}_4^2 is only three dimensional, and the 3R_1 will become points in this space.

There are two advantageous ways of choosing a basis set of invariants. The first was used in I and forms invariants by scaling with respect to the tangent variable τ . It is shown below (IV C) that this procedure enables one to plot the lines of tricritical points as points at the corners of a tetrahedron in the space of invariants. The 2R_2 become the six lines joining the vertices of the tetrahedron. First, however, we analyze the situation in the space of invariants formed by scaling with respect to \mathcal{R} , because this reduces the problem to a form similar to the already familiar tricritical point of a metamagnet which has been fully treated in I.

B. Analysis in the Space of Invariants of \mathcal{G}_4 Formed by Scaling with Respect to \mathcal{R}

We form invariants by scaling with respect to \mathcal{R} ,

$$y_{40} \equiv x_{40} / |x_{42}|^{1/a_{42}} \equiv G_{\mathcal{R}},$$

$$y_{41} \equiv x_{41} / |x_{42}|^{a_{41}/a_{42}} \equiv H_{2\mathcal{R}},$$

$$y_{42} \equiv x_{43} / |x_{42}|^{a_{43}/a_{42}} \equiv H_{\mathcal{R}},$$

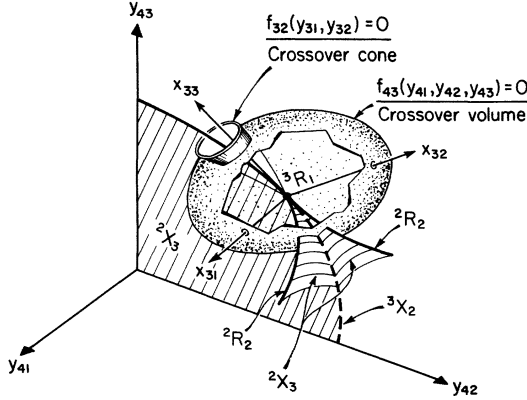


FIG. 2. Plot of the region around one of four 3R_1 in the space of invariants (y_{41}, y_{42}, y_{43}) of \mathcal{G}_4 . The boundaries of regions of validity for double- and triple-power scaling are indicated.

$$y_{43} \equiv x_{44} / |x_{42}|^{a_{44}/a_{42}} \equiv \tau_{\mathcal{R}}. \quad (4.4a)$$

The scaling hypothesis of Eq. (4.1) requires that

$$y_{40} = f_3(y_{41}, y_{42}, y_{43}) \text{ or } G_{\mathcal{R}} = f_3(H_{2\mathcal{R}}, H_{\mathcal{R}}, \tau_{\mathcal{R}}). \quad (4.4b)$$

A representation of the various critical spaces close to the 4R_0 in the invariant (y_{41}, y_{42}, y_{43}) -space is given in Fig. 2 (for $\mathcal{R} < 0$).⁵ That figure is similar to Fig. 14 of I, which shows the situation for constant $\mathcal{R} < 0$. This similarity is clearly to be expected because one can always choose a particular value of $\mathcal{R} \neq 0$ in the variables of Fig. 2.

The various coexistence volumes 2X_3 map into surfaces in Fig. 2 with a conformation close to the tricritical point that is entirely analogous to the conformation in Fig. 14 of I. The 2X_3 end in 2R_2 , which are the same as before, in regions that are close to the 4R_0 .

We can now proceed entirely by analogy with the scaling hypothesis made at tricritical points in I, bearing in mind that the variables being used are not H_2, H, T , but the invariant variables $H_{2\mathcal{R}}, H_{\mathcal{R}}, \tau_{\mathcal{R}}$.

