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Systematic Application of Generalized Homogeneous Functions to Static Scaling, Dynamic Scaling, and Universality*

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A function $f(x_1, x_2, \ldots, x_n)$ is a generalized homogeneous function (GHF) if we can find numbers $a_1, a_2, \ldots, a_n$ such that for all values of the positive number λ, $f(\lambda^{a_1}x_1, \lambda^{a_2}x_2, \ldots, \lambda^{a_n}x_n) = \lambda^{a_f} f(x_1, x_2, \ldots, x_n)$. We organize the properties of GHFs in four theorems. These are used to systematically examine the consequences of various scaling hypotheses. An advantage of this approach is that the same formalism may be used to treat thermodynamic functions, static correlation functions, dynamic correlation functions, and "universality." The simple case of thermodynamic scaling (two independent variables) is first generalized to static and dynamic correlation functions (three and four variables), and then to scaling with a parameter (for which the critical subspace becomes higher dimensional). In this last case, where a second GHF hypothesis is made, the necessity of crossover lines is demonstrated. The assumption of homogeneity is clearly separated from any extra assumptions that may also be called scaling (or "strong scaling"), but are independent of and different from that of homogeneity. One practical insight gained from the present approach is that *all experimentally measured exponents are expressible as the ratio of two scaling powers*, a_f (which refers to the function) and a_j (which refers to the path of approach to the critical point). A second practical advantage is that, since a GHF can be scaled with respect to any of its arguments, one can immediately write a variety of scaling functions for each type of scaling hypothesis. The GHF approach thereby permits data to be plotted in a variety of convenient fashions, and is found to facilitate computation of the relevant scaling functions (in particular, the GHF approach led directly to the recent calculation of the Heisenberg model scaling function by Milošević and Stanley).

I. INTRODUCTION AND OUTLINE OF PRESENT WORK

Scaling hypotheses have been made in a wide variety of situations, and have been applied near the critical point successively to thermodynamic functions,[1–8] static correlation functions,[1,3,9–29] and dynamic correlation functions.[30–32] In some cases the full consequences of the scaling hypothesis made may not have been appreciated, while in other cases the scaling hypothesis has been formulated in such a fashion that it led to relations that are not in accord with calculations on model systems.[3,19]

In this work we state the scaling hypotheses for static and dynamic correlation functions, as well as for thermodynamic functions, in terms of very simple statements involving the use of generalized homogeneous functions. Generalized homogeneous functions (GHFs) have appeared before in connection with scaling,[1,3,13,16,32] but their systematic investigation in order to consider all consequences of the scaling hypothesis is, we believe, new and worthwhile. Accordingly, we concern ourselves with the properties of GHFs and with their application to systems of physical interest. We shall see not only that a certain "unity" among the various previous approaches is obtained, but also that we can make certain *additional* predictions, some of which (such as ways of plotting experimental data) are relatively trivial, while others (such as a method of actually calculating the scaling function directly from an interaction Hamiltonian using high-temperature series expansions) lead to altogether new and useful information.

In Sec. II we define a GHF and systematically assemble the properties of GHFs that are useful in the context here considered.[33] These properties are stated for functions of only two variables, while the generalization to functions of n variables and the detailed proofs of these properties are relegated to Appendix A, which we encourage the serious reader not to skip.

In Sec. III the scaling hypothesis is formulated in terms of GHFs for the case of thermodynamic functions. We show how every critical-point exponent can be simply and directly expressed in terms of two unknown quantities and how these unknown quantities can be eliminated among groups of three exponents to yield the three-exponent relations or *thermodynamic scaling laws*. We also examine the question of scaling functions and show how the initial GHF assumption leads to the prediction of an arbitrary number of scaling functions, the most useful of which is, to the best of our knowledge, hitherto unpublished. We also show that one could have equally well assumed the existence of a scaling function and derived as a consequence the GHF property, so that the GHF approach is fully equivalent to the scaling-function approach.

In Sec. IV we proceed to consider the GHF assumption for the static correlation function and we relate our approach to previous work. In particular, we show that the two-exponent relations predicted by certain other approaches[3] (but not borne out by model calculations)[19] are, in fact, not predicted at all by our approach—the correction terms to these relations are of precisely such a form as to reduce them to familiar thermodynamic scaling laws.

In Sec. V we consider the GHF hypothesis for dynamic correlation functions and we show that this hypothesis directly implies the dynamic scaling hypotheses made by previous authors. The GHF approach has the virtue that it is trivially generalized to nonzero magnetic field, and that one can easily see how the scaling hypotheses for the static correlation functions and thermodynamic functions are immediate consequences.[34]

In both Secs. IV and V critical-point exponents and scaling functions are derived in a systematic fashion using the properties of GHFs. Some of these scaling functions have not appeared in the literature before.

In Sec. VI we use the GHF formalism to discuss scaling with a parameter. The shapes of the critical line and of the crossover line between different regions of scaling are treated. The simultaneous validity of two types of scaling behavior is shown to require the existence of crossover lines, thereby restricting the region of validity of the scaling about the critical line.

II. USEFUL PROPERTIES OF GENERALIZED HOMOGENEOUS FUNCTIONS

In this section we assemble the properties of GHFs that are of use in the context of this work. We shall state these properties for functions $f(x_1, x_2)$ with *two* independent variables, since in the most elementary applications (thermodynamic functions of a simple magnet, for example) the appropriate functions are functions of only two variables. These properties are assembled in the form of "definitions" and "theorems" in order to demonstrate most clearly the properties of GHFs and the power of stating the scaling hypothesis in terms of them. The proofs of our statements, although not difficult, are relegated to Appendix A, in order that the reader may pass directly to subsequent sections concerning the applications of these functions to systems of physical interest.

A. Definitions of Generalized Homogeneous Functions and Subclass of Scale-Invariant Functions

Definition 1. A function $f(x_1, x_2)$ is a GHF if there exist two numbers a_1, a_2 such that for all positive λ,

$$f(\lambda^{a_1} x_1, \lambda^{a_2} x_2) = \lambda^{a_f} f(x_1, x_2) \quad . \tag{2.1}$$

We refer to a_1, a_2 as the scaling powers of the variables x_1, x_2, respectively, while the number a_f is called the scaling power of the function $f(x_1, x_2)$; the numbers a_1, a_2, and a_f are not all zero.

There are really only two *independent* scaling powers, as all scaling powers in (2.1) can be changed by an arbitrary factor p ($p \neq 0$). To demonstrate this assertion, we set $\lambda = \bar{\lambda}^p$ in Eq. (2.1), and observe that if (2.1) is valid for all positive λ, it is valid for $\lambda = \bar{\lambda}^p$ (where $\bar{\lambda}$ is positive). Hence we have

$$f(\bar{\lambda}^{pa_1} x_1, \bar{\lambda}^{pa_2} x_2) = \bar{\lambda}^{pa_f} f(x_1, x_2) \ . \tag{2.2}$$

The number a_f is either zero, in which case we call the function $f(x_1, x_2)$ a scale-invariant function (for reasons that will become clear below) or else $a_f \neq 0$, in which case we can set $p = 1/a_f$ in (2.2) and express $f(x_1, x_2)$ as a GHF with scaling power unity. Thus, a GHF is either a scale-invariant function

$$f(\lambda^{a_1} x_1, \lambda^{a_2} x_2) = f(x_1, x_2) \tag{2.3a}$$

or else we can choose a_1 and a_2 such that

$$f(\lambda^{a_1} x_1, \lambda^{a_2} x_2) = \lambda f(x_1, x_2) \quad . \tag{2.3b}$$

For example, the functions $x_1 x_2^{-2} + x_1^2 x_2^{-4}$ and $x_1^2 + x_1 x_2^2$ are both GHFs. The scaling powers may be chosen to be $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{4}$ in both cases, but the first function is scale invariant ($a_f = 0$) and the second is not ($a_f = 1$).[35] Additional examples of GHFs are shown in Table I.

Functions $f(x_1, x_2)$ which satisfy relations of the form

$$f[g_1(\lambda) x_1, g_2(\lambda) x_2] = g_f(\lambda) f(x_1, x_2) \quad , \tag{2.4}$$

where the functions $g_i(\lambda)$ and $g_f(\lambda)$ possess inverses, are also GHFs. The proof that the form (2.4) implies $g_i(\lambda) = \lambda^{a_i}$ and $g_f(\lambda) = \lambda^{a_f}$ is given in Appendix A.[36]

The generalization of all the above statements to functions of more than two variables is obvious and for functions of n variables there will be n independent scaling powers ($a_1, a_2, \ldots, a_n$) and, as

TABLE I. Examples of generalized homogeneous functions. A function $f(x_1, x_2)$ is a GHF if there exist two numbers a_1, a_2 such that for all positive λ, $f(\lambda^{a_1} x_1, \lambda^{a_2} x_2) = f(x_1, x_2)$. A homogeneous function is a special case of a GHF for which $a_1 = a_2$. For these examples, $a_f = 1$.

$f(x_1, x_2)$	a_1	a_2
(1) $x_1^2 + x_2^4$	$\frac{1}{2}$	$\frac{1}{4}$
(2) $x_1 x_2^2 + x_2^4$	$\frac{1}{2}$	$\frac{1}{4}$
(3) $x_1^p x_2^q + x_1^r$; p, q, r arbitrary	$1/r$	$(r-p)/qr$
(4) $x_1^p x_2^q + x_2^s$; p, q, r arbitrary	$(s-q)/ps$	$1/s$
(5) $x_1^p x_2^q + x_1^{p'} x_2^{q'} + \cdots + x_1^r$, where $(r-p)/qr = (r-p')/q'r = \cdots$	$1/r$	$(r-p)/qr$
(6) $x_1^p x_2^q + x_1^{p'} x_2^{q'} + \cdots + x_2^s$	$(s-q)/ps$	$1/s$
(7) $x_1^p x_2^q + x_1^r + x_2^s$, where $p/r + q/s = 1$	$1/r$	$1/s$
(8) $x_1^p x_2^q + x_1^{p'} x_2^{q'} + \cdots + x_1^r + x_2^s$, where $p/r + q/s = 1$, $p'/r + q'/s = 1$, and so forth	$1/r$	$1/s$
(9) $x_1^4 + x_2^4 + x_1 x_2^2$	Not a GHF	
(10) $\sin[1/(x_1^2 + x_2^2)]$	Not a GHF	

we shall see below, n independent critical-point exponents.

We conclude this section with the observation that if $f(x_1, x_2)$ is a GHF with scaling powers a_1, a_2, and a_f and $g(x_1, x_2)$ is a GHF with scaling powers a_1, a_2, and a_g, then (a) the product fg is a GHF with scaling power $a_f + a_g$, (b) the quotient f/g is a GHF with scaling power $a_f - a_g$, (c) the sum $f \pm g$ is a GHF if and only if $a_f = a_g$.

B. Derivatives and Legendre Transforms of Generalized Homogeneous Functions

Theorem 1. If $f(x_1, x_2)$ is a GHF with scaling power a_f [cf. Eq. (2.1)], the partial derivative

$$f^{(j,k)}(x_1, x_2) \equiv \frac{\partial^j}{\partial x_1^j} \frac{\partial^k}{\partial x_2^k} f(x_1, x_2) \tag{2.5}$$

is also a GHF

$$f^{(j,k)}(\lambda^{a_1} x_1, \lambda^{a_2} x_2) = \lambda^{a_f - j a_1 - k a_2} f^{(j,k)}(x_1, x_2) , \tag{2.6}$$

with scaling power $a_f - j a_1 - k a_2$.

For example, the function $f(x_1, x_2) = x_1^2 + x_1 x_2^2$ may be considered a GHF with $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, and $a_f = 1$ (or as a GHF with $a_1 = 2$, $a_2 = 1$, and $a_f = 4$); its derivative $f^{(1,0)}(x_1, x_2) = 2x_1 + x_2^2$ is a GHF with $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, and scaling power $1 - \frac{1}{2} = \frac{1}{2}$ (or a GHF with $a_1 = 2$, $a_2 = 1$, and scaling power $4 - 2 = 2$).

Theorem 2. Let $f(x_1, x_2)$ be a GHF with scaling power a_f, and let

$$\bar{f}(\bar{x}_1, x_2) \equiv f(x_1, x_2) - x_1 \bar{x}_1 \tag{2.7}$$

denote the Legendre transform of $f(x_1, x_2)$, in which the conjugate variable $\bar{x}_1 \equiv f^{(1,0)}(x_1, x_2)$ replaces x_1 as independent variable. Then the function $\bar{f}(\bar{x}_1, x_2)$ is also a GHF, with the *same* scaling power as the original function ($a_{\bar{f}} = a_f$); the scaling power of the variable $\bar{x}_1$ is given by $a_{\bar{1}} = a_f - a_1$:

$$\bar{f}(\lambda^{a_f - a_1} \bar{x}_1, \lambda^{a_2} x_2) = \lambda^{a_f} \bar{f}(\bar{x}_1, x_2) \quad . \tag{2.8}$$

Theorem 2 implies that *all* Legendre transforms of a GHF are also GHFs, and they all have the same scaling power. The scaling power of the transformed variables will, of course, change in accordance with theorem 1, but note that the procedure of Legendre transformation is internally consistent in the sense that, for the reverse transformation from $\bar{f}(\bar{x}_1, x_2)$ to $f(x_1, x_2)$, one finds that the scaling power of x_1 is $a_f - (a_f - a_1) = a_1$.

Theorem 2 is proved in Appendix A and it is easily shown that all 2^n forms of a function of n variables that are equivalent under Legendre transformation are also GHFs of the same scaling power.

Together theorems 1 and 2 have a remarkable consequence: Since all thermodynamic functions are related by a combination of Legendre transforms and derivatives, if one thermodynamic potential is a GHF, then all thermodynamic functions are GHFs (this statement will be illustrated in detail in Sec. III A and Appendix B).[37]

C. Mathematical Form of Generalized Homogeneous Function: Scaling Functions, Power-Law Singularities, and Exponents

The next theorem demonstrates the equivalence of our approach to those which begin with statements about scaling functions[38]; it also shows how to relate the scaling functions to the physical functions explicitly, how any variable can be used to scale, and how to evaluate the critical-point exponents of an arbitrary function directly in terms of the scaling parameters—by inspection!

Theorem 3. A function $f(x_1, x_2)$ is a GHF with scaling power a_f and independent variable scaling powers a_1, a_2 if there exists some function $g_\pm(u)$ such that

$$f(x_1, x_2) = |x_1|^{a_f/a_1} g_{\mathrm{sgn}\, x_1}(x_2/|x_1|^{a_2/a_1}) \tag{2.9a}$$

or, equally, if there exists some function $h_\pm(u)$ such that

$$f(x_1, x_2) = |x_2|^{a_f/a_2} h_{\mathrm{sgn}\, x_2}(x_1/|x_2|^{a_1/a_2}) \quad , \tag{2.9b}$$

where $\mathrm{sgn}\, x \equiv x/|x|$ denotes the signature of x. Conversely, if $f(x_1, x_2)$ is a GHF, then (2.9a) and (2.9b) follow, where the functions $g_\pm(u)$ and $h_\pm(u)$ in Eqs. (2.9) are given in terms of $f(x_1, x_2)$ by the simple relations

$$g_\pm(u) = f(\pm 1, u) \quad , \tag{2.10a}$$

$$h_\pm(u) = f(u, \pm 1) \quad . \tag{2.10b}$$

Note that the presence of the absolute-value signs in Eqs. (2.9) and the $\pm$ subscripts arises from the fact that we require λ to be positive in our defining equation (2.1).

Note also that *either* variable x_1 or x_2 can be used to form functions such as $g_\pm$ and $h_\pm$ that remain unchanged when $x_i \to \lambda^{a_i} x_i$ for all i.

