LETTER TO THE EDITOR

Backbone and elastic backbone of percolation clusters obtained by the new method of 'burning'

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Abstract. We introduce a method of obtaining information about percolation clusters in which the clusters are 'burned' in different ways. This method yields the backbone and the elastic backbone, their number of loops, shortest path and other quantities. It works in any dimension and can be applied also to other clusters. We obtain accurate estimates for the backbone exponents in two and three dimensions.

Suppose we choose two points P_1 and P_2 separated by a distance comparable to the correlation length ξ. If we pass a current from P_1 to P_2 then the set of current carrying bonds is called the backbone and the remaining bonds are called dangling ends. One can also define the backbone as the intersection of all the self-avoiding walks between P_1 and P_2 (Shlifer *et al* 1979, Hong and Stanley 1983a, b).

The investigation of the backbone of clusters—especially in the case of percolation clusters—has been of interest for a long time. Possible applications are the conductivity of random systems (Skal and Shklovski 1975, de Gennes 1976, Daoud 1983) and the flow of fluids in porous media (Stanley and Coniglio 1984). From the theoretical point of view, the backbone has provided an understanding of the general structure of percolation structures within the framework of links, nodes and blobs (Coniglio 1981, 1982, Pike and Stanley 1981, Stanley 1977, Hong and Stanley 1983a, Stanley and Coniglio 1984). The exponents of the backbone such as the exponent β_B:

$$\rho_S^B(p) \sim (p - p_c)^{\beta_B}, \tag{1}$$

where $\rho_S^B(p)$ is the mean density of sites of the backbone of the fractal dimension $D_B = d - \beta_B/\nu$, have been calculated numerically by different methods: large-cell renormalisation (Shlifer *et al* 1979), Monte Carlo (Kirkpatrick 1978, Pike and Stanley 1981), ε-expansion (Harris 1983, Harris and Lubensky 1983), position–space renormalisation group (Hong and Stanley 1983b), and series expansions (Hong and Stanley 1983a). The error bars are quite big, however, and it is important to find effective algorithms for obtaining backbones in order to improve the error estimates for these exponents. The availability of backbones is also important in testing the argument of Stanley and Coniglio (1984) stating the non-validity for backbones of the conjecture of Alexander and Orbach (1982). (See Hong *et al* (1984) for the case of ants on backbones.)

We now introduce a new fractal object, which we call the elastic backbone. It has been proposed that the elasticity of disordered structures—a gel at the sol–gel transition point, for instance—can be described by the electrical conductivity (de Gennes 1976) and thus related to the normal backbone. However, elasticity can be described by small springs in the bonds of a cluster. It is difficult to imagine that *more* than just the shortest paths connecting the two points at which a force is applied should contribute to this elasticity. This can be seen by considering figure 1, where a piece of the backbone is shown before and after the elongation due to the force F. We thus introduce as a new object *the 'elastic backbone'* of a cluster as being the *sites that lie on the union of all the shortest paths between two points* P_1 *and* P_2 on the cluster. If the distance between P_1 and P_2 goes to infinity, we expect that the densities ρ_S^E of sites or ρ_L^E of loops or ρ_C^E of cutting sites of the elastic backbone should become independent of P_1 and P_2. The question of critical behaviour of ρ_S^E, ρ_L^E and ρ_C^E will be studied in this paper.
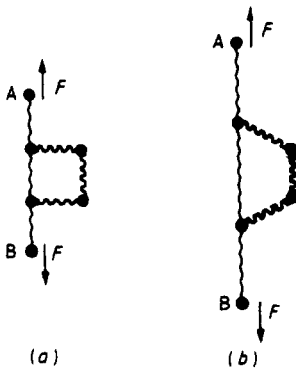

Figure 1. Supposing that the angles of the object of springs shown in (a) can change freely, we show in (b) the elongation of the object due to the application of the force F at A and B.