The directions of the second scaling hypothesis for the "invariant space metamagnet" (Fig. 2) can be chosen in an analogous fashion to the ordinary metamagnet of I. The direction x_{31} is chosen in the y_{41} direction and points out of the coexistence volume 2X_3 , the direction x_{32} lies in the $y_{42} - y_{43}$ plane and is not parallel to the critical surface 2R_2 , and the direction x_{33} is parallel to the critical surface 2R_2 . More precisely,

$$x_{31} = y_{41}, \quad x_{3j} = R_{ji} y_{4i} \quad (i, j = 2, 3) \quad (4.5)$$

and R_{ji} is an appropriate nonsingular linear transformation and summation over i is implied. The

scaling hypothesis now requires the invariant part of the Gibbs potential $x_{30} \equiv y_{40} = G / |x_{42}|^{1/a_{42}} \equiv F_3(x_{31}, x_{32}, x_{33})$ to be an invariant equation under the group \mathcal{G}_3 :

$$\mathcal{G}_3: x_{3i}^i = \lambda^{a_{3i}} x_{3i} \text{ with } a_{30} = 1 \quad (i = 0, 1, 2, 3). \quad (4.6)$$

The result of the simultaneous validity of the groups \mathcal{G}_3 and \mathcal{G}_4 (i.e., the product group $\mathcal{G}_3 \otimes \mathcal{G}_4$) is that the Gibbs potential is expressible as a "double-power scaling function"⁴

$$y_{30} = f_2(y_{31}, y_{32}), \quad (4.7)$$

where

$$y_{3i} \equiv x_{3i} / x_{33}^{a_{3i}/a_{33}} \quad (i = 0, 1, 2). \quad (4.8)$$

Equation (4.7) is valid within a double crossover region, bounded by the region of validity of \mathcal{G}_4 and also a crossover surface in the y_{41} -invariant space—given by

$$f_{43}(y_{41}, y_{42}, y_{43}) = 0. \quad (4.9)$$

Within this region all thermodynamic functions are representable in the space y_{31}, y_{32} of Fig. 3. Here appropriate functions are invariant under the product group $\mathcal{G}_4 \otimes \mathcal{G}_3$.

Figure 3 is entirely analogous to Fig. 15 of I, and, as in that case, the three lines of critical points in Fig. 2 map into three points.

The 2R_2 are thus predicted to be representable in regions close to both the 3R_1 and 4R_0 by the equations

$$y_{31} = k_{31}^i, \quad y_{32} = k_{32}^i, \quad (4.10)$$

where $i = 1, 2, 3$. For the 2R_2 shown in Fig. 2 of I, we have $y_{31} = 0$ and $y_{32} = -k_{32}$. Similarly the equation of the 3X_2 is $y_{31} = 0, y_{32} = \text{const}$. Appropriate regions of other 2X_3 will map into lines in the y_{31}, y_{32} plane. In particular the 2X_3 in the $H_2 = 0$ hyperplane is restricted to an appropriate range of y_{32} for $y_{31} = 0$.

Near each 2R_2 , a third group of scaling equations

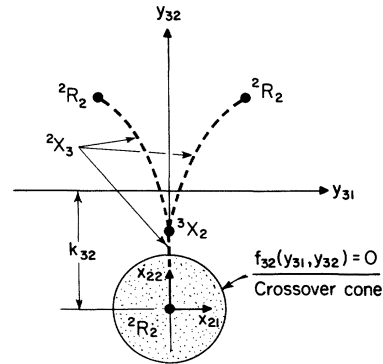


FIG. 3. Plot of the three 2R_2 of Fig. 2 in the space (y_{31}, y_{32}) , invariants of the group $\mathcal{G}_4 \otimes \mathcal{G}_3$. The crossover cone of Fig. 2 has become the circle in Fig. 3.