Theorem 3 also generates an explicit form for the scaling function, which has led directly to a method for the direct calculation of scaling functions for any system (e.g., the Heisenberg model) for which high-temperature series expansions exist.[39,40]

The following corollary to theorem 3 will be used in Sec. III B to show that the exponents of all the thermodynamic functions are simply calculable in terms of the scaling powers a_i, while elimination of the (unspecified) parameters a_i leads to the relations among critical-point exponents known as "scaling laws."[41]

Corollary to theorem 3. All GHFs have power-law singularities at the origin when it is approached along one of the principal axes. The exponent for the path of approach $|x_i| \to 0$ is given by a_f/a_i, the ratio of the scaling power of the function to the scaling power of the variable that is approaching zero; for a function of n variables, this result becomes

$$f(0, \ldots, x_i, \ldots, 0) \sim |x_i|^{a_f/a_i} f(0, \ldots, 1, \ldots, 0) \quad . \tag{2.11}$$

In particular, we note from Eq. (2.11) that the ratio of the critical-point exponents corresponding to two different paths of approach $x_i \to 0$ and $x_j \to 0$ to the origin are independent of the function under consideration; i.e., for any function we have

$$\frac{\text{exponent for the path } x_i \to 0}{\text{exponent for the path } x_j \to 0} = \frac{a_j}{a_i} \quad . \tag{2.12a}$$

Equally, we see that the ratio of the critical-point exponents of two different functions $f(x_1, \ldots)$ and $g(x_1, \ldots)$ is path independent; i.e., for any two paths $x_i \to 0$ and $x_j \to 0$, we have

$$\frac{\text{exponent of } f \text{ for the path } x_i \to 0}{\text{exponent of } g \text{ for the path } x_i \to 0} = \frac{\text{exponent of } f \text{ for the path } x_j \to 0}{\text{exponent of } g \text{ for the path } x_j \to 0} = \frac{a_f}{a_g} \quad . \tag{2.12b}$$

D. Bridge between Dynamic and Static Scaling: Theorem on Fourier Transforms

The final theorem enables one to relate the dynamic and static scaling hypotheses. The relationship between the dynamic structure factor and the static correlation functions involves Fourier transforms, and theorem 4 is used to show the equivalence between scaling statements for each. The fluctuation-dissipation relation between the static pair correlation function and the isothermal susceptibility (a thermodynamic function) is also a special case of a Fourier transform, so that theorem 4 will also be used to relate the scaling hypothesis for the pair correlation function to the scaling hypothesis for the thermodynamic functions.

Theorem 4. Let $f(x_1, x_2)$ be a GHF with scaling power a_f and let

$$\hat{f}(\hat{x}_1, x_2) \equiv \int f(x_1, x_2) e^{x_1 \cdot \hat{x}_1} d^d x_1 \tag{2.13}$$

denote the Fourier transform in which the conjugate variable $\hat{x}_1$ replaces x_1 as independent variable, and d is the dimensionality of the variable. Then the function $\hat{f}(\hat{x}_1, x_2)$ is also a GHF, with scaling power $\hat{a}_f = a_f - d\hat{a}_1$ and the scaling power of the transformed variable $\hat{x}_1$ is given by $\hat{a}_1 = -a_1$,

$$\hat{f}(\lambda^{\hat{a}_1} \hat{x}_1, \lambda^{a_2} x_2) = \lambda^{a_f - d\hat{a}_1} \hat{f}(\hat{x}_1, x_2) \quad . \tag{2.14}$$

E. Homogeneous Functions

A homogeneous function is a very special case of a GHF for which $a_1 = a_2 (= a_3, \ldots, = a_n$ for functions of n variables); a GHF for $n = 1$ is automatically a homogeneous function. Thus, by Eq. (2.2) homogeneous functions can be specified entirely by a *single* scaling power, in contrast to the case of GHFs, where n scaling powers are needed ($n > 1$). If we arrange that $a_1 = a_2 = \ldots a_n = 1$, then a_f is this power and it is called the *degree* of the homogeneous function.

Although the set of functions which are GHFs is vastly larger than the set of functions that are homogeneous (cf. Table I), nevertheless GHFs form a rather small class of functions. However, if we ask instead for the class of functions which may be closely approximated by some GHF near the *origin* $x_1 = x_2 = 0$, then a very large number of functions fit the definition [notable exceptions are functions possessing essential singularities and functions with numerous cross terms such as $f(x_1, x_2) = x_1^4 + x_2^4 + x_1 x_2^2$)]. This is the key to the success of the scaling hypothesis, which is taken up in Sec. III.

III. THERMODYNAMIC FUNCTIONS

In this section we consider only thermodynamic functions near the critical point, and we formulate the scaling hypothesis for these functions. In subsequent sections we shall consider static and dynamic correlation functions, and we shall see that the scaling hypothesis for the latter implies the scaling hypothesis presented here.

Because of theorems 1–3 of Sec. II, it does not matter which function we choose to formulate our GHF hypothesis about—but it does matter which variables we work with.

A. GHF Hypothesis for Thermodynamic Functions

1. Statement

We shall formulate the hypothesis for a magnetic system first, and we shall consider all variables in units such that at the critical point the variable is zero. Thus, we change the temperature variable to

$$\tau \equiv T - T_c \tag{3.1a}$$

and the entropy variable to

$$\sigma \equiv S - S_c , \tag{3.1b}$$

where S_c denotes the entropy at the critical point; there is no change of variable necessary for the magnetization or the magnetic field, since $M = H = 0$ at the critical point.

In general, the Gibbs potential may be split into two parts, one regular at the critical point and the other singular. The scaling hypothesis is made about the singular part, and we shall mean by the symbol $G(\tau, H)$ only the singular part. A discussion of this point is presented in Appendix B.

We now state the scaling hypothesis for a simple magnet: *Close to the critical point the singular part of the Gibbs potential per spin is asymptotically* a GHF; i.e., there exist two numbers a_τ and a_H such that, for all positive λ,

$$G(\lambda^{a_\tau}\tau, \lambda^{a_H} H) = \lambda G(\tau, H) . \tag{3.2}$$

2. Equivalent Statement in Terms of Other Thermodynamic Potentials

Theorem 2 of Sec. II B implies that all $2^n - 1 = 3$ Legendre transforms of (3.2) are also GHFs. Thus, for the Helmholtz potential $A \equiv G + MH$, we have

$$A(\lambda^{a_\tau}\tau, \lambda^{a_M} M) = \lambda A(\tau, M) ; \tag{3.3a}$$

for the internal energy $U \equiv A + TS$, we have

$$U(\lambda^{a_\sigma}\sigma, \lambda^{a_M} M) = \lambda U(\sigma, M) ; \tag{3.3b}$$

and for the enthalpy $E \equiv U - MH$, we have

$$E(\lambda^{a_\sigma}\sigma, \lambda^{a_H} H) = \lambda E(\sigma, H) \tag{3.3c}$$

(but see the details in Appendix B). Here the scaling powers of the "conjugate variables" magnetization and entropy are not independent of a_τ and a_H but are determined by theorem 2,

$$a_M = 1 - a_H \tag{3.4a}$$

and

$$a_\sigma = 1 - a_\tau . \tag{3.4b}$$

The reader will note from (3.2) that we have chosen the scaling powers a_τ and a_H such that the scaling power of the Gibbs potential is unity, and by theorem 2 it follows that the scaling powers of all four thermodynamic *potentials* are unity. In Appendix C we find bounds on the scaling powers of the four *variables*.

For systems other than simple magnets, we must choose our independent variables quite carefully. For a simple fluid, if we assume that the singular part of the Gibbs potential is a GHF in the variables T and P, then we obtain false predictions (such as the prediction that C_P and C_V vary near T_c as the same power of τ). In general, we should choose variables one of which is in the direction of the "coexistence surface" and the other of which is out of the surface; these are called weak and strong directions, respectively (for detailed definitions and examples, see Griffiths and Wheeler[42]). In the case of a magnet, the coexistence surface is in the H-T plane and is that portion of the T axis from $T = 0$ to $T = T_c$; hence, the variable τ is parallel to and the variable H is perpendicular to the coexistence surface. On the other hand, for a one-component fluid, the coexistence surface in the T-P plane is the vapor pressure curve, which near the critical point is parallel to neither the T nor the P axes. Hence the scaling hypothesis is made not in terms of the variables T and P but rather in terms of different variables.[43]

A brief discussion concerning the application of the scaling hypothesis to model systems is presented in Appendix D.

B. Critical-Point Exponents for all Thermodynamic Functions in Terms of a_τ, a_H

Since every thermodynamic function can be expressed as some partial derivative of some thermodynamic potential, it follows from theorems 1 and 2 that all thermodynamic functions satisfy the scaling hypothesis also; moreover, theorems 1 and 2 provide a simple expression of the scaling power of an arbitrary thermodynamic function directly in terms of the initial parameters a_τ, a_H of Eq. (3.3a).

For example, the magnetization per spin $M(T,H) = -(\partial G/\partial H)_T$ is seen by theorem 2 to obey

$$M(\lambda^{a_\tau}\tau, \lambda^{a_H} H) = \lambda^{a_M} M(\tau, H) , \tag{3.5}$$

with $a_M = 1 - a_H$ by Eq. (2.6). Applying now the corollary to theorem 3, we find

$$M(\tau, 0) \sim |\tau|^{a_M/a_\tau} \quad (\tau \to 0, \quad H = 0) \tag{3.6a}$$

and

$$M(0, H) \sim |H|^{a_M/a_H} \quad (H \to 0, \quad \tau = 0) . \tag{3.6b}$$

The critical-point exponents β and δ are defined by the relations $M(\tau, 0) \sim |\tau|^\beta$ and $M(0,H) \sim |H|^{1/\delta}$, whence we have from (3.6a) and (3.6b) that

$$\beta = a_M/a_\tau = (1 - a_H)/a_\tau \tag{3.7a}$$

TABLE II. Predictions of the GHF hypothesis (3.2) for the power-law dependences for all four thermodynamic potentials and for a variety of other thermodynamic functions obtained by partial differentiation of the potentials. These expressions illustrate the utility of theorem 2 and the corollary to theorem 3 in affording an expression of all thermodynamic-function critical-point exponents in terms of two unspecified scaling powers a_τ and a_H.

$$
\begin{array}{ll}
G(\tau,H) \sim |\tau|^{1/a_\tau} & U(\sigma,M) \sim |\sigma|^{1/a_\sigma} \\
\qquad \sim |H|^{1/a_H} & \qquad \sim |M|^{1/a_M} \\
A(\tau,M) \sim |\tau|^{1/a_\tau} & E(\sigma,H) \sim |\sigma|^{1/a_\sigma} \\
\qquad \sim |M|^{1/a_M} & \qquad \sim |H|^{1/a_H}
\end{array}
$$

$$
\begin{aligned}
\sigma(\tau,H) \sim G^{(1,0)}(\tau,H) &\sim |\tau|^{a_\sigma/a_\tau} \Rightarrow 1-\alpha = a_\sigma/a_\tau = (1-a_\tau)/a_\tau \\
&\sim |H|^{a_\sigma/a_H} \Rightarrow \psi = a_\sigma/a_H = (1-a_\tau)/a_H \\[1ex]
M(\tau,H) \sim G^{(0,1)}(\tau,H) &\sim |\tau|^{a_M/a_\tau} \Rightarrow \beta = a_M/a_\tau = (1-a_H)/a_\tau \\
&\sim |H|^{a_M/a_H} \Rightarrow \delta^{-1} = a_M/a_H = (1-a_H)/a_H \\[1ex]
\chi_T(\tau,H) \sim G^{(0,2)}(\tau,H) &\sim |\tau|^{a_\chi/a_\tau} \Rightarrow -\gamma' = -\gamma = a_\chi/a_\tau = (1-2a_H)/a_\tau \\
&\sim |H|^{a_\chi/a_H} \Rightarrow 1-\delta^{-1} = a_\chi/a_H = (1-2a_H)/a_H \\[1ex]
C_H(\tau,H) \sim G^{(2,0)}(\tau,H) &\sim |\tau|^{a_C/a_\tau} \Rightarrow -\alpha' = -\alpha = a_C/a_\tau = (1-2a_\tau)/a_\tau \\
&\sim |H|^{a_C/a_H} \Rightarrow -\Phi = a_C/a_H = (1-2a_\tau)/a_H \\[1ex]
C_M(\tau,M) \sim A^{(2,0)}(\tau,M) &\sim |\tau|^{a_C/a_\tau} \Rightarrow -\alpha' = -\alpha = a_C/a_\tau = (1-2a_\tau)/a_\tau \\
&\sim |M|^{a_C/a_M} \Rightarrow -\Phi\delta = a_C/a_M = (1-2a_\tau)/a_M
\end{aligned}
$$

and

$$\delta^{-1} = a_M/a_H = (1-a_H)/a_H . \tag{3.7b}$$

This simple procedure can readily be applied to obtain the critical-point exponents for any thermodynamic function directly in terms of a_τ and a_H, and examples of the more common thermodynamic functions are given in Table II. The reader is encouraged to work through a few examples from this table in order to appreciate that *the critical-point exponents for an arbitrary function can be obtained essentially by inspection*! This is because each exponent is the ratio of two numbers, the numerator being dependent solely upon the function under consideration and the denominator being dependent upon the path of approach to the critical point; thus the reader can verify the validity of Eqs. (2.12a) and (2.12b).

For example, to illustrate (2.12a), note that if we divide the exponent for the path $|\tau| \to 0$ by the exponent for the path $H \to 0$, considering successively $S(\tau,H)$, $M(\tau,H)$, $\chi_T(\tau,H)$, and $C_H(\tau,H)$, we find

$$\frac{1-\alpha}{\psi} = \frac{\beta}{\delta^{-1}} = \frac{-\gamma}{\delta^{-1}-1} = \frac{-\alpha}{-\phi} = \frac{a_H}{a_\tau} , \tag{3.8a}$$

where the exponents used are all defined in Table II. Equivalently, we can illustrate (2.12b) by considering, say, the two functions $M(\tau,H)$ and $C_H(\tau,H)$ for the paths $|\tau| \to 0$, $|H| \to 0$,

$$\frac{\beta}{-\alpha} = \frac{\delta^{-1}}{-\phi} = \frac{a_M}{a_{C_H}} \left(= \frac{1-a_H}{1-2a_\tau} \right) . \tag{3.8b}$$

Equations (3.8a) and (3.8b) provide relations among the critical-point exponents; these relations are generically termed "scaling laws." Some of the more useful scaling laws are shown in Table III. Note that we can eliminate the unknown scaling powers a_τ, a_H from any two critical-point exponents that are linearly independent.[44] If one chooses, for example, the exponents β and δ, one finds

$$a_\tau = [\beta(\delta+1)]^{-1} \tag{3.9a}$$

and

$$a_H = \delta/(\delta+1) , \tag{3.9b}$$

as the reader can verify from Eqs. (3.7a) and (3.7b). Since all critical-point exponents can be expressed in terms of a_τ, a_H, it follows that all exponents can equally be written in terms of β, δ or in terms of any other pair of linearly indepen-

TABLE III. Relations among critical-point exponents (scaling laws) which are predicted by the scaling hypothesis for (a) thermodynamic functions, (b) static correlation functions, and (c) dynamic correlation functions. These relations are not all independent of one another, and in fact there are 2, 3, and 4 independent exponents for cases (a), (b), and (c), respectively, corresponding to the number of independent scaling powers appearing in Eqs. (3.2), (4.2), and (5.2). For example, if one wished, one could express all thermodynamic-function exponents in terms of, say, β and δ; all static correlation-function exponents in terms of β, δ, and ν; and all dynamic correlation-function exponents in terms of β, δ, ν, and z.