To construct the elastic backbone (and also the full backbone), we propose a new method which we call 'burning'. We first select two endpoints P_1 and P_2, which we choose to be as close as possible to diagonally opposite corners of a surrounding hypercube. The first step is to burn the cluster from P_1: at time t_1 site P_1 burns and gets the value 1; at the next time unit t_2, all the neighbours of P_1 burn and get the value 2; at time t_{i+1}, all the neighbours of sites burning at time t_i that have not yet burned are burned and get the value $i+1$. This procedure was first presented by Stauffer (1984) as a means of describing forest fires (of fractal shape). The first burning of the cluster imposes an ordering to the sites of the cluster. One obtains the burning time T_P, which is the number of time units needed to burn all the sites, the length of the shortest path l_p (Middlemiss *et al* 1980), which is the number of time units needed to burn P_2, and the number of loops L_P of the cluster, which is the number of times that one tries to burn a site P_i that is already burned in the same time unit. The ordering of the cluster obtained in this first burning will be used in the next steps and is thus kept in mind. The second step is a burning starting at P_2 and with the additional condition that only those sites can be burned at time t_{i+1} which, in the ordering of the previous step, have a smaller value than those burned at time t_i. Due to this condition, only sites on the elastic backbone are now burned. Simultaneously one obtains the

number of sites S_E, of loops L_E, and of cutting sites C_E of the elastic backbone; C_E is the number of time steps that only one site is burning. Also the maximal number M_E of sites that burn simultaneously can be obtained, and it is in some sense a measure of the broadness of the elastic backbone. Finally, the backbone is obtained by growing it from the elastic backbone. So before starting the third step the 'growing' backbone is equal to the elastic backbone. In the third step one burns from the sites P_i, which were the sites at which loops were closed in the first burning. In this third burning again a site can only be burned at t_{i+1} if its value (in the ordering from the first burning) is smaller than the value of the sites burned at t_i but additionally the growing backbone cannot burn. Now the question is asked if starting at a given P_i the elastic backbone is reached at only one site or at several sites. In the case of several sites all the sites that have been burned from P_i are added to the growing backbone. This is done over and over again until no more parts can be attached to the growing backbone by burning from the P_i. Then the backbone is complete. One can finally obtain the number of sites S_B or loops L_B of the backbone, for instance, by burning the backbone in the same way as in the first step. We note that the backbone constructed in this way depends on P_1 and P_2 in the same fashion as described above for the elastic backbone and also for the backbone the distance between P_1 and P_2 should ideally go to infinity.

We applied the above method to the largest (or incipient infinite) cluster of site percolation at p_c on a square lattice and a simple cubic lattice in finite boxes of edge length L. The endpoints P_1 and P_2 were determined to be as close as possible to diagonally opposing corners, but if their Euclidean distance was smaller than a number α the cluster was not considered and a new percolation configuration was chosen. This procedure prevents, due to statistical fluctuations, too small or deformed clusters being taken as prototypes for a spanning or 'infinite' clusters in a finite box. The parameter α can be changed and the results should not depend on α in the limit of large L. If α is too close to the length of the diagonal of the box, however, too many rejections of clusters will slow down the algorithm. The number of realisations (different samples taken) varies in two dimensions from 450 000 for $L = 10$ to 100 for $L = 500$ and in three dimensions from 500 for $L = 6$ to 100 for $L = 40$.

In two dimensions we use $p_c = 0.592\,77$ (Gebele 1983). Our results are shown in figure 2. We also tested for α dependence; within error bars the results are the same for all values of α. We find very straight lines for the sites of the backbone and the slope gives, because of finite size scaling and

$$S_B(p_c) = L^d \rho_S^B(p_c) \sim L^{d - \beta_B / \nu} = L^{D_B}, \tag{2}$$

the fractal dimension of the backbone $D_B = 1.60 \pm 0.05$. Note that the question of whether $\nu_B = \frac{4}{3} = \nu$ (percolation) does not arise in equation (2). The points for the sites of the elastic backbone, of the shortest path and the cutting sites of the elastic backbone also lie on straight lines although with larger statistical deviations. They all have more or less the same slope of about 1.1. In the large L limit it cannot be excluded that the slope(s) might go to unity, as may occur for the shortest path, but without extrapolating possible curvatures hidden in the error bars our data yield $D_E = 1.10 \pm 0.05$ for the fractal dimension of the elastic backbone.