is valid. For each 2R_2 , we pick two directions x_{21} and x_{22} , the strong and weak directions of Griffiths and Wheeler, in the y_{3i} invariant plane, and define $x_{20} \equiv y_{30} = x_{30}/x_{33}^{1/a_{33}}$. We now require $x_{20} = F_2(x_{21}, x_{22})$ to be an invariant equation under the group \mathcal{G}_2 of transformations:

$$\mathcal{G}_2: x'_{2i} = \lambda^{a_{2i}} x_{2i}, \quad i = 0, 1, 2 \quad (4.11)$$

where $a_{20} = 1$. This implies that the Gibbs potential near a^4R_0 , a^3R_1 , and a^2R_2 is expressible as a triple-power scaling function:

$$x_{20}/x_{22}^{1/a_{22}} = F_1(x_{21}/x_{22}^{a_{21}/a_{22}}). \quad (4.12)$$

Thus all the data from the appropriate four-dimensional region should collapse into a line. The crossover cone for the validity of the triple-power scaling function should also be representable in the space (y_{31}, y_{32}) of Fig. 3 as

$$f_{32}(y_{31}, y_{32}) = 0. \quad (4.13)$$

We illustrate this result for the special case of $H_2 = 0$ (for $\mathcal{R} < 0$). For the $H_2 = 0$ hyperplane, the invariant mapping of the various CRS and CXS are given in Fig. 4. Within the appropriate region (a wedge within a circle), Eq. (4.12) applies. Since $x_{21} = 0$ for this case, all data collapse to a point

$$x_{20} = C x_{22}^{1/a_{22}}, \quad (4.14)$$

which can be rewritten (from definitions of x_{20} , x_{30} , y_{40})

$$G = C x_{42}^{1/a_{42}} x_{33}^{1/a_{33}} x_{22}^{1/a_{22}}. \quad (4.15)$$

This "triple-power scaling law" is a special case of the general triple-power scaling function. It is valid within the wedge between the curves $y_{32} = x_{32}x_{33}^{-a_{32}/a_{33}} = \text{const.}$ and the circle $f_{43}(y_{42}, y_{43}) = 0$ as shown in Fig. 4.

In conclusion, the systematic application of the heirarchy of scaling hypotheses gives predictions

of the shapes of the CRS and CXS within the various crossover regions of the product groups. Finally, if we set $y_{41} = y_{42} = 0$ in Eq. (4.4b) we obtain the scaling hypothesis about the critical point $T_c(\mathcal{R} = 0)$. If \mathcal{R} is allowed to vary we simply obtain the well-known double-power scaling law with a parameter⁶ for $\mathcal{R} > 0$. This double-power law is valid under the product group $\mathcal{G}_4 \otimes \mathcal{G}_2$ where \mathcal{G}_2 is the symmetry group about the line $T_c(\mathcal{R})$ of Fig. 2 of I.

This is an explicit example of a case where a product group $\mathcal{G}_j \otimes \mathcal{G}_i$ with $|j - i| \neq 1$ may be appropriate.

C. Analysis Using the Tangent Variable to Form Invariants

We now form a different set of functionally independent absolute invariants \bar{y}_{4i} of \mathcal{G}_4 by scaling with respect to the tangent variable $x_{44} \equiv \tau - T_{NO}$:

$$\bar{y}_{4i} = x_{4i}/|x_{44}|^{a_{4i}/a_{44}}, \quad i = 0, 1, \dots, 3. \quad (4.16)$$

The scaling hypothesis of Eq. (4.1) requires that

$$\bar{y}_{40} = \bar{f}_3(\bar{y}_{41}, \bar{y}_{42}, \bar{y}_{43}). \quad (4.17)$$

In the space of invariant field variables $(\bar{y}_{41}, \bar{y}_{42}, \bar{y}_{43})$, the four 3R_1 again map into points: two in the $\bar{y}_{41} = 0$ plane for $\bar{y}_{42} < 0$ and two in the $\bar{y}_{43} = 0$ plane for $\bar{y}_{42} > 0$. These four points form the corners of a tetrahedron, as in Fig. 5. In this representation we are explicitly able to show the way that the 2R_2 connect the different 3R_1 .

In Fig. 5, the curves joining the vertices of the tetrahedron are the scaled representation of the 2R_2 . Each 3R_1 is joined by these curves to each of the other 3R_1 . Similarly, the 4X_1 on the T axis of Fig. 2 becomes the point at the origin of Fig. 5 and the four 3X_2 become curves joining the origin to the 3R_1 . Similarly, for the 2X_3 which are not depicted in Fig. 5.