(a) Thermodynamic scaling laws

$$
\begin{gathered}
\alpha + 2\beta + \gamma = 2 \\
\alpha + \beta(\delta+1) = 2 \\
(2-\alpha)\, \zeta + 1 = (1-\alpha)\delta, \quad \text{where } \zeta \equiv \delta\psi - 1 \\
\gamma(\delta+1) = (2-\alpha)(\delta-1) \\
\gamma = \beta(\delta-1) \quad [\text{or } \beta\delta = \beta + \gamma] \\
\Phi + 2\psi - 1/\delta = 1 \\
\Phi\beta\delta = \alpha \\
\psi\beta\delta = 1-\alpha \\
\delta = (2-\alpha+\gamma)/(2-\alpha-\gamma) \\
\Delta = \beta + \gamma = \beta\delta \\
\Delta = 1 + \tfrac{1}{2}(\gamma - \alpha)
\end{gathered}
$$

(b) Static correlation-function scaling laws

$$
\begin{gathered}
\gamma = (2-\eta)\,\nu \\
(\delta-1)/\delta = (2-\eta)\,\mu \\
\nu/\mu = (1-\alpha)\psi = \beta\delta = \alpha/\varphi = \Delta
\end{gathered}
$$

(c) Dynamic correlation-function scaling laws

$$
\begin{gathered}
x = \nu z \\
y = \mu z \\
x = \Delta y
\end{gathered}
$$

dent exponents (cf. Table IV). In particular, one can then use Table II to express any exponent in terms of a_τ, a_H and use Table IV to express a_τ, a_H in terms of any pair of linearly independent exponents.

The Essam-Fisher gap exponents Δ_n' and Δ_n describe the fashion in which the ratio of two functions diverges[45]; as $T \to T_c^-$,

$$\frac{G^{(0,n+1)}(\tau, H=0)}{G^{(0,n)}(\tau, H=0)} \sim |\tau|^{-\Delta_n'} , \tag{3.10a}$$

while for $T \to T_c^+$,

$$\frac{G^{(0,n+2)}(\tau, H=0)}{G^{(0,n)}(\tau, H=0)} \sim |\tau|^{-2\Delta_n} . \tag{3.10b}$$

Theorem 1 tells us that $G^{(0,n)}$ is a GHF with scaling power $1-na_H$, so that the quotient in (3.10a) is a GHF with scaling power

$$[1-(n+1)a_H]-(1-na_H) = -a_H ,$$

independent of n. Hence combining (2.11) and the definitions of (3.10), we have

$$\Delta_n' = \Delta_n \equiv \Delta = a_H / a_\tau . \tag{3.11}$$

One can clearly generalize the concept of a gap exponent by the relations

$$\frac{G^{(m,n+1)}(\tau, 0)}{G^{(m,n)}(\tau, 0)} \sim |\tau|^{-\Delta_{mn}^T} \tag{3.12a}$$

and

$$\frac{G^{(m+1,n)}(0, H)}{G^{(m,n)}(0, H)} \sim |H|^{-\Delta_{mn}^H} . \tag{3.12b}$$

These generalized gap exponents are equal to the original Essam-Fisher gap exponents, as clearly

$$\Delta_{mn}^T = (\Delta_{mn}^H)^{-1} \equiv \Delta = a_H / a_\tau . \tag{3.13}$$

C. Equation of State near Critical Point: "Scaling Functions"

The equation of state is a functional relationship among three thermodynamic variables, as for example $M = M(\tau, H)$. In this section we demonstrate that the GHF assumption (3.2) leads to predictions concerning thermodynamic functions for all values of their arguments near the critical point (not just their critical-point exponents): In particular, we shall show by straightforward application of theorem 3 that the equation of state near the critical point is of a very special form.

We shall illustrate the generality of applying theorem 3 to obtain scaling functions by obtaining systematically three scaling relations for the M-T-H equation of state $M = M(\tau, H)$. A similar analysis can be applied to any other triplet of variables. For example, if measurements were made instead at constant entropy, the functions that should be considered are the enthalpy and internal energy.[46] The scaling hypothesis in our form makes definite predictions for such a case from Eqs. (3.3b) and (3.3c) both for the critical-point exponents to be expected and for the correct ways to plot data in order to obtain scaling functions.

1. First M-τ-H Scaling Function H_τ versus M_τ

We illustrate the present approach by deriving what was historically probably the first scaling function. Since $H(\tau, M) = (\partial A/\partial M)_\tau$, it follows from (3.3a) and theorem 1 that $H(\tau, M)$ is a GHF with scaling power $a_H = 1 - a_M$,

$$H(\lambda^{a_\tau}\tau, \lambda^{a_M} M) = \lambda^{a_H} H(\tau, M) . \tag{3.14}$$

Setting $\lambda = |\tau|^{-1/a_\tau}$ in (3.14) leads to

$$H(\tau, M)/|\tau|^{a_H/a_\tau} = H(\mathrm{sgn}\tau, M/|\tau|^{a_M/a_\tau}) , \tag{3.15}$$

where the notation sgnτ denotes the signature of τ. We write (3.15) in the form

$$H_\tau = \mathfrak{F}^{(1)}_{\mathrm{sgn}\tau}(M_\tau) , \tag{3.16}$$

where we have defined the scaled quantities

$$H_\tau \equiv H/|\tau|^{a_H/a_\tau} = H/|\tau|^{\beta\delta} \tag{3.17a}$$

and

$$M_\tau \equiv M/|\tau|^{a_M/a_\tau} = M/|\tau|^{\beta} . \tag{3.17b}$$

The second equalities in (3.17) follow from (3.7) or from Table II [we could write $\beta\delta = \Delta$ using (3.11)].

The quantities H_τ and M_τ defined in (3.17) are sometimes called scale invariant quantities, as they are invariant under the scale transformation

$$\tau \to \lambda^{a_\tau}\tau, \quad H \to \lambda^{a_H} H, \quad M \to \lambda^{a_M} M . \tag{3.18}$$

Equation (3.16) says that all M-H-τ data points near the critical point should asymptotically approach two curves (rather than an infinite family of curves); all data taken for $T < T_c$ (sgn$\tau = -1$) will collapse onto one branch of the function $\mathfrak{F}^{(1)}_{\mathrm{sgn}\tau}(x)$, while all data for $T > T_c$ (sgn$\tau = +1$) will collapse onto the other branch. This "scaling function" is sketched in Fig. 1(a).

The "data-collapsing" prediction of (3.16) has been borne out by measurements on a number of systems[47–49] and this provides experimental confirmation of the GHF hypothesis (since theorem 3 is an if-and-only-if statement).

2. Second M-τ-H Scaling Function: H_M versus τ_M

Choosing $\lambda = |M|^{-1/a_M}$ in (3.14), we find

$$H(\tau, M)/|M|^{a_H/a_M} = H(\tau/|M|^{a_\tau/a_M}, 1) \tag{3.19}$$

or

$$H_M = \mathfrak{F}^{(2)}(\tau_M) , \tag{3.20}$$

where

$$H_M \equiv H/|M|^{a_H/a_M} = H/|M|^{\delta} \tag{3.21a}$$

TABLE IV. Expressions for the scaling powers a_τ, a_H in terms of pairs of critical-point exponents; these relations may be verified using Table II. This table emphasizes that no pair of exponents is more fundamental than any other pair.

Exponents	a_τ	a_H
(α,β)	$(2-\alpha)^{-1}$	$(2-\alpha-\beta)/(2-\alpha)$
(α,γ)	$(2-\alpha)^{-1}$	$(2-\alpha+\gamma)/2(2-\alpha)$
(α,δ)	$(2-\alpha)^{-1}$	$\delta/(\delta+1)$
(β,γ)	$(2\beta+\gamma)^{-1}$	$(\beta+\gamma)/(2\beta+\gamma)$
(β,δ)	$[\beta(\delta+1)]^{-1}$	$\delta/(\delta+1)$
(γ,δ)	$(\delta-1)/\gamma(\delta+1)$	$\delta/(\delta+1)$

and

$$\tau_M \equiv \tau/\left|M\right|^{a_\tau/a_M} = \tau/\left|M\right|^{1/\beta} . \tag{3.21b}$$

Note that we have omitted the subscript sgn$M$ from the second scaling function $\mathfrak{F}^{(2)}(x)$ because the two branches of $\mathfrak{F}^{(2)}(x)$ are identical by symmetry (H is an odd function of M). The function $\mathfrak{F}^{(2)}(x)$ is often called the Griffiths scaling function[5] and denoted by $h(x)$; it is sketched in Fig. 1(b).

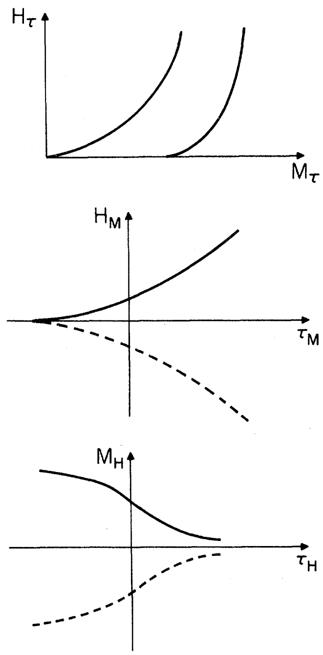

FIG. 1. Sketches of the three possible scaling functions for interpretation of M-H-τ equation-of-state data near the critical point, where $\tau \equiv T - T_c$. The three functions shown, $\mathfrak{F}_\pm^{(1)}$, $\mathfrak{F}^{(2)}$, and $\mathfrak{F}^{(3)}$, are defined in Table V, part (a) and in Eqs. (3.16), (3.20), and (3.24). A dashed line indicates that data are not taken for this branch of the scaling function due to the fact that M is an odd function of H, and accordingly the subscript $\pm$ is used only in the scaling function $\mathfrak{F}_\pm^{(1)}$ [cf. Table V, part (a)].

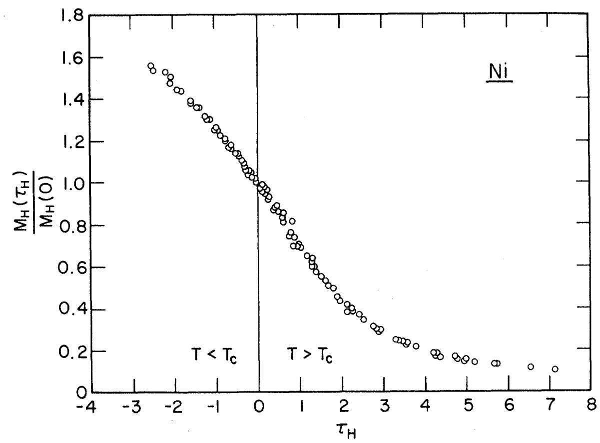

FIG. 2. Plot of the experimental data of Kouvel and Comly (Ref. 47) on the ferromagnetic metal nickel, plotted using the coordinate system of Fig. 1(c). The data appear to lie smoothly upon a single curve and this curve is the scaling function $M_H = \mathfrak{F}^{(3)}(\tau_H)$ of Eq. (3.24) and Table V, part (a). These data are compared with the function $\mathfrak{F}^3(x)$, calculated for the Heisenberg model, in Refs. 39 and 40.

3. Third M-τ-H Scaling Function: M_H versus τ_H

The functions $\mathfrak{F}^{(1)}(x)$ and $\mathfrak{F}^{(2)}(x)$ have one disadvantage. Small values of τ in the first case, and small values of M in the second case, lead to huge values of the scaled variables—to get all the data onto a single plot generally necessitates the use of log-log paper, and a *consequent loss of precision*. In fact, data are taken for the critical isotherm ($\tau = 0$) to a point at infinity in Fig. 1(a), and cannot be plotted at all!

A third way of plotting M-τ-H data does not suffer from these defects, and to the best of our knowledge is new. It has recently been used with good advantage in the actual calculation of scaling functions for the Heisenberg model.[39,40]

We return to the scaling hypothesis (3.2) and observe that since $M(\tau, H) = (\partial G/\partial H)_\tau$,

$$M(\lambda^{a_\tau}\tau, \lambda^{a_H}H) = \lambda^{a_M} M(\tau, H) , \tag{3.22}$$

where $a_M = 1 - a_H$ by theorem 1. Setting $\lambda = |H|^{-1/a_H}$ in (3.22), we have

$$M(\tau,H)/\left|H\right|^{a_M/a_H} = M(\tau/\left|H\right|^{a_\tau/a_H}, 1) \tag{3.23}$$

or

$$M_H = \mathfrak{F}^{(3)}(\tau_H) , \tag{3.24}$$

where

$$M_H \equiv M/\left|H\right|^{a_M/a_H} = M/\left|H\right|^{1/\delta} \tag{3.25a}$$

and

$$\tau_H \equiv \tau/\left|H\right|^{a_\tau/a_H} = \tau/\left|H\right|^{1/\beta\delta} . \tag{3.25b}$$

Again because of M-H symmetry, we need not introduce a subscript sgn$H$ on the scaling function $\mathfrak{F}^{(3)}(x)$. Experimental data are found to fall evenly (cf. Fig. 2) upon a single smooth curve, which in fact is just the curve of the magnetization as a function of temperature in a fixed external field [cf. Eq. (3.23)]. Of course the reason that the data do not fall at extreme regions of this new "scaling plot" is that in practice one never makes measurements for $H=0$ (cf. the detailed data tabulation of Ref. 47, or the plots in Refs. 39 and 40).

The definitions of the three scaling functions $\mathfrak{F}^{(j)}(x)$ are summarized in Table V, part (a).

IV. STATIC CORRELATION FUNCTIONS

In this section we formulate the scaling hypothesis for the static pair correlation function. We shall see that the scaling hypothesis presented here implies the scaling hypothesis for thermodynamic functions presented in Sec. III.

A. GHF Hypothesis for the Pair Correlation Functions

1. Statement

We shall formulate the hypothesis for a magnetic system, in which case the pair correlation function is defined by

$$C_2(\tau, H, \vec{r}) \equiv \langle s_0 s_{\vec{r}} \rangle - \langle s_0 \rangle \langle s_{\vec{r}} \rangle , \tag{4.1}$$

where s_0 and $s_{\vec{r}}$ are the z components of spins situated at sites 0 and $\vec{r}$ of a lattice and the angular brackets denote a thermal average.[50] Note from (4.1) that we express the temperature dependence of C_2 in terms of the variable τ defined in (3.1a).

We now state the scaling hypothesis for a simple magnet: *Close to the critical point and for large r the pair correlation function $C_2(\tau, H, \vec{r})$ is asymptotically* a GHF; i.e., there exist three numbers b_τ, b_H, and b_r such that for all positive λ,

$$C_2(\lambda^{b_\tau}\tau, \lambda^{b_H}H, \lambda^{b_r}\vec{r}) = \lambda C_2(\tau, H, \vec{r}) . \tag{4.2}$$

2. Equivalent Statement in Terms of Structure Factor

According to theorem 4, we could as well have made the scaling hypothesis about the structure factor, which is the spatial Fourier transform of $C_2(\tau, H, \vec{r})$; clearly $C_2(\tau, H, \vec{r})$ is a GHF for large $\vec{r}$ if and only if $S(\tau, H, \vec{q})$ is a GHF for small q, and by (2.14), we have[51]

$$S(\lambda^{b_\tau}\tau, \lambda^{b_H}H, \lambda^{b_q}\vec{q}) = \lambda^{1-db_q} S(\tau, H, \vec{q}) , \tag{4.3a}$$

where

$$b_q = -b_r . \tag{4.3b}$$

3. Relation between Correlation-Function Scaling Parameters and Thermodynamic-Function Scaling Parameters

The scaling powers of the variables τ, H, and $\vec{r}$ are chosen in Eq. (4.2) so that the scaling power

TABLE V. (a) Summary of scaling functions for M-H-τ data, as predicted by Eqs. (3.16), (3.20), and (3.24). (b) Summary of scaling functions for static structure factor data, as predicted by Eqs. (4.37). (c) Summary of scaling functions for the characteristic frequency data, as predicted by Eqs. (5.13). The scaled variables are uniquely defined in terms of the scaling powers (except that, of course, the ratio of two a's is equal to the ratio of two b's—e.g., $a_H/a_\tau = b_H/b_\tau$). However, the exponents shown for the scaled variables are not unique, and may be related to other exponents using the scaling laws. The two exponents β and δ were arbitrarily chosen for case (a), while the three exponents γ, ν, and μ were chosen for (b) and x, y, z for (c). Note that we need introduce only one new exponent (e.g., ν) in going from (a) to (b) (corresponding to the introduction of b_q), since $\gamma = \beta(\delta - 1)$ and $\mu = \nu/\beta\delta$, and similarly in going from (b) to (c) (say z, corresponding to b_ω), since $x = \nu z$ and $y = \nu z/\beta\delta$. Note that we have retained the subscript sgnτ only on $\mathfrak{F}^{(1)}$, $\mathfrak{G}^{(1)}$, and $\mathfrak{K}^{(1)}$ because other functions are perforce symmetric.