In three dimensions we work at $p_c = 0.3117$ (Heermann and Stauffer 1981). Our results are shown in figure 3 for $\alpha = L$. The statistical errors are indicated for $L = 10$, where two independent runs are shown. All quantities seem to be critical as they show straight lines over a wide range with a slope different from one (compare with the broken line) and a curvature that would give a slope one for large values of L seems
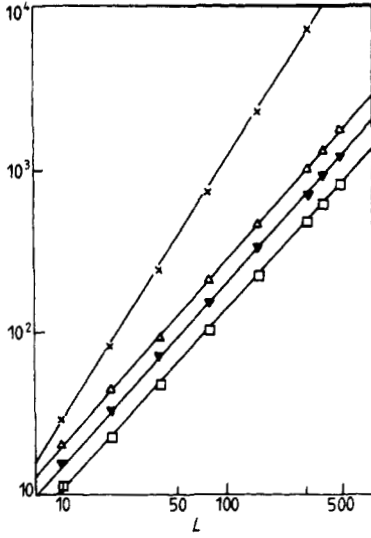
Figure 2. Log–log plot showing dependence on L of the number of sites in the backbone $S_B(\times)$, the elastic backbone $S_E(\triangle)$, the shortest path $l_p(\blacktriangledown)$ and red bonds *in* the elastic backbone $S_{red}(\square)$ for $p_c = 0.59277$.
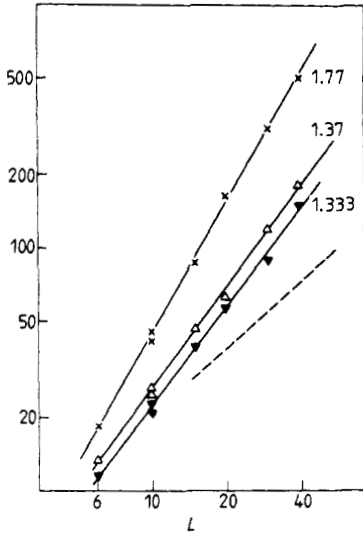

Figure 3. Log–log plot of the number of sites in the backbone $S_B(\times)$, the number of sites in the elastic backbone $S_E(\triangle)$, and the number of sites on one shortest path $l_p(\blacktriangledown)$ against the system length L for three dimensions at $p_c = 0.3117$. The broken line has slope 1.

extremely unlikely. The fractal dimension of the backbone is estimated to be $D_B = 1.77 \pm 0.07$ and the fractal dimension of the elastic backbone is, as in the two-dimensional case, equal to the fractal dimension of the shortest path and has a value of $D_E = 1.35 \pm 0.05$. In figure 4 we show the number of loops of the largest cluster,
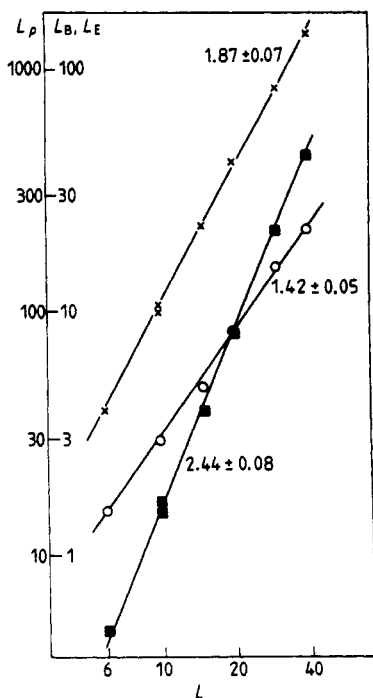
Figure 4. Log–log plot of the number of loops of the largest cluster L_p(■) backbone L_B(×) and elastic backbone L_E(○) against the system length in three dimensions at $p_c = 0.3117$.

the backbone and the elastic backbone. Within their statistical error bars, the three slopes agree with the corresponding fractal dimensions. This shows that for all the three objects the site and bond problems yield the same fractal dimension because of Euler's relation.