The curve joining the two 3R_1 for $\bar{y}_{42} < 0$ is the 2R_2 on top of the "mountain" in Fig. 2 of I. The

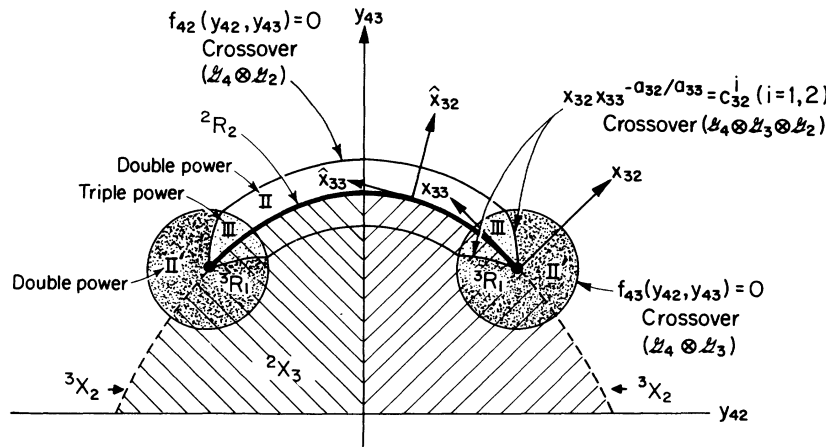


FIG. 4. Full (y_{43}, y_{42}) plane of Fig. 2 showing both 3R_1 for $y_{41} = 0$. They are connected by a 2R_2 near which there is a region where the groups \mathcal{G}_4 and \mathcal{G}_2 are valid, but \mathcal{G}_3 is not. The crossover surfaces and the groups whose ranges of validity are thereby limited, are indicated.

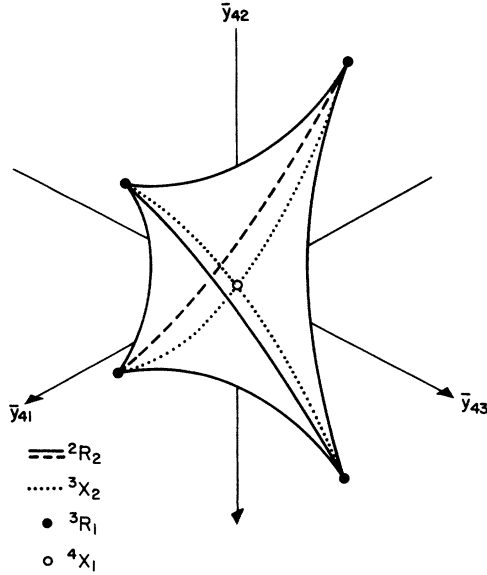


FIG. 5. A schematic representation of the complete system of CRS and CXS near the 4R_0 , obtained by scaling with respect to the tangent variable. Two of the 3R_1 for $\bar{y}_{42} > 0$ are in the plane $\bar{y}_{43} = 0$ while those for $\bar{y}_{42} < 0$ (also shown in Fig. 4) are in the plane $\bar{y}_{41} = 0$. The total system is topologically equivalent to a tetrahedron. Not shown are the various 2X_3 which join the 3X_2 to the 2R_2 . The values of \bar{y}_{4i} are mapped as $\bar{y}_{4i}/(\bar{y}_{4i} + c_i)$ where c_i are constants to ensure that all pertinent points are mapped within a finite region.

other curves leading away from these two 3R_1 (for $\bar{y}_{42} < 0$) to the other two 3R_1 (for \bar{y}_{42}) are the wings of Fig. 14 of I, which extend into the region $H_2 \neq 0$.

Thus the wings for $\mathcal{R} < 0$ are joined by continuously connected surfaces to the wings for $\mathcal{R} > 0$ and are thus sections of the same 2R_2 . There are in reality six 2R_2 : one for $\mathcal{R} > 0$, one for $\mathcal{R} < 0$, and four "wings."