Function	Dependent variable	Independent variable(s)		Definition of scaling function
		(a)		
$H_\tau = \mathfrak{F}^{(1)}_{\mathrm{sgn}}(M_\tau)$	$H_\tau \equiv H/\lvert\tau\rvert^{a_H/a_\tau} = H/\lvert\tau\rvert^{\beta\delta}$	$M_\tau \equiv M/\lvert\tau\rvert^{a_M/a_\tau} = M/\lvert\tau\rvert^{\beta}$		$\mathfrak{F}^{(1)}_{\mathrm{sgn}\tau}(u) \equiv H(\mathrm{sgn}\tau, u)$
$H_M = \mathfrak{F}^{(2)}(\tau_M)$	$H_M \equiv H/\lvert M\rvert^{a_H/a_M} = H/\lvert M\rvert^{\delta}$	$\tau_M \equiv \tau/\lvert M\rvert^{a_\tau/a_M} = \tau/\lvert M\rvert^{1/\beta}$		$\mathfrak{F}^{(2)}(u) \equiv H(u, 1)$
$M_H = \mathfrak{F}^{(3)}(\tau_H)$	$M_H \equiv M/\lvert H\rvert^{a_H/a_H} = M/\lvert H\rvert^{1/\delta}$	$\tau_H \equiv \tau/\lvert H\rvert^{a_\tau/a_H} = \tau/\lvert H\rvert^{1/\beta\delta}$		$\mathfrak{F}^{(3)}(u) \equiv M(u, 1)$
		(b)		
$S_\tau = \mathfrak{G}^{(1)}_{\mathrm{sgn}\tau}(H_\tau, q_\tau)$	$S_\tau \equiv S/\lvert\tau\rvert^{b_S/b_\tau} = S/\lvert\tau\rvert^{-\gamma}$	$H_\tau \equiv H/\lvert\tau\rvert^{b_H/b_\tau} = H/\lvert\tau\rvert^{\nu/\mu}$	$q_\tau \equiv q/\lvert\tau\rvert^{b_q/b_\tau} = q/\lvert\tau\rvert^{\nu}$	$\mathfrak{G}^{(1)}_{\mathrm{sgn}\tau}(u, v) \equiv S(\mathrm{sgn}\tau, u, v)$
$S_H = \mathfrak{G}^{(2)}(\tau_H, q_H)$	$S_H \equiv S/\lvert H\rvert^{b_S/b_H} = S/\lvert H\rvert^{-\gamma\mu/\nu}$	$\tau_H \equiv \tau/\lvert H\rvert^{b_\tau/b_H} = \tau/\lvert H\rvert^{\mu/\nu}$	$q_H \equiv q/\lvert H\rvert^{b_q/b_H} = q/\lvert H\rvert^{\mu}$	$\mathfrak{G}^{(2)}(u, v) \equiv S(u, 1, v)$
$S_q = \mathfrak{G}^{(3)}(\tau_q, H_q)$	$S_q \equiv S/\lvert q\rvert^{b_S/b_q} = S/\lvert q\rvert^{-\gamma/\nu}$	$\tau_q \equiv \tau/\lvert q\rvert^{b_\tau/b_q} = \tau/\lvert q\rvert^{1/\nu}$	$H_q \equiv H/\lvert q\rvert^{b_H/b_q} = H/\lvert q\rvert^{1/\mu}$	$\mathfrak{G}^{(3)}(u, v) \equiv S(u, v, 1)$
		(c)		
$\omega^c_\tau = \mathfrak{K}^{(1)}_{\mathrm{sgn}\tau}(H_\tau, q_\tau)$	$\omega^c_\tau \equiv \omega^c/\lvert\tau\rvert^{b_\omega/b_\tau} = \omega^c/\lvert\tau\rvert^{x}$	$H_\tau \equiv H/\lvert\tau\rvert^{b_H/b_\tau} = H/\lvert\tau\rvert^{x}$	$q_\tau \equiv q/\lvert\tau\rvert^{b_q/b_\tau} = q/\lvert\tau\rvert^{x/z}$	$\mathfrak{K}^{(1)}_{\mathrm{sgn}\tau}(u, v) \equiv \omega^c(\mathrm{sgn}\tau, u, v)$
$\omega^c_H = \mathfrak{K}^{(2)}(\tau_H, q_H)$	$\omega^c_H \equiv \omega^c/\lvert H\rvert^{b_\omega/b_H} = \omega^c/\lvert H\rvert^{y}$	$\tau_H \equiv \tau/\lvert H\rvert^{b_\tau/b_H} = \tau/\lvert H\rvert^{y/x}$	$q_H \equiv q/\lvert H\rvert^{b_q/b_H} = q/\lvert H\rvert^{y/z}$	$\mathfrak{K}^{(2)}(u, v) \equiv \omega^c(u, 1, v)$
$\omega^c_q = \mathfrak{K}^{(3)}(\tau_q, H_q)$	$\omega^c_q \equiv \omega^c/\lvert q\rvert^{b_\omega/b_q} = \omega^c/\lvert q\rvert^{z}$	$\tau_q \equiv \tau/\lvert q\rvert^{b_\tau/b_q} = \tau/\lvert q\rvert^{z/x}$	$H_q \equiv H/\lvert q\rvert^{b_H/b_q} = H/\lvert q\rvert^{z/y}$	$\mathfrak{K}^{(3)}(u, v) \equiv \omega^c(u, v, 1)$

of the pair correlation function is unity. In Sec. III we chose the scaling power of the Gibbs potential to be unity. Thermodynamic functions are obtainable from correlation functions, and hence we would expect there to be some relation between the a's and b's. The bridge between thermodynamics and the pair correlation function is provided by the fluctuation-dissipation relation[52] (which is a particular case of a Fourier transform),

$$\chi_T(\tau, H) = \int d\vec{r}\, C_2(\tau, H, \vec{r}) \tag{4.4a}$$

$$= S(\tau, H, q=0). \tag{4.4b}$$

We may apply theorem 4 to (4.4a), or Eq. (4.3) to (4.4b), obtaining

$$\chi_T(\lambda^{b_\tau}\tau, \lambda^{b_H}H) = \lambda^{1+db_r}\chi_T(\tau, H) \tag{4.5a}$$

$$= \lambda^{1-db_q}\chi_T(\tau, H). \tag{4.5b}$$

Equation (4.3b) shows that (4.5a) and (4.5b) are equivalent. Now $\chi_T(\tau, H) = -G^{(0)}(\tau, H)$; hence upon twice differentiating Eq. (3.2) and applying theorem 2, we find that the GHF hypothesis for the thermodynamic functions yields

$$\chi_T(\lambda^{a_\tau}\tau, \lambda^{a_H}H) = \lambda^{1-2a_H}\chi_T(\tau, H) . \tag{4.6}$$

Comparing (4.5a) and (4.6), we obtain

$$a_\tau/(1-2a_H) = b_\tau/(1+db_r) \tag{4.7a}$$

and

$$a_H/(1-2a_H) = b_H/(1+db_r) , \tag{4.7b}$$

since the scaling powers of a function are unique, up to an arbitrary factor p [cf. Eq. (2.2)]. From (4.7b) we see that

$$b_H/a_H = 1 + db_r + 2b_H , \tag{4.8}$$

and one can similarly show that

$$b_x/a_x = (1+db_r)/(1-2a_H) = 1 + db_r + 2b_H \tag{4.9}$$

for all variables x [$x = H$ in Eq. (4.8)]. Indeed, according to theorem 3, the critical-point exponent for the path $H \to 0$ for any function f will be given equally by a_f/a_H or by b_f/b_H, whence (4.9) follows.

Equation (4.9) forms the bridge between the scaling powers for the static pair correlation function and the thermodynamic functions treated in Sec. III.

B. Critical-Point Exponents in Terms of b_τ, b_H, and b_r

In parallel to Sec. III B, we shall here utilize theorems 1–4 in constructing expressions for the critical-point exponents for functions related to the pair correlation function. As in Sec. III B, we shall see that by eliminating the unknown scaling powers we can obtain numerous relations among the critical-point exponents (scaling laws).

1. Generalized Correlation Length $\xi_\phi(\tau, H)$ and the Exponents ν_ϕ, μ_ϕ

The critical point is characterized by a dramatic increase in the *range* of the pair correlation function $C_2(\tau, H, \vec{r})$ and to this end we define the family of functions[53]

$$[\xi_\phi(\tau, H)]^{2\phi} \equiv [\int d\vec{r}\, |r|^{2\phi}\, C_2(\tau, H, \vec{r})]/ [\int d\vec{r}\, C_2(\tau, H, \vec{r})] . \tag{4.10}$$

The reader should note that $\xi_\phi(\tau, H)$ has the dimensions of length and we shall call it the generalized correlation length; it reduces to a common definition of the correlation length ξ for $\phi = 1$ (the second moment),

$$\xi_1(\tau, H) \equiv \xi(\tau, H) . \tag{4.11}$$

We proceed to demonstrate that $\xi_\phi(\tau, H)$ is a GHF with scaling power b_r. To begin with, the denominator of (4.10) is a GHF with scaling power $(1+db_r)$ as given by Eqs. (4.4) and (4.5). The integrand of the numerator is a GHF with scaling power $(2\phi b_r + 1)$ (cf. the discussion at the end of Sec. II A). By theorem 4 then, the numerator of (4.10a) is a GHF with scaling power $(2\phi b_r + 1 + db_r)$. Thus, the right-hand side of (4.10a) is a GHF with scaling power $(2\phi b_r)$ and hence $\xi_\phi(\tau, H)$ is a GHF with scaling power b_r,

$$\xi_\phi(\lambda^{b_\tau}\tau, \lambda^{b_H}H) = \lambda^{b_r}\xi_\phi(\tau, H) . \tag{4.12}$$

Applying now the corollary to theorem 3, we find

$$\xi_\phi(\tau, 0) \sim |\tau|^{b_r/b_\tau} \quad (\tau \to 0,\ H = 0) \tag{4.13a}$$

and

$$\xi_\phi(0, H) \sim |H|^{b_r/b_H} \quad (H \to 0,\ \tau = 0) . \tag{4.13b}$$

The critical-point exponents ν_ϕ and μ_ϕ, defined by the relations[53]

$$\xi_\phi(\tau, 0) \sim |\tau|^{-\nu_\phi} \tag{4.14a}$$

and

$$\xi_\phi(0, H) \sim |H|^{-\mu_\phi} , \tag{4.14b}$$

describe the fashion in which the generalized correlation length ξ_ϕ diverges at the critical point; we note that $\nu_1 \equiv \nu$ and $\mu_1 \equiv \mu$, the conventional exponents. Comparing (4.13) and (4.14) we have[54]

$$\nu_\phi = \nu = -b_r/b_\tau \quad (\text{all } \phi) \tag{4.15a}$$

and

$$\mu_\phi = \mu = -b_r/b_H \quad (\text{all } \phi) . \tag{4.15b}$$

Other authors have used the symbols ν_c or ϵ to denote $\mu\delta$.[11,20,21]

2. *Long-Range Decay Exponent* η_ϕ

At the critical point ($\tau=0$, $H=0$) the correlation length $\xi_\phi(0,0)=\infty$ but the correlation function $C_2(0,0,r)$ of (4.1) still decays to zero as $r\to\infty$. This decay is measured by the exponent η, defined through the relation

$$X_\phi(\tau=0,\ H=0,\ \bar{r})\sim\bar{r}^{\,2+\phi-\eta_\phi}\ , \tag{4.16a}$$

where the function X_ϕ is defined by

$$X_\phi(\tau,H,\bar{r})\equiv\int_0^{\bar{r}}|\,r\,|^{\phi}\,C_2(\tau,H,r)\,r^{d-1}\,dr\ . \tag{4.16b}$$

The exponent normally called η is defined by Eqs. (4.16a) and (4.16b) with $\phi=0$; thus $\eta_0\equiv\eta$.[55]

By theorem 2, the function $X_\phi(\tau,H,R)$ is a GHF with scaling power $1+(d+\phi)b_r$,

$$X_\phi(\lambda^{b_\tau}\tau,\lambda^{b_H}H,\lambda^{b_r}\bar{r})=\lambda^{1+(d+\phi)b_r}X_\phi(\tau,H,\bar{r})\ . \tag{4.17}$$

Setting $\tau=H=0$ and choosing $\lambda=|\,\bar{r}\,|^{-1/b_r}$, we find

$$X_\phi\,(0,0,\bar{r})=|\,\bar{r}\,|^{(1/b_r+d+\phi)}\,X_\phi(0,0,1)\ . \tag{4.18}$$

Comparing (4.18) with (4.16a) we have

$$\eta_\phi=\eta=2-d-1/b_r\ . \tag{4.19}$$

We shall denote the new exponents ν, μ, and η by the generic term *correlation-function critical-point exponents*, to distinguish them from the *thermodynamic-function critical-point exponents* considered in Sec. III.

3. *Scaling Laws Predicted by GHF Hypothesis*

Equations (4.15a), (4.15b), and (4.19) express ν, μ, and η, respectively, in terms of the correlation function scaling powers b_τ, b_H, and b_r. We can, in analogy to Eqs. (3.9), solve for b_τ, b_H, and b_r in terms of the exponents ν, μ, and η, with the result

$$b_\tau=1/\nu\,(d-2+\eta)\ , \tag{4.20a}$$

$$b_H=1/\mu\,(d-2+\eta)\ , \tag{4.20b}$$

and

$$b_r=-1/(d-2+\eta)\ . \tag{4.20c}$$

However, we *cannot* eliminate the three unknown scaling powers to obtain a single relationship among the three correlation-function critical-point exponents ν, μ, and η.

We can nevertheless obtain numerous relationships between the correlation-function critical-point exponents and the exponents for the thermodynamic functions. One method of obtaining such relations is by utilizing equations such as (2.12a), in which case we obtain from (4.14) and (4.15), in parallel with (3.8a), the relations

$$\frac{\nu}{\mu}=\frac{1-\alpha}{\psi}=\frac{\beta}{\delta^{-1}}=\frac{-\gamma}{\delta^{-1}-1}=\frac{-\alpha}{-\phi}\ . \tag{4.21}$$

A second, perhaps somewhat more systematic, method is to utilize Eq. (4.9) to express the a_x (and hence, by Table II and the discussion in Sec. III B, to express the exponents for all thermodynamic functions) directly in terms of the correlation-function scaling powers b_x. Then we utilize Eqs. (4.20) to express the b_x in terms of the correlation-function critical-point exponents.