The values that we find for the fractal dimension of the backbone of percolation, in two dimensions: 1.60 and in three dimensions 1.77, are in fair agreement with the results of series expansion $(1.80 \pm 0.05$ in $d = 2$, 1.83 ± 0.08 in $d = 3$, Hong and Stanley 1983a), Monte Carlo simulation (1.55–1.62 in 2D and 1.8–2.1 in 3D, Kirkpatrick 1978; 1.72 in 2D, Li and Strieder 1982), and PSRG (1.55 for $d = 2$, Hong and Stanley 1983b) and large-cell renormalisation ($\sim$1.62 in 2D).

In summary, we have introduced a new critical object in percolation: the elastic backbone which, at least in three dimensions, has a non-trivial fractal dimension of about 1.35. It seems to be equivalent to the shortest path in the cluster. However, the elastic backbone does not explain the experimental discrepancy by nearly a factor of two between the conductivity and elasticity exponents (Adam *et al* 1981) because in addition to the effect shown in figure 1 a real gel will also have retrieving forces in the angles and volumes enclosing nearly incompressible fluids so that any deformation will contribute to the elasticity. We proposed a fast algorithm which obtains for any cluster not only the elastic backbone but also the usual backbone. The fractal dimension that we get for the usual backbone agrees with numerical work of other authors.

Note added in proof. In two dimensions our result agrees well with recent calculations by Puech and Rammal (1983).

References

Adam M, Delsanti M, Durand D, Hild G and Munch J P 1981 *Pure Appl. Chem.* **53** 1489
Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L625
Coniglio A 1981 *Phys. Rev. Lett.* **46** 250
—— 1982 *J. Phys. A: Math. Gen.* **15** 3829
Daoud M 1983 *J. Physique Lett* to appear
de Gennes P G 1976 *J. Physique Lett.* **37** L1
Gebele T 1984 *J. Phys. A: Math. Gen.* **17** L51
Harris A B 1983 *J. Phys. A: Math. Gen.* **16** L365
Harris A B and Lubensky T C 1983 *Preprint*
Heermann D W and Stauffer D 1981 *Z. Phys.* B **44** 339
Hong D C, Havlin S, Herrmann H J and Stanley H E 1984 *Preprint*
Hong D C and Stanley H E 1983a *J. Phys. A: Math. Gen.* **16** L475
—— 1983b *J. Phys. A: Math. Gen.* **16** L525
Kirkpatrick S 1978 *AIP Conf. Proc.* **40** 99
Li P S and Strieder W 1982 *J. Phys. C: Solid State Phys.* **15** L1235
Middlemiss K M, Whittington S G and Gaunt D S 1980 *J. Phys. A: Math. Gen.* **13** 1835
Puech L and Rammal R 1983 *J. Phys. C: Solid State Phys.* **16** L1197
Pike R and Stanley H E 1981 *J. Phys. A: Math: Gen.* **14** L169
Shlifer G, Klein W, Reynolds P J and Stanley H E 1979 *J. Phys. A: Math. Gen.* **12** L169
Skal A and Shlovski B I 1975 *Sov. Phys.–Semicond.* **8** 1029
Stanley H E 1977 *J. Phys. A: Math. Gen.* **10** L211
Stanley H E and Coniglio A 1984 *Phys. Rev.* B **29** 522
Stauffer D 1984 *Introduction to Percolation Theory* ch 1 (London: Taylor and Francis) to be published