Now one can proceed to make a scaling hypothesis about the 3R_1 as before. First we choose a suitable set of variables which are linear combinations of the \bar{y}_{4i} :

$$\bar{x}_{3j} = \bar{R}_{ji} (\bar{y}_{4i} - k_{4i}), \quad (4.18)$$

where $i, j = 1, 2, 3$ and summation over i is implied. Then the Gibbs function will be expected to be given by an invariant equation under the group \mathcal{G}_3 of transformations

$$\bar{x}'_{3j} = \lambda^{a_{3j}} x_{3j}, \quad j = 0, 1, 2, 3 \quad (4.19)$$

and $a_{30} = 1$. However this new transformation must inevitably give the same results as when the scaling was performed with respect to the groups $\mathcal{G}_4 \otimes \mathcal{G}_3$ in Sec. IV B. The analysis will be the same as for that case.

V. SCALING LAWS AND SCALING FUNCTIONS

The consequences of the existence of valid equations of scaling for critical points of higher order are very similar to those for a 3R_0 .

In a region where a direct product of r groups of transformations are valid, it is possible to form scaling functions with simultaneous scaling by r variables. This will give data collapsing by r dimensions, and this should make the testing of the scaling hypothesis in the vicinity of a critical subspace of any order quite possible: the maximum value for r is $(\Theta - 1)$ because in the vicinity of a critical point of order Θ there should be regions where a direct product of $(\Theta - 1)$ groups of transformations are valid (e.g., the crossover region near a 3R_0 of I and region III in Fig. 4). Now it should be possible to plot all the data from the Θ -dimensional region in terms of one variable.

There should be larger regions where one group \mathcal{G}_1 is not valid and the data should be expressible in terms of two appropriately scaled variables. Then data collapse onto a surface, which should be easy to verify.

For each set of scaling powers $a_{\Theta i}$ at a critical subspace of order Θ , there will exist a different set of exponents and scaling laws relating them. For a point of order Θ , there will therefore be Θ independent exponents, and if more than Θ are defined, there must be scaling laws relating them.

In a system like the one considered in Sec. IV, there will therefore be four independent exponents at the 4R_0 , three at each of the four 3R_1 , and two at each of the 2R_2 , giving 28 possible exponents before the symmetry is taken into account. The symmetries show that the 3R_1 are all equivalent because we have reflections $H \leftrightarrow -H$ and $H_2 \leftrightarrow -H_2$ and the pseudoreflexion $\mathcal{R} \leftrightarrow -\mathcal{R}$, $H \leftrightarrow H_2$. These symmetries show that there are only two sorts of 2R_2 . Thus there are only $4 + 3 + 2 + 2 = 11$ scaling powers we need to know to evaluate every possible exponent for the system.

In fact for this system we shall see that we know seven scaling powers certainly and we can estimate two more. The only pair that we do not know are the scaling powers for the "wings." (see Ref. 6 of I for definition of "wings").

At the 4R_0 the exponents are the same as for the two-dimensional Ising model, and this gives $a_{41} = a_{43} = \frac{1}{16}$, $a_{42} = \frac{7}{8}$, and $a_{44} = \frac{1}{2}$, where the first two are equal by the pseudoreflexion symmetry and exponent a_{42} is evaluated from our knowledge of the crossover exponent $\gamma_4 = a_{42}/a_{44} = \varphi_4$.

For the 3R_1 , we know the scaling powers from our knowledge of the exponents γ and γ_{st} (describing the divergence of the functions χ and χ_{st}), and also from the crossover φ_3 describing the shape of the 2R_2 for $H_2 = 0$ approaching the 3R_1 .

Work by Harbus and Stanley⁷ estimates the scaling power a_{32} to be $\frac{2}{3}$. Tentative calculations⁸ based on the shape of the phase boundary and the amplitude function of the staggered susceptibility along the phase boundary give rough estimates of $a_{31} \cong \frac{11}{12}$ and $a_{33} \cong \frac{1}{3}$.