For example, using this procedure we can express $-\gamma$, the susceptibility exponent for the path $\tau\to 0$ in terms of the correlation-length exponents ν and η,

$$\gamma=\frac{2a_H-1}{a_\tau}=2\,\frac{b_H}{b_\tau}-\frac{1+db_r+2b_H}{b_\tau}=(2-\eta)\nu \tag{4.22a}$$

or, alternatively, since $\gamma=\gamma'$ and $\nu=\nu'$,

$$\gamma'=(2-\eta)\nu'\ . \tag{4.22b}$$

Similarly, we can express $-(\delta-1)/\delta$, the susceptibility exponent for the path $H\to 0$, in terms of μ and η,

$$\frac{\delta-1}{\delta}=\frac{2a_H-1}{a_H}=2-\frac{1+db_r+2b_H}{b_H}=(2-\eta)\mu\ . \tag{4.23}$$

4. *Relations Not Predicted by GHF Hypothesis: Two-Exponent Relations*

This section concerns the exponent relations

$$d\nu=2-\alpha \tag{4.24a}$$

and

$$(\delta-1)/(\delta+1)=(2-\eta)/d\ . \tag{4.24b}$$

When the scaling hypothesis was first proposed, relations (4.24) were predicted to hold.[1,3,56] This is because the initial formulations involved assumptions different from (and in addition to) our assumption, Eq. (3.2). Equations (4.24) are borne out by two exactly soluble models, the $d=2$ Ising model ($\nu=1$, $\alpha=0$, $\delta=15$, and $\eta=\frac{1}{4}$) and by the $d=3$ spherical model ($\nu=1$, $\alpha=-1$, $\delta=5$, and $\eta=0$). However, numerical calculations for other model systems (such as the $d=3$ Ising and $d=3$ Heisenberg models) revealed a small but not easily ignorable failure of (4.24), the left-hand sides being 1–2% larger than the right-hand sides. Also numerous experiments have led to results for which the predictions of (4.24) are outside experimental error. Attempts have been made to explain the breakdown of (4.24) by suggesting that logarithmic terms might contribute[57] or by introducing a second length.[58]

One reason for the fact that historically there has been concern over the breakdown of Eqs. (4.24) is that the initial arguments[1,3] favoring (4.24) were appealing. A second reason is that Eqs. (4.24) correspond, respectively, to the

Josephson[59] and the Buckingham-Gunton-Stell[60] inequalities taken as equalities. The GHF hypothesis for thermodynamic functions led to scaling laws that corresponded to rigorous exponent inequalities with the $\geq$ sign replaced by an = sign, and the same was anticipated here.

The purpose of this section is first, to demonstrate clearly that relations (4.24) do not follow from making the GHF hypothesis (3.2); and second, to show that our formulation results in (4.24) only if a second assumption—in addition to Eq. (3.2)—is made. If it is not made, then we will find correction terms which change the relations (4.24) back into standard thermodynamic equalities (scaling laws), and hence without the second assumption, (4.24) are not valid.

a. Why Eqs. (4.24) *do not arise from the GHF assumption.* In general we would expect that the elimination of the three powers b_τ, b_H, and b_r to lead to relations among any four of the exponents, with at least two of these exponents being correlation-function exponents [see, e.g., Eq. (4.21)]. Since two thermodynamic exponents can often be expressed in terms of a third as in the example $(2-\eta)\nu=\beta(\delta-1)=\gamma$, we expect to find four- or three-exponent equalities, but never relations involving only two exponents, and therefore relations (4.24) can not be expected from the outset.

b. How Eqs. (4.24) *do arise if a "second assumption" is made.* Our formulation (3.2) can be made equivalent to the early work if we make a second assumption [in addition to Eq. (3.2)], namely, that the pair correlation function

$$C_2(\tau, H, \vec{r}) \equiv \langle s_0 s_{\vec{r}} \rangle - \langle s_0 \rangle^2$$

should have the same scaling power as the square of the magnetization, $M^2 \propto \langle s_0 \rangle^2$ [cf. Eq. (3.1)]. The second term in the definition of C_2 does indeed behave like M^2. However since $\lim_{r\to\infty} C_2(0,0,\vec{r}) = 0$, this behavior must be exactly cancelled by a term in $\langle s_0 s_{\vec{r}} \rangle$. Hence there is no *a priori* reason why C_2 should have the scaling properties of M^2 and in fact it is our contention that it need not.

Mathematically this second assumption would lead one to conclude that when $\epsilon \to \lambda^{b_\epsilon}\epsilon$, $H \to \lambda^{b_H} H$, then C_2 and M^2 should transform with the same power of λ; that is to say

$$\begin{aligned} C_2(\lambda^{b_\epsilon}\epsilon, \lambda^{b_H} H, \lambda^{b_r}\vec{r}) &= \lambda C_2(\epsilon, H, \vec{r}) \longleftrightarrow M^2(\lambda^{b_\epsilon}\epsilon, \lambda^{b_H} H) \\ &= \lambda^{2b_M} M^2(\epsilon, H)\ , \end{aligned} \tag{4.25a}$$

where

$$2b_M = 1\ , \tag{4.25b}$$

thereby eliminating one of original three undetermined scaling parameters, since from theorem 3, and Eq. (4.5a),

$$b_M = 1 + db_r + b_H\ . \tag{4.26}$$

Hence the exponents μ, ν, and η can be related in additional fashions. The additional relations are Eqs. (4.24); these are the Josephson and Buckingham-Gunton-Stell inequalities taken as equalities. To prove this assertion, we consider (4.24a) and (4.24b) *seriatum*.

First we evaluate the right-hand side of (4.24a):

$$2-\alpha = \frac{1}{a_\tau} = \frac{b_M + b_H}{b_\tau} = -\frac{db_r}{b_\tau} + \frac{2b_M - 1}{b_\tau}\ , \tag{4.27}$$

where we have used Table II and Eqs. (4.9) and (4.26). If $b_M = \frac{1}{2}$, then (4.24a) follows directly from (4.27) on using (4.15a). If, on the other hand, we do not make this second assumption then we find on combining $\beta = b_M / b_\tau$ and (4.20c) with (4.27) that[61]

$$d\nu = 2 - \alpha + X\ , \tag{4.28}$$

where

$$X \equiv (d - 2 + \eta)\nu - 2\beta \tag{4.29}$$

is a "correction term." One can show that[62]

$$X \geq 0\ , \tag{4.30a}$$

so that the Josephson inequality

$$d\nu \geq 2 - \alpha \tag{4.30b}$$

is consistent with the GHF assumption. The fact that violations of (4.24a) are consistent with (4.30b) supports the GHF hypothesis (3.2).

Note also that (4.28) can be combined with (4.22a) to yield

$$2 - \alpha = 2\beta + \gamma\ , \tag{4.31}$$

which is just the Rushbrooke equality (cf. Table III), a consequence of thermodynamic scaling. The validity of (4.31) further supports the scaling hypothesis.

The second problematic equality is (4.24b), and so we inspect $(\delta-1)/(\delta+1)$, using $\delta = a_H/a_M$ (cf. Table I) and Eqs. (4.9) and (4.26),

$$\frac{\delta-1}{\delta+1} = \frac{b_H - b_H}{b_H + b_M} = -\frac{1+db_r}{1+db_r+2b_H}\ . \tag{4.32}$$

Now the second assumption, $b_M = \frac{1}{2}$, leads to $1 + 2b_H = -2db_r$ from (4.26) and thence to (4.24b) on using (4.20c). When the analysis of (4.32) is carried through fully, we obtain

$$\frac{\delta-1}{\delta+1} = \frac{2-\eta}{2\beta\delta/\nu - 2 + \eta} = \frac{\gamma}{2\beta\delta - \gamma}\ . \tag{4.33}$$

Multiplying this out gives the Widom equality

$$\beta(\delta-1) = \gamma \tag{4.34}$$

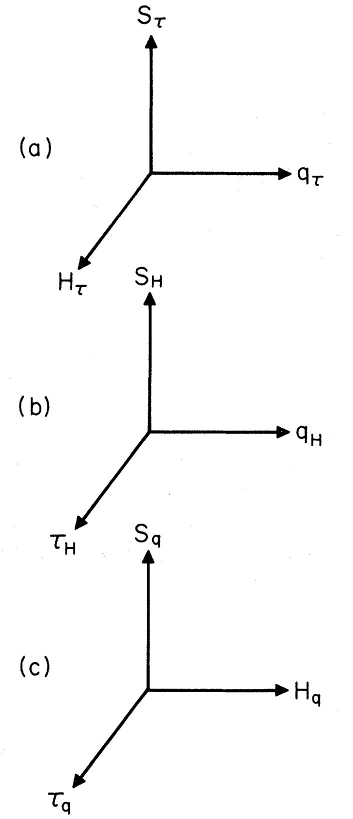

FIG. 3. Coordinate systems which can be used to test the validity of the scaling hypothesis (4.2) for the static correlation function. If data are found to collapse onto a single surface in *any* of the three coordinate systems, then it follows from theorem 3 that (4.2) is valid. The surface will appear as a line in any two-dimensional section of the space [e.g., $H=0$ in (a)]. (a)–(c) denote scaling of the structure factor $S(\tau, H, q)$ with respect to τ, H, and q, respectively [cf. Eqs. (4.37)]. The surfaces are described by the functions $\mathcal{G}_\pm^{(1)}(H_\tau, q_\tau)$, $\mathcal{G}^{(2)}(\tau_H, q_H)$, and $\mathcal{G}^{(3)}(\tau_q, H_q)$; these functions, as well as the scaled variables H_τ, q_τ, etc., are defined in Table V, part (b). Calculation of the scaling functions is underway (Ref. 66).

and its validity (cf. Table III) supports the validity of the GHF hypotheses (3.2) and (4.2).

C. Scaling Functions for Static Correlation Function

According to theorem 3, any GHF can be scaled with respect to any of its arguments. In this section we have seen that the pair correlation function $C_2(\tau, H, r)$, its spatial Fourier transform $S(\tau, H, q)$, and its normalized ϕth moment $\xi_\phi(\tau, H)$ are all GHFs according to the scaling hypothesis. Of these three functions, the structure factor $S(\tau, H, q)$ is probably the most directly measurable, as it is proportional to the total scattering cross section.

Before discussing the scaling properties of the structure factor, we remark that the generalized correlation length $\xi_\phi(\tau, H)$ has the same scaling properties as $M(\tau, H)$, providing we let $a_M \to b_\tau$, $a_\tau \to b_\tau$, and $a_H \to b_H$ in our analysis of Sec. IIIC (cf. Table V).

For $S(\tau, H, q)$, however, we have three independent variables. The appropriate scaling functions are now functions of two scaled variables and data are predicted to collapse onto a surface rather than onto a single curve (cf. Figs. 1–3).[63] Applying theorem 3 successively to each variable in $S(\tau, H, q)$, we obtain

$$\frac{S}{|\tau|^{b_S/b_\tau}} = S\left(\mathrm{sgn}\tau, \frac{H}{|\tau|^{b_H/b_\tau}}, \frac{q}{|\tau|^{b_q/b_\tau}}\right), \tag{4.35a}$$

$$\frac{S}{|H|^{b_S/b_H}} = S\left(\frac{\tau}{|H|^{b_\tau/b_H}}, 1, \frac{q}{|H|^{b_q/b_H}}\right), \tag{4.35b}$$

$$\frac{S}{|q|^{b_S/b_q}} = S\left(\frac{\tau}{|q|^{b_\tau/b_q}}, \frac{H}{|q|^{b_H/b_q}}, 1\right), \tag{4.35c}$$

where, from theorem 4 and Eqs. (4.3) and (4.20c),

$$b_S \equiv b_\chi = 1 - db_q = 1 - d/(d-2+\eta). \tag{4.36}$$

Equations (4.35) may be written in the form

$$S_\tau = S(\mathrm{sgn}\tau, H_\tau, q_\tau) \equiv \mathcal{G}^{(1)}_{\mathrm{sgn}\tau}(H_\tau, q_\tau), \tag{4.37a}$$

$$S_H = S(\tau_H, 1, q_H) \equiv \mathcal{G}^{(2)}(\tau_H, q_H), \tag{4.37b}$$

$$S_q = S(\tau_q, H_q, 1) \equiv \mathcal{G}^{(3)}(\tau_q, H_q), \tag{4.37c}$$

where

$$S_\tau \equiv S/|\tau|^{b_S/b_\tau} = S/|\tau|^{-\gamma}, \tag{4.38a}$$

$$S_H \equiv S/|H|^{b_S/b_H} = S/|H|^{-\gamma/\Delta}, \tag{4.38b}$$

$$S_q \equiv S/|q|^{b_S/b_q} = S/|q|^{-\gamma/\nu} \tag{4.38c}$$

represent the structure factor scaled, respectively, with respect to the variables τ, H, and q. The independent variables H_τ and τ_H are defined in (3.17a) and (3.25b), while the remaining variables q_τ, q_H, τ_q, and H_q are all defined in Table V, part (b).

Experimental data of Smith *et al.*[64] are being analyzed[65] to test the predictions of Eqs. (4.37) and calculation of the appropriate scaling functions $\mathcal{G}^{(1)}$, $\mathcal{G}^{(2)}$, and $\mathcal{G}^{(3)}$ is underway.[66]

V. DYNAMIC CORRELATION FUNCTIONS

In this section we formulate the scaling hypothesis for the dynamic pair correlation function, or, equivalently, for the dynamic structure factor. We shall demonstrate that the hypothesis presented here implies the scaling hypotheses presented in Secs. III and IV. Throughout this section, H refers to the ordering field; this is a magnetic field for a ferromagnet and a staggered magnetic field for an antiferromagnet.

A. GHF Hypothesis for the Dynamic Pair-Correlation Function

1. Statement

The dynamic pair correlation function is defined in analogy to (4.1),

$$C_2(\tau, H, \vec{r}, t) \equiv \langle s_0(t=0) s_{\vec{r}}(t)\rangle - \langle s_0(0)\rangle\langle s_{\vec{r}}(t)\rangle \quad . \tag{5.1}$$

The dynamic scaling hypothesis still takes the same form: *Close to the critical point, and for large* $|\vec{r}|$ *and* t, *the dynamic pair correlation function* $C_2(\tau, H, \vec{r}, t)$ *is asymptotically a* GHF; i.e., one can find four numbers b_τ, b_H, b_r, and b_t such that, for all positive λ,

$$C_2(\lambda^{b_\tau}\tau, \lambda^{b_H}H, \lambda^{b_r}\vec{r}, \lambda^{b_t}t) = \lambda\, C_2(\tau, H, \vec{r}, t) \quad . \tag{5.2}$$

The b's introduced in (5.2) are the same as the b's of Sec. IV, since

$$C_2(\tau, H, \vec{r}) = C_2(\tau, H, \vec{r}, t=0)$$

[cf. the definitions (4.1) and (5.1)], and indeed the validity of (5.2) implies the validity of (4.2), which in turn implies the validity of (3.2). Thus, dynamic scaling leads to all of static scaling.

2. Equivalent Statement in Terms of Structure Factor

A large number of experimental quantities are directly expressible in terms of the dynamic structure factor, which is the Fourier transform in space and in time of the pair correlation function,

$$\mathcal{S}(\tau, H, \vec{q}, \omega) \equiv \int d\vec{r} \int_{-\infty}^{\infty} dt\, C_2(\tau, H, \vec{r}, t) e^{-i(\vec{q}\cdot\vec{r}-\omega t)} \quad . \tag{5.3}$$

According to theorem 4 of Sec. II D, homogeneity statements about (5.1) or (5.3) are equivalent, and so the dynamic structure factor is a GHF,

$$\mathcal{S}(\lambda^{b_\tau}\tau, \lambda^{b_H}H, \lambda^{b_q}\vec{q}, \lambda^{b_\omega}\omega) = \lambda^{1-db_q-b_\omega}\, \mathcal{S}(\tau, H, \vec{q}, \omega) \, , \tag{5.4a}$$

where $b_q = -b_r$ as in Eq. (4.3b) and

$$b_\omega = -b_t \quad . \tag{5.4b}$$

B. Recovery of Halperin-Hohenberg Hypotheses, when $H=0$

We can easily obtain from (5.2)—or, equivalently, from (5.4)—both of the dynamic scaling hypotheses proposed by Halperin and Hohenberg. Let us assume that the ω dependence of the dynamic structure factor is dominated as the critical point is approached by a large central peak (about $\omega=0$). Then a parameter of significance is the half-width ω^c, which becomes zero at the critical point. This is defined as the frequency such that half the area under a curve of structure factor versus frequency lies between $-\omega^c$ and $+\omega^c$,

$$\int_{-\omega^c}^{\omega^c} \mathcal{S}(\tau, H, q, \omega)\, d\omega = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathcal{S}(\tau, H, q, \omega)\, d\omega \quad . \tag{5.5}$$

Equivalently, on multiplying both sides of (5.5) by

$$\tau^{(b_S - b_\omega)/b_\tau} \equiv \tau^{(1-db_q-2b_\omega)/b_\tau} \, , \tag{5.6}$$

we have

$$\int_{-\omega_\tau^c}^{\omega_\tau^c} \mathcal{S}(1, H_\tau, q_\tau, \omega_\tau)\, d\omega_\tau = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathcal{S}(1, H_\tau, q_\tau, \omega_\tau)\, d\omega_\tau \, , \tag{5.7}$$

where

$$H_\tau \equiv H/|\tau|^{b_H/b_\tau}, \quad q_\tau \equiv q/|\tau|^{b_q/b_\tau}, \quad \omega_\tau \equiv \omega/|\tau|^{b_\omega/b_\tau} \quad . \tag{5.8}$$

By inspection of (5.7) we see that ω_τ^c depends on the variables H_τ and q_τ and therefore it is a function of them:

$$\omega_\tau^c \equiv \omega^c/|\tau|^{b_\omega/b_\tau} = \omega_\tau^c(H_\tau, q_\tau) \quad . \tag{5.9}$$

Using the corollary to theorem 3, it follows from (5.9) that ω^c is a GHF of the same scaling power as ω,

$$\omega^c(\lambda^{b_\tau}\tau, \lambda^{b_H}H, \lambda^{b_q}q) = \lambda^{b_\omega}\omega^c(\tau, H, q) \quad . \tag{5.10}$$

On setting $H=0$ in (5.10), and changing variables from temperature τ to inverse correlation length $\kappa = |\tau|^{b_q/b_\tau} = \tau^\nu$, we have

$$\omega^c(\lambda^{b_q}\kappa, 0, \lambda^{b_q}q) = \lambda^{b_\omega}\omega^c(\kappa, 0, q) \quad . \tag{5.11}$$

Equation (5.11) explicitly shows that ω^c is a *homogeneous* function of κ and q, which is the first of the two dynamic scaling hypotheses of Ref. 31.

To obtain the second statement proposed as a hypothesis in Ref. 31, we define the shape function $F(x)$ as a sort of "normalized" dynamic structure factor in which the total area is unity and the characteristic frequency ω^c corresponds to an argument of unity,

$$F(\omega/\omega^c) \equiv \omega^c\, \mathcal{S}(\tau, H, q, \omega)/\int_{-\infty}^{\infty} \mathcal{S}(\tau, H, q, \omega)\, d\omega$$
$$= \omega_\tau^c\, \mathcal{S}(1, H_\tau, q_\tau, \omega_\tau)/\int_{-\infty}^{\infty} \mathcal{S}(1, H_\tau, q_\tau, \omega_\tau)\, d\omega_\tau . \tag{5.12}$$

From (5.12) we see that $\int_{-\infty}^{\infty} F(\Omega)\, d\Omega = 1$, where $\Omega \equiv \omega/\omega^c$. Equation (5.12) shows that for $H=0$, F depends on q and τ again only in the invariant form proposed previously: it depends on q, τ only as q/κ. We can again extend this form of the hypothesis to nonzero H: q and H enter in—and only in—the invariant combinations q_τ, H_τ.

C. Critical-Point Exponents and Scaling Functions

A quantity of special experimental interest is the characteristic frequency ω^c defined in (5.5). This is a function of the three variables τ, H, and $\vec{q}$. The scaling functions in this case are therefore to be formed and treated analogously to the scaling functions in Sec. IV. This means that (if we restrict our-

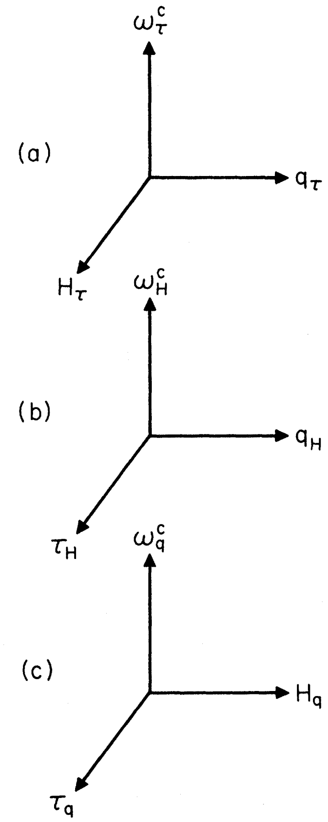

FIG. 4. Coordinate systems which can be used to test the validity of the scaling hypothesis (5.2) for the dynamic correlation function [strictly speaking, the characteristic frequency defined in Eq. (5.5)]. If data are found to collapse upon a surface, then (5.2) must be valid. (a)–(c) denote scaling of the characteristic frequency $\omega^c(\tau, H, q)$ with respect to τ, H, and q, respectively [cf. Eqs. (5.13)]. The surfaces described are the functions defined in Table V, part (c). In particular, the data of Ref. 67 correspond to a $H=0$ section of (a), and accordingly the data are found to collapse upon a single line.

selves to scaling with respect to intensive variables) we will have three possible three-dimensional plots. These will have principal axis triads (cf. Fig. 4) $(\omega_\tau^c, H_\tau, q_\tau)$, $(\omega_H^c, \tau_H, q_H)$, and $(\omega_q^c, \tau_q, H_q)$, where $\omega_\tau^c \equiv \omega^c/|\tau|^{b_\omega/b_\tau}$, etc. From Eq. (5.10) it is seen that

$$\omega_\tau^c \equiv \omega^c/|\tau|^{b_\omega/b_\tau} = \omega^c(\mathrm{sgn}\tau, H_\tau, q_\tau) , \qquad (5.13a)$$

$$\omega_H^c \equiv \omega^c/|H|^{b_\omega/b_H} = \omega^c(\tau_H, 1, q_H) , \qquad (5.13b)$$

$$\omega_q^c \equiv \omega^c/|q|^{b_\omega/b_q} = \omega^c(\tau_q, H_q, 1) . \qquad (5.13c)$$

From Eqs. (5.13a)–(5.13c) we have three new scaling functions [cf. Table V, part (c) and Fig. 4] and three new critical-point exponents:

$$x \equiv b_\omega/b_\tau , \qquad (5.14a)$$

$$y \equiv b_\omega/b_H , \qquad (5.14b)$$

$$z \equiv b_\omega/b_q . \qquad (5.14c)$$

These new exponents x, y, z of the characteristic frequency are related by scaling laws. Being for the same function, these scaling laws are of the type given by Eq. (2.12a). They are

$$x/y = b_H/b_\tau = \Delta , \qquad (5.15a)$$

$$y/z = b_q/b_H = \mu , \qquad (5.15b)$$

$$x/z = b_q/b_\tau = \nu . \qquad (5.15c)$$

Care has to be taken in applying these, because in experimental determinations of z for $T > T_c$, $\log_{10}\omega^c$ is usually plotted against $\log_{10}\tau$. Often this is disguised as a plot of $\log_{10}\omega^c$ against $\log_{10}\kappa = \nu \log_{10}\tau$. The slope of these graphs is then used to evaluate z. In fact, the slope is given by x/ν and this is only equal to z if scaling is valid and (5.15c) holds. The point is that in checking the equalities (5.15) care must be taken to determine the precise nature of the exponent data.

In Sec. IV D values of b_ω are discussed for various models. These values can be used in conjunction with (5.14) and (5.15) to find the values of the other exponents in these models.

Equation (5.13a) has been studied in zero-ordering field ($H=0$) for the antiferromagnet MnF_2 by Schulhof, Heller, Nathans, and Linz,[67] who plot $\omega_\tau^c \equiv \omega_c/\tau^{b_\omega/b_\tau} \equiv \omega_c/\kappa^z$ vs $q_\tau \equiv q/\tau^{b_q/b_\tau} = q/\kappa$. They found data collapsing onto two curves; one for $T < T_c$ and the other for $T > T_c (\mathrm{sgn}\tau = \pm 1)$.

D. Values of New Scaling Parameter b_ω for Various Systems

The original way of expressing the characteristic frequency was to write $\omega^c(q) \equiv q^z f(\vec{q}/\kappa)$, where q is the wave vector and κ is the inverse correlation length, and $z = b_\omega/b_q$ from (5.14c). Detailed analyses of various models[31] predict values for z and thereby enable one to eliminate b_ω in particular cases. Since these values of z are different for different systems, there is no system-independent value for b_ω.[68]

For liquid-helium and isotropic antiferromagnets the predictions of the hydrodynamic theories are that $z = \frac{3}{2}$ or

$$b_\omega = \tfrac{3}{2} b_q . \qquad (5.16)$$

This has been confirmed by experiments on MnF_2.[67]

For ferromagnets, previous work[31] predicts $z = -3\beta/\nu$. Using (5.14c) and Tables II and III, we have

$$b_\omega = 3b_q - b_M . \qquad (5.17)$$

Using (4.26), (5.17) becomes

$$b_\omega = (3+d)b_q - 1 - b_H . \qquad (5.18)$$

Now *if* the "second assumption," Eq. (4.25b), that

$2b_M = 1$ is valid, *then* (5.18) and (4.26) imply that for $d = 3$[69]

$$b_\omega = b_H . \tag{5.19}$$

The reader should note that the well-known equation

$$z = \tfrac{1}{2}(d + 2 - \eta) \tag{5.20}$$

follows from (4.26) and (5.18) *only* if the "second assumption," Eq. (4.25b), is made.

For gas-liquid critical points the hydrodynamic prediction[31] is

$$z = 2 + (\gamma - a)/\nu , \tag{5.21}$$

where a is the thermal-conductivity exponent. The mode-mode coupling[70] hypothesis predicts $(\gamma - a) = \nu$ for $d = 3$ and so in this case we obtain $z = 3$. Thus, for fluid systems the prediction of hydrodynamics and mode-mode coupling is

$$b_\omega = 3b_q . \tag{5.22}$$

VI. GENERALIZED HOMOGENEOUS FUNCTIONS AND UNIVERSALITY HYPOTHESIS

The universality hypothesis has arisen from attempts to answer the question: "On what features of an interaction Hamiltonian do critical-point exponents depend?" A universal class is a set of models of systems for which the critical-point exponents are equal. Recent studies have been made on model Hamiltonians containing parameters chosen such that, by varying the parameter, the universal class can be altered; these studies have led workers to conclude that a second kind of scaling hypothesis is possible, namely, one for which these parameters are also scaled.[71–75] The main results of this second kind of hypothesis are discussed in Secs. VIA–VIC, and contact with previous work is made where appropriate. We shall see that (i) there exists a consistency condition on the variation of critical temperature with the parameter, and (ii) it is possible to apply theorem 3 of Sec. IIC to each GHF hypothesis successively (if appropriate conditions are satisfied), and determine the power law behavior of two variables, one of which is the new parameter.

A. Dependence of T_c upon a Parameter

To illustrate point (i) above, let us consider a magnetic system with independent variables T, H, and some parameter R. Although the arguments we shall present are quite general, the reader may find it convenient, for the sake of concreteness, to consider a specific system, for example, the Ising Hamiltonian with lattice anisotropy (different coupling strengths in different lattice directions),

$$\mathcal{H}_{\text{lat anis}} = -\sum_{\langle ij \rangle}^{xy} J_{xy} S_i^z S_j^z - \sum_{\langle ij \rangle}^{z} J_z S_i^z S_j^z$$

$$= -J_{xy} \Big(\sum_{\langle ij \rangle}^{xy} S_i^z S_j^z + R \sum_{\langle ij \rangle}^{z} S_i^z S_j^z\Big), \tag{6.1}$$

where $R \equiv J_z/J_{xy}$. In Eq. (6.1), the first summation is over pairs of nearest-neighbor sites i and j whose relative displacement vector $\vec{r}_{ij}$ has no z component, while the second summation is over all other pairs of nearest-neighbor sites. Thus, when $R = 0$, this system is a simple quadratic lattice of dimensionality $d = 2$, while for any finite value of $R > 0$, it is a lattice of dimensionality $d = 3$.

Series expansions have recently been calculated[76] for arbitrary values of R, and their analysis supports the predictions of the universality hypothesis[77,74] that critical-point exponents change discontinuously from their $d = 2$ values for $R = 0$ (e.g., $\gamma = \frac{7}{4}$) to their $d = 3$ values for $R > 0$ (e.g., $\gamma = \frac{5}{4}$). For small nonzero values of R, although the limiting values of the exponents are three dimensional, the region over which the three-dimensional behavior is valid shrinks as R becomes smaller.[76] The critical temperature $T_c(R)$ was also found to depend on R in a singular fashion.

These findings are all interpretable in terms of simple statements about GHFs, and that is the purpose of the present section. We emphasize that the arguments of this section are more general than the Hamiltonian (6.1), and that we believe our discussion is somewhat more general and more simple than previous treatments.

For a system described by a Hamiltonian of the form of (6.1), we expect that the critical temperature T_c will be a continuous function of the parameter $R \equiv J_z/J_{xy}$. Hence if we were to apply the treatment of Secs. III–V to a real system with a fixed value of R, we would be making the various GHF hypotheses in the temperature variable

$$\tau_R \equiv T - T_c(R). \tag{6.2a}$$

For example, Eq. (3.2) is a GHF hypothesis in the variables τ_R and H,

$$G(\lambda^{a_\tau} \tau_R,\ \lambda^{a_H} H;\ R) = \lambda G(\tau_R,\ H; R). \tag{6.3a}$$

We shall show below [cf. Eq. (6.21)] that (6.3a) is useful only for τ_R and H in a restricted range.