Of the 2R_2 , two have the familiar exponents of the three-dimensional Ising model, giving⁶ $a_{21} = \frac{5}{6}$ and $a_{22} = \frac{8}{15}$ while the other four 2R_2 are "wings" and possess exponents about which we know nothing at all.⁹

The scaling laws are obtained by eliminating the a_{ij} between expressions for different exponents. Exponents defined for paths of approach to a set of equivalent 0R_d (e.g., the 3R_1 above) will thus be related by scaling laws, but equalities between exponents for different sets of 0R_d (i.e., not related by symmetry) should be regarded as fortuitous and not as scaling laws. This will remain true for the critical spaces of any system.

VI. SUMMARY AND CONCLUSIONS

We have shown above how to make a scaling hypothesis at a critical point of arbitrary order, in the case where the dimensionality of the spaces of critical points of order θ is θ less than the total number of dimensions available; i.e., Eq. (1.1) of paper I is satisfied. We have also shown how the choice of variables for hypotheses at points of lower order is decided by the groups of transformations about points of higher order. In the example given in Sec. IV it was shown that there could be regions

where different direct products of groups of scaling transformations were valid, e.g. $\mathcal{G}_4 \otimes \mathcal{G}_3$ and $\mathcal{G}_4 \otimes \mathcal{G}_2$.

In the particular example treated, there was a very high degree of symmetry about the temperature axis, and instead of having different scaling powers for every CRS (giving 28 possible exponents) only nine independent exponents were found—two at each of two sorts of 2R_2 , three at the four 3R_1 , and only two at the 4R_0 because of the symmetry and the fact $\gamma_4 = \varphi_4$.

The same symmetry that reduced the number of exponents from 28 to 9 was operative in making the lines of tricritical points intersect on the $T=0$ axis. In a four-dimensional space there is nothing to force lines of tricritical points to intersect except a symmetry¹⁰; it is of interest to note that in complex fluid mixtures¹¹ critical points of higher order can occur when the Hamiltonians do not possess symmetries. The order θ for such a critical point is equal to the number of phases in equilibrium there as is the case here but the number of scaling fields at such a critical point is usually not equal to θ .

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³T. S. Chang, A. Hankey, and H. E. Stanley, Phys. Rev. B 8, 346 (1973).

⁴F. Harbus, A. Hankey, H. E. Stanley, and T. S. Chang Phys. Rev. B (to be published).

⁵Usually this is the "tangent" variable with smallest a_{0r} , but it may be advantageous to scale with respect to a different variable. See, for example, Sec. IV B.

⁶Scaling functions may change their values or functional forms as the scaled variables change signs.

⁷Clearly, the result for $\mathcal{R} > 0$ will be the same as for $\mathcal{R} < 0$ with γ_{41} and γ_{42} interchanged.

⁸A. Hankey and H. E. Stanley, Phys. Rev. B 6, 3515 (1972), and references contained therein.

⁹F. Harbus and H. E. Stanley, Phys. Rev. Lett. 29, 58 (1972).

¹⁰F. Harbus, H. E. Stanley, and T. S. Chang, Proceedings of the Thirteenth International Conference on Low Temperature Physics, Boulder, Colo., August, 1972 (unpublished).

¹¹There has been a suggestion by P. Kortman [Phys. Rev. Lett. 29, 1449 (1972)] that the exponents for the wings are classical, but this may not be justified.

¹²Tricritical points possess a reflection symmetry in the strong variable $[x_s \rightarrow -x_s]$ and this almost forces the three lines to intersect—but not quite, e.g., the He⁴ phase diagram. Another operative fact here is that x_s is also a strong direction for the wings.

¹³R. B. Griffiths and B. Widom, Phys. Rev. A (to be published); A. Hankey, T. S. Chang, and H. E. Stanley, Stanford Linear Accelerator Center Technical Report, May, 1973 (unpublished).