A natural question is: "Can the parameter R scale as well?" We make the hypothesis that if we express the Gibbs potential as a function of the temperature variable

$$\tau_0 \equiv T - T_c\,(R = 0), \tag{6.2b}$$

then this function, call it $\overline{G}$, is a GHF—i.e., we can find three numbers $\bar{a}_\tau$, $\bar{a}_H$, and $\bar{a}_R$ such that for all positive λ,

$$\overline{G}(\lambda^{\bar{a}_\tau} \tau_0,\ \lambda^{\bar{a}_H} H,\ \lambda^{\bar{a}_R} R) = \lambda\, \overline{G}\,(\tau_0,\ H,\ R). \tag{6.3b}$$

It is important to emphasize that the scaling powers a_τ, a_H of Eq. (6.3a) are not the same as the barred scaling powers $\bar{a}_\tau$, $\bar{a}_H$ of (6.3b). In fact, they are not even proportional to one another, for if they were proportional then certain critical point exponents (e.g., Δ) would of necessity be equal. Indeed, for our lattice-anisotropy example of Eq. (6.1), the exponents derived from (6.3a) (i.e., from the unbarred scaling powers) should be those appropriate to a $d=3$ (three-dimensional) lattice, while those exponents derived from (6.3b) (i.e., from the barred scaling powers) should be those appropriate to a $d=2$ lattice. The latter statement follows from setting $R=0$ in Eq. (6.3b), whereupon the potential $\bar{G}(\tau_0, H, 0)$ denotes the Gibbs potential of a $d=2$ lattice.[78]

Now we show that the simultaneous validity of (6.3a) and (6.3b) results in a relation between the new scaling powers $\bar{a}_R$, $\bar{a}_\tau$ and the variation of $T_c(R)$ with R. This relation can be obtained by applying theorem 3 successively to the magnetization functions

$$M(\tau_R, H; R) \equiv \frac{\partial}{\partial H} G(\tau_R, H; R)$$

and

$$\bar{M}(\tau_0, H, R) \equiv \frac{\partial}{\partial H} \bar{G}(\tau_0, H, R).$$

We obtain, following Sec. IIC, the result that, in zero field,

$$M(\tau_R, 0; R) = |\tau_R|^\beta M(1, 0; R) \tag{6.4a}$$

and

$$\bar{M}(\tau_0, 0, R) = |\tau_0|^{\bar{\beta}} \bar{M}(1, 0, R/|\tau_0|^\varphi), \tag{6.4b}$$

where $\beta = (1-a_H)/a_\tau$, $\bar{\beta} = (1-\bar{a}_H)/\bar{a}_\tau$, and

$$\varphi \equiv \bar{a}_R/\bar{a}_\tau. \tag{6.5}$$

Now both magnetization functions are zero at $T = T_c(R)$—i.e., both (6.4a) and (6.4b) must be zero. If $R \neq 0$, then the critical point is at $T = T_c(R)$ and $\tau_R = 0$. Hence either (i) $\tau_0 = 0$, implying that $T_c(R) = T_c(0)$ for all R, or else (ii) the function $\bar{M}(1, 0, R/|\tau_0|^\varphi)$, of (6.4b) must have a zero for some fixed value of R—call it R_0. Possibility (ii) is physically interesting and yields a condition for the simultaneous validity of Eqs. (6.3a) and (6.3b), namely,

$$R/|\tau_0|^\varphi = R_0 \quad [T = T_c(R)], \tag{6.6a}$$

which implies that

$$T_c(R) - T_c(0) = K R^{1/\varphi}, \tag{6.6b}$$

where the constant K is defined by $K \equiv R_0^{-1/\varphi}$. Equation (6.6b) is also a necessary condition for the simultaneous validity of Eqs. (6.3a) and (6.3b) because the possibility (i) corresponds to simply $K=0$.

Note that (6.6b) provides a condition on the sign of the new scaling power $\bar{a}_R$. Equation (6.6b) implies that

$$\varphi \equiv \bar{a}_R/\bar{a}_\tau > 0, \tag{6.7a}$$

and since $\bar{a}_\tau > 0$ also (cf. Appendix C), we have

$$\bar{a}_R > 0. \tag{6.7b}$$

Condition (6.7) on φ and $\bar{a}_R$ seems to have been overlooked by some authors.[75]

We now provide a simple geometric argument that leads to (6.6b). Consider the coordinate system shown in Fig. 5(a), where the abscissa denotes $\tau^{1/\bar{a}_\tau}$ and the ordinate denotes $R^{1/\bar{a}_R}$. Suppose that we have a system characterized by a parameter R_1, as shown by the horizontal dashed line in Fig. 5(a). Lowering the temperature corresponds to moving to the left along the dashed line. At a point Q we reach the critical temperature $T_c(R)$ for this system. Now from (6.4) we see that the Gibbs potential in zero field is a homogeneous function of the variables plotted on the abscissa and ordinate of Fig. 5(a), and therefore if any thermodynamic function is singular at point Q, it is singular along the entire line that passes through the origin and point Q. Accordingly, it follows that

$$T_c(R) - T_c(0) \propto R^{\bar{a}_\tau/\bar{a}_R}, \tag{6.8}$$

where the exponent in (6.8) is just φ^{-1} on account of the definition (6.5). Note that (6.8) again illustrates the corollary to theorem 3 that an exponent is simply the ratio of the scaling power of the function to the scaling power of the variable that is approaching zero.

B. Power-Law Behavior in Two Variables

In this section we study how functions vary when both τ and R approach zero. The GHF approach gives, in a simple fashion, the double power law behavior originally suggested for spin anisotropy.[71]

Again, we treat for the sake of specificity, the lattice-anisotropy Hamiltonian of Eq. (6.1), and we make the scaling hypotheses (6.2) and (6.3). Consider, e.g., the susceptibility function, for which (6.2) and (6.3) imply, respectively, the relations

$$\chi_T(\lambda^{a_\tau}\tau_R, \lambda^{a_H}H; R) = \lambda^{a_\chi}\chi_T(\tau_R, H; R) \tag{6.9}$$

and

$$\bar{\chi}_T(\lambda^{\bar{a}_\tau}\tau_0, \lambda^{\bar{a}_H}H, \lambda^{\bar{a}_R}R) = \lambda^{\bar{a}_\chi}\bar{\chi}_T(\tau_0, H, R), \tag{6.10}$$

where $a_\chi = 1 - 2a_H$ and $\bar{a}_\chi = 1 - 2\bar{a}_H$.

Next we apply theorem 3 of Sec. IIC to (6.9) and (6.10), with the results

$$\chi_T(\tau_R, H; R) = |\tau_R|^{-\gamma}\chi_T\left(1, \frac{H}{|\tau_R|^{a_H/a_\tau}}; R\right) \tag{6.11}$$

and

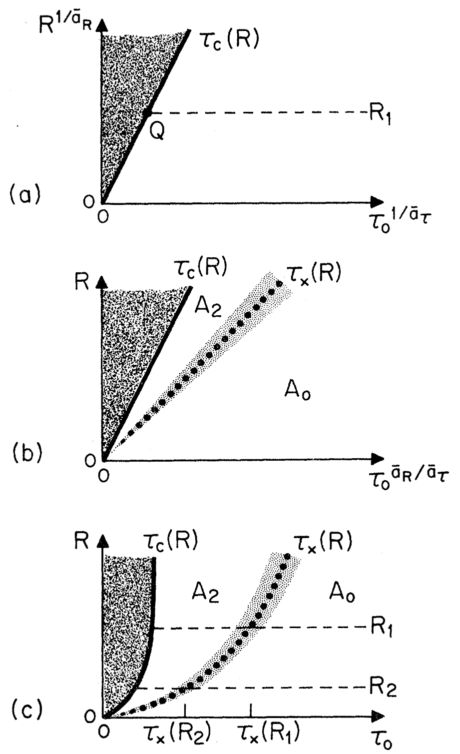

FIG. 5. Important regions in scaling with a parameter. (a) A plot of $R^{1/\bar a_R}$ vs $\tau_0^{1/\bar a_\tau}$. Note that the line of critical points $\tau_c(R)$ which is determined by scaling is straight in this plot due to Eq. (6.3b). [Since the barred functions are homogeneous functions of the fully scaled variables, if a system is singular at a point Q, it will be singular along the entire ray labeled $\tau = \tau_c(R)$.] The shaded region to the left of this line is a coexistence surface. (b) R vs $\tau_0^{\bar a_R/\bar a_\tau}$; this is exactly analogous to Halperin-Hohenberg diagram (Ref. 31), where there q is plotted vs $\kappa \equiv |\tau|^{b_q/b_\tau}$. Note that we include the crossover line $\tau_x(R)$ which is also straight. A_2 is the region where two scaling hypotheses hold [Eqs. (6.3a) and (6.3b)] and $R \neq 0$ behavior is predicted. A_0 is the region where only the hypothesis about $\tau = 0$, $R = 0$ is valid [Eq. (6.3(b)] and hence $R = 0$ behavior is predicted. (c) R vs τ, assuming that $a_R > a_\tau$. In this case there is a relatively large region of A_2 ($R \neq 0$ behavior) as $\tau_c(R)$ is approached along R_1 and R_2. This would not be the case if $\bar a_\tau > \bar a_R$. In general $\varphi > 1$ so $\bar a_R > \bar a_\tau$. For lattice anisotropy, system 1 (with $R = R_1$) might correspond to Rb_2NiF_4 and system 2 (with $R = R_2$) to K_2NiF_4, with $\tau_x(R_2)$ being below the limits of experimental resolution. For spin anisotropy, system 1 might be FeF_2 and system 2, MnF_2.

$$\bar\chi_T(\tau_0, H, R) = |\tau_0|^{-\bar\gamma}\, \bar\chi_T\left(1,\ \frac{H}{|\tau_0|^{\bar a_H/\bar a_\tau}},\ \frac{R}{|\tau_0|^{\bar a_R/\bar a_\tau}}\right), \tag{6.12}$$

where $-\gamma \equiv a_x/a_\tau$ and $-\bar\gamma \equiv \bar a_x/\bar a_\tau$ but χ_T and $\bar\chi_T$ are equal at equal values of τ, H, and R. If we choose $H = 0$ and express (6.6b) as $\tau_R = \tau_0 - KR^{1/\varphi}$, then equating the right-hand sides of (6.12) and (6.11) gives

$$\bar\chi_T(1, 0, R/|\tau_0|^\varphi) = |\tau_0|^{\bar\gamma}(\tau_0 - KR^{1/\varphi})^{-\gamma}\chi_T(1, 0; R) . \tag{6.13}$$

This functional equation demands that the right-hand side be a function of $R/|\tau_0|^\varphi$ only, and thereby determines the functional form of both $\chi_T(1,0;R)$ and $\chi_T(1,0,R/|\tau_0|^\varphi)$. To see this latter statement, we rewrite (6.13), in the form

$$\bar\chi_T(1, 0, u^\varphi) = |\tau_0|^{\bar\gamma-\gamma}(1 - Ku)^{-\gamma}\chi_T(1, 0; R) , \tag{6.14}$$

where we have introduced the notation

$$u \equiv R^{1/\varphi}/|\tau_0| . \tag{6.15}$$

Then, since the right-hand side of (6.14) is a function of u, we have

$$\chi_T(1, 0; R) = R^{-(\bar\gamma-\gamma)/\varphi} \tag{6.16}$$

and, on using (6.16) in (6.14),

$$\bar\chi_T(1, 0, u^\varphi) = u^{-(\bar\gamma-\gamma)}(1 - Ku)^{-\gamma} . \tag{6.17}$$

This double-power-law analysis can be carried out for any function. However, its utility is in a sense limited, because we cannot expect (6.16) to be valid for arbitrarily small R. If we combine (6.12), (6.13), and (6.16), we find

$$\bar\chi_T(\tau_0, 0, R) = (\tau_0 - KR^{1/\varphi})^{-\gamma} R^{-(\bar\gamma-\gamma)/\varphi} , \tag{6.18}$$

and this equation predicts singular behavior as $R \to 0$ even for $\tau_0 \neq 0$, which is absurd because χ for $R = 0$ should be singular only for $\tau_0 = 0$. The best way to circumvent this and to make the hypothesis internally consistent is to restrict the range of validity of Eq. (6.3a). We therefore introduce a "crossover temperature" $\tau_x(R)$ defined such that the condition

$$\tau_R < \tau_x(R) \tag{6.19}$$

determines the limit of validity for the scaling hypothesis (6.3a). Of course, this $\tau_x(R)$ is not precisely defined, and one should speak of a crossover *region*. But by the same argument as led to (6.8), we can see that $\tau_x(R)$

$$\tau_x(R) \sim R^{\bar a_\tau/\bar a_R} . \tag{6.20}$$

Hence we expect the range of validity of (6.3a) to be subject to the condition

$$\tau_R^{1/\bar a_\tau} < \text{const} \times R^{1/\bar a_R} , \tag{6.21}$$

and we indicate this by region A_2 in Fig. 5(b).

Condition (6.21) means that (6.18) is not valid for R arbitrarily small and we do not arrive at the previous inconsistency. We have therefore demonstrated that condition (6.21), which was previously introduced simply on physical grounds, is necessary for the internal consistency of the theory.

At temperatures closer to the critical temperature than $\tau_x(R)$, we therefore expect the system described by (6.1) to behave as a three-dimensional system ($d=3$), whereas further away the system will display two-dimensional behavior [see Fig. 5(c)]. Although $\tau_x(R)$ is strictly a region, rather than a well-defined temperature, we can identify $\tau_x(R)$ with the center of the region [cf. Fig. 5(b)]. In practice, this region is rather narrow. This prediction of scaling analysis is consistent with two very recent findings:

(i) Series expansions (numerical experiments)[79] for three-dimensional lattices with arbitrary $R(\equiv J_z/J_{xy})$ indicate two-dimensional exponents far from $T_c(R)$ and three-dimensional exponents close to $T_c(R)$, with the region of three-dimensional behavior shrinking as $R\to 0$ *in accordance with the predictions of* (6.21). This represents the first detailed confirmation of (6.21), since we can vary R freely.

(ii) Experiments[80] on layered magnetic compounds with extremely small values of R indicate what might well be interpreted as a crossover from two- to three-dimensional exponents as $T\to T_c(R)$. Unlike the case of the numerical experiments described in (i), one cannot vary R at will in the laboratory, though perhaps with pressure one could vary R over enough of a range to test the explicit predictions of (6.21).

C. Comparison between Change of Lattice Dimensionality and Change of Spin Symmetry

The principal features of an interaction Hamiltonian—with reasonable short-range interactions—which determine critical behavior are thought to be lattice dimensionality d and spin dimensionality (or symmetry) D. In the previous sections we discussed in some detail the problem of change of lattice dimensionality as a means of illustrating the general principles of scaling with a parameter. In this section we consider very briefly the problem of changing the spin symmetry.[18] Thus, instead of the Hamiltonian (6.1) with lattice anisotropy, we consider a Hamiltonian with spin anisotropy,

$$\mathcal{H}=-\sum_{ij} J_{ij}\left[S_i^x S_j^x+S_i^y S_j^y+(1-R)S_i^z S_j^z\right], \tag{6.22}$$

where the spins are unit vectors $[(S_i^x)^2+(S_i^y)^2+(S_i^z)^2=1]$ so that the limiting case $R=0$ corresponds to $D=3$, the classical Heisenberg model, while $R=1$ corresponds to $D=2$, the classical planar or plane-rotator model. The reader can generalize (6.22) to more complicated systems if he wishes.

In discussing the scaling properties of (6.22), we must be very careful about which components of the magnetic field scale. If $R=0$, there is full xyz symmetry and all components of H should scale. However, if $R\neq 0$, then the x,y interactions are stronger than the z interactions and near the critical point the spins will tend to be in the xy plane. Hence there will only be critical fluctuations in the x and y components of the magnetization. Therefore, we expect that χ_{xx} and χ_{yy} diverge, but not χ_{zz}, where

$$\chi_{\alpha\beta}\equiv\frac{\partial M_\beta}{\partial H_\alpha}\ . \tag{6.23}$$

We will therefore assume that for fixed nonzero R, only the x and y components of magnetic field scale and hence we shall find that derivatives with respect to H_z of the Gibbs potential—such as M_z and χ_{zz}—do not scale.

Thus, the scaling hypothesis analogous to (6.3a) and (6.3b) are

$$C_2^{ii}(\lambda^{b_\tau}\tau_R,\ \lambda^{b_H}H_j,\lambda^{b_r}\vec{r};R)=\lambda C_2^{ii}(\tau_R,\ H_j,\ \vec{r};R)\ , \tag{6.24}$$

where $i,\ j=x$ or y, and

$$C_2^{ii}(\lambda^{\bar{b}_\tau}\tau_0,\ \lambda^{\bar{b}_H}\vec{H},\ \lambda^{\bar{b}_r}\vec{r},\ \lambda^{\bar{b}_R}R)=\lambda C_2^{ii}(\tau_0,\ \vec{H},\ \vec{r},\ R), \tag{6.25}$$

where $i=x,\ y$, or z. The behavior of τ_R and the double-power-law scaling can now be obtained for the appropriate functions by following the procedure illustrated for lattice anisotropy in Secs. VIA and VIB.

Note that we have made the scaling hypotheses (6.24) and (6.25) for the correlation function rather than for the Gibbs potential. This is possible because of theorem 4 and the relation $\chi^{ii}=\int C_2^{ii}\times(\tau,\ H,\ \vec{r})\,d\vec{r}$ and the arguments presented in Appendix B that scaling for the susceptibility implies scaling for the Gibbs potential.[81]

It is interesting to contrast the two cases of lattice anisotropy (change of lattice dimension) and spin anisotropy (change of spin symmetry), and this is shown schematically in Table VI. In the former, the higher dimensional symmetry dominates, because the dimension of interacting blocks of spins is the dominating factor, while in the latter case the lower dimensional spin symmetry dominates, because the fluctuating spin vectors are constrained to a lower dimensional symmetry.

VII. SUMMARY AND CONCLUSIONS

By stating the scaling hypotheses in all situations in terms of GHFs, and by precisely stating the relevant properties of these functions, the possibilities and limitations of the scaling hypotheses are clearly defined and their consequences can be easily found. For example, in Sec. VI we found that the restrictions on the scaling parameter $\bar{a}_R$ and the limited range of validity of one of the scaling hypotheses were consequences of the GHF hypotheses alone (and require no other pseudophysical arguments).

TABLE VI. Comparison of the lattice anisotropy (change of lattice dimension) and spin anisotropy (change of spin symmetry) cases.

Fixed R	Change of lattice dimension	Change of spin symmetry
$R=0$	two dimensional	three dimensional
$R\neq 0,\ \tau>\tau_x(R)$	two dimensional	three dimensional
$R\neq 0,\ \tau<\tau_x(R)$	three dimensional	two dimensional

A principal advantage of the present approach is that the same formalism may be used in an extremely wide variety of cases.

(i) The treatment of thermodynamic scaling (with two independent variables) went over immediately to the treatment of static correlation-function scaling (with three independent variables) and to the scaling of the dynamic correlation functions (with four independent variables). In particular, for the case of static correlation functions, it was seen that additional[3,20,21,23,24] assumptions (besides homogeneity) are avoided, while for the dynamic correlation functions, the two hypotheses of Halperin and Hohenberg[31] are reduced to a single GHF statement.

(ii) For more complex systems (than simple magnets),[43] the critical point becomes a *critical subspace*, as in, e.g., Sec. VI, where $T_c=T_c(R)$ is a line in the τ-H-R space. We saw that the GHF approach requires no generalization and treatment of multicomponent systems is thus straightforward.[36]

(iii) The problem arising when the critical subspaces of (ii) intersect requires particular care, but is nevertheless amenable to the GHF approach. For example, at tricritical points the shapes of the three intersecting critical lines are determined by the GHF hypothesis, and this enables one to establish scaling hypotheses on all three critical lines, and at the tricritical point itself, in a fully self-consistent fashion. The shapes of the boundaries between regions of different scaling behavior can also be determined from scaling as in Sec. VI.[82]

In addition to the above conceptual advantages, the GHF approach is of considerable practical utility. For example, among the advantages (illustrated at various places in the text) are the following:

(a) The GHF approach enables one to write down by inspection a simple expression, $a_{\text{function}}/a_{\text{path}}$, for each critical-point exponent of an arbitrary function.

(b) The GHF approach suggests further tests of the scaling hypothesis by predicting plots of appropriate scaled variables (cf. Figs. 1–5) since a GHF may be scaled with respect to any of its independent variables.

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APPENDIX A: PROOFS OF THEOREMS 1–4 OF SECTION II

We demonstrate the proofs of theorems 1–4 for a function of two variables $f(x_1,\ x_2)$. In all cases the generalization to a function of n different variables is straightforward.

Theorem 1. If $f(x_1,\ x_2)$ is a GHF with scaling power a_f, the partial derivative $f^{(j,k)}(x_1,x_2)$ defined in Eq. (2.5) is also a GHF with scaling power $a_f-ja_1-ka_2$.

Proof. The defining equation of a GHF is Eq. (2.1),

$$f(\lambda^{a_1}x_1,\lambda^{a_2}x_2)=\lambda^{a_f}f(x_1,\ x_2). \tag{A1}$$

Differentiating each side, and using Eq. (2.5) gives

$$\lambda^{(ja_1+ka_2)}f^{(j,k)}(\lambda^{a_1}x_1,\ \lambda^{a_2}x_2)=\lambda^{a_f}f^{(j,k)}(x_1,x_2), \tag{A2}$$

which may be rewritten

$$f^{(j,k)}(\lambda^{a_1}x_1,\ \lambda^{a_2}x_2)=\lambda^{(a_f-ja_1-ka_2)}f^{(j,k)}(x_1,x_2). \tag{A3}$$

But this shows that $f^{(j,k)}$ is a GHF with scaling power $(a_f-ja_1-ka_2)$. QED.

Theorem 2. Let $f(x_1,x_2)$ be a GHF with scaling power a_f, then the Legendre transform $\bar{f}(\bar{x}_1,x_2)$ defined in Eq. (2.7) is also a GHF with scaling power a_f. The scaling power of the transformed variable is $\bar{a}_1=a_f-a_1$.

Proof. By theorem 1, $\bar{x}_1\equiv(\partial f/\partial x_1)$ obeys the equation

$$\bar{x}_1(\lambda^{a_1}x_1,\ \lambda^{a_2}x_2)=\lambda^{a_f-a_1}\bar{x}_1(x_1,x_2), \tag{A4}$$

so we have

$$\bar{a}_1=a_f-a_1. \tag{A5}$$

Scaling both sides of Eq. (2.7) gives

$$\bar{f}[\bar{x}_1(\lambda^{a_1}x_1,\lambda^{a_2}x_2),\ \lambda^{a_2}x_2]=f(\lambda^{a_1}x_1,\lambda^{a_2}x_2)-\bar{x}_1(\lambda^{a_1}x_1,\lambda^{a_2}x_2)\lambda^{a_1}x_1, \tag{A6}$$

and using (A4), (A5), and (A1) this becomes

$$\bar{f}(\lambda^{\bar{a}_1}\bar{x}_1,\ \lambda^{a_2}\ x_2)=\lambda^{a_f}[f(x_1,x_2)-\bar{x}_1x_1]=\lambda^{a_f}\bar{f}(\bar{x}_1,x_2). \tag{A7}$$

Hence we can find consistent powers $\bar{a}_1,\ a_2$ for the variables $x_1,\ x_2$ such that $\bar{f}(\bar{x}_1,x_2)$ obeys the defining equation (A1) and is a GHF. QED. This theorem can be extended to treat a transform of a function $f(x_1, \ldots, x_n)$ with respect to any combination of its variables.

Theorem 3. A function $f(x_1, x_2)$ is a GHF if and only if there exists a function $g_\pm(u)$ such that

$$f(x_1, x_2) = |x_1|^{a_f/a_1} g_{\mathrm{sgn}x_1}(x_2/|x_1|^{a_2/a_1}) \tag{A8}$$

or equally a function $h_\pm(u)$ such that

$$f(x_1, x_2) = |x_2|^{a_f/a_2} h_{\mathrm{sgn}x_2}(x_1/|x_2|^{a_1/a_2}). \tag{A9}$$

Proof. If $f(x_1, x_2)$ is a GHF, then the parameter λ can be chosen to have any particular value for each value of x_1 (or x_2) and the defining equation will still be true. In particular, we can let $\lambda^{a_1}|x_1| = 1$ and Eq. (A1) becomes

$$f(\pm 1, x_2/|x_1|^{a_2/a_1}) = |x_1|^{-a_f/a_1} f(x_1, x_2) \tag{A10}$$

and so

$$f(x_1, x_2) = |x_1|^{a_f/a_1} f(\pm 1, x_2/|x_1|^{a_2/a_1}) , \tag{A11}$$

which is exactly of the form (A8). Similarly, the choice $\lambda^{a_2}|x_2| = 1$ will yield (A9). The "if" part of the statement is trivial to prove since the functions defined by (A8) and (A9) clearly satisfy (A1) and are therefore GHFs. QED.

Theorem 4. Let $f(x_1, x_2)$ be a GHF of scaling power a_f. Then the Fourier transform $f(x_1, x_2)$ defined in Eq. (2.13) is a GHF with scaling power $a_{\hat{f}} = a_f - da_1$. The scaling power of the transformed variable is $\hat{a}_1 = -a_1$.

Proof. Consider Eq. (2.13),

$$\hat{f}(\hat{x}_1, x_2) \equiv \int f(x_1, x_2) e^{i\hat{x}_1 \cdot x_1} d^d x_1 . \tag{A12}$$

The right-hand side of (A12) will only be a GHF if we choose $e^{i\hat{x}_1 \cdot x_1}$ to be invariant under the transformation $\hat{x}_1 \to \lambda^{\hat{a}_1} x_1$, $x_1 \to \lambda^{a_1} x_1$. Hence, we choose

$$\hat{a}_1 = -a_1 . \tag{A13}$$

This choice is entirely consistent with the point of view that $\hat{x}_1$ has a physical dimension inverse to that of x_1. We now find that Eq. (A12) transforms as

$$\hat{f}(\lambda^{\hat{a}_1}\hat{x}_1, \lambda^{a_2}x_2) = \int f(\lambda^{a_1}x_1, \lambda^{a_2}x_2) e^{i\hat{x}_1 \cdot x_1} d^d(\lambda^{a_1}x_1) , \tag{A14}$$

and since we are integrating over all space, the limits of the integrals do not matter. Equations (A1) and (A13) make (A14) obey

$$f(\lambda^{\hat{a}_1}\hat{x}_1, \lambda^{a_2}x_2) = \lambda^{a_f + da_1} \int f(x_1, x_2) e^{i\hat{x}_1 \cdot x_1} d^d x_1$$
$$= \lambda^{a_f - d\hat{a}_1} \hat{f}(\hat{x}_1, x_2) , \tag{A15}$$

and this shows that we can find a scaling power $\hat{a}_1 = -a_1$ such that $\hat{f}(\hat{x}_1, x_2)$ obeys an equation like (A1). $\hat{f}$ is therefore a GHF with scaling power $a_f - d\hat{a}_1$. QED.

Powers need no generalization. The statement that Eq. (2.4) implies that $g_i(\lambda) = \lambda^{a_i}$ and $g_f(\lambda) = \lambda^{a_f}$ is important because it precludes a generalization of Eq. (2.1) [or (A1)]. The proof is very simple: If $g_i(\lambda)$ possesses an inverse [which intuitively it must since it is single valued and continuous], then we can define the set of functions $F^i_j(\lambda) \equiv g_i^{-1}\{g_j(\lambda)\}$. Equation (A1) becomes

$$f[\lambda x_1, F^1_2(\lambda)x_2] = F^1_f(\lambda) f(x_1, x_2) . \tag{A16}$$

This can be written in succession for $\lambda_1\lambda_2$, and this gives

$$f[\lambda_1\lambda_2 x_1, F^1_2(\lambda_1)F^1_2(\lambda_2)x_2] = F^1_f(\lambda_1)F^1_f(\lambda_2) f(x_1, x_2) . \tag{A17}$$

For physical systems the scaling factors in these equations should be unique. So writing $\lambda = \lambda_1\lambda_2$ in (A16) and comparing with (A17) yields

$$F^1_j(\lambda_1)F^1_j(\lambda_2) = F^1_j(\lambda_1\lambda_2) . \tag{A18}$$

But the solutions of the functional equation (A18) are known[33] to be

$$F^1_j(\lambda) = \lambda^{p_j} \tag{A19}$$

for some power p_j. Equation (A16) is therefore really of the form (A1) and our assertion is demonstrated.

APPENDIX B: RELATIONS BETWEEN SCALING HYPOTHESES FOR DIFFERENT FUNCTIONS

Legendre Transform Theorem for Variables Not Tending to Zero at the Critical Point

All GHFs tend to zero or infinity at the zero of their arguments. The thermodynamic potentials remain finite at the critical point, because the extensive quantities, their first derivatives, remain finite. Hence the part which satisfies scaling must go to zero. Since G_{tot} is not zero at the critical point we divide G_{tot} into regular and singular parts. If symmetry is to be maintained, the same should be done for all thermodynamic potentials.

For a Legendre transform with respect to a variable which is zero at the critical point (e.g., M or H), theorem 2 is directly applicable. However, the variables entropy and temperature tend to nonzero values (T_c and S_c) and we have to make allowances for this. We will consider the transformation between Gibbs potential and enthalpy to illustrate the modified theorem; it can equally well be stated for the Helmholtz potential and the internal energy.

The Gibbs potential $G_{\mathrm{tot}}(T, H)$ assumes a value

$$G_c = G_{\mathrm{tot}}(T_c, 0) \tag{B1a}$$

at the critical point. We can define the regular part of the Gibbs potential to be

$$G_{\rm reg}(T,H)=G_c-S_c\tau\ , \tag{B2a}$$

where S_c is the value of the entropy at the critical point and

$$\tau=T-T_c\ . \tag{B3a}$$

Equations (B2a) and (B3a) are consistent with the definition of entropy $S\equiv-(\partial G/\partial T)_H$. Similarly, if we consider the total enthalpy $E_{\rm tot}$, it must assume a finite value at the critical point given by

$$E_c=E_{\rm tot}(S_c,0)\ . \tag{B1b}$$

In particular $E_c=G_c+T_cS_c$. The regular part of the enthalpy is similarly defined to be

$$E_{\rm reg}(S,H)=E_c+T_c\sigma\ , \tag{B2b}$$

where T_c is the value of the temperature at the critical point and

$$\sigma=S-S_c\ . \tag{B3b}$$

Equations (B2b) and (B3b) are consistent with the definition $T\equiv(\partial E/\partial S)_H$.

The following theorem is now rigorously true.

Theorem 2*A*. The singular part of the Gibbs potential $G_s(\tau,H)$, defined as the difference between $G_{\rm tot}(T,H)$ and $G_{\rm reg}(T,H)$,

$$G_s\equiv G_{\rm tot}(T,H)-G_{\rm reg}(T,H)\ , \tag{B4a}$$

is related by Legendre transform with respect to the scaling variables τ and σ to the singular part of the enthalpy, defined as the difference between $E_{\rm tot}$ and $E_{\rm reg}$,

$$E_s\equiv E_{\rm tot}(S,H)-E_{\rm reg}(S,H)\ . \tag{B4b}$$

Considering G_s as a function of τ and H, and E_s as a function of σ and H, we have

$$G_s(\tau,H)=E_s-\sigma\frac{\partial E_s}{\partial\sigma}\ , \tag{B5a}$$

$$E_s(\sigma,H)=G_s-\tau\frac{\partial G_s}{\partial\tau} \tag{B5b}$$

The proof of theorem 2A is absolutely straightforward, since it follows directly from the identification of $E_{\rm tot}$ as the Legendre transform of $G_{\rm tot}$ (with respect to T) and vice versa. The only trick is that σ,τ must be identified as the first derivatives of G_s and E_s, respectively (this is essential to get the right scaling behavior of these variables),

$$\frac{\partial G_s}{\partial\tau}=-\sigma\ , \tag{B6a}$$

$$\frac{\partial E_s}{\partial\sigma}=\tau\ . \tag{B6b}$$

The important point of the theorem is that although G and E are related by transforms with respect to the physical variables T and S, the singular parts G_s and E_s are related by a Legendre transform with respect to the scaling variables τ and σ. Theorem 2 is now applicable.

Theorem 2*A*, as emphasized above, is absolutely self-consistent. However, it cannot be exactly applied to a physical system, and certain equations must be regarded as approximations, the validity of which improves as the critical point is approached. For a particular class of cases, which are treated below, the approximations are not valid, and no Legendre transform theorem exists.

Precisely, if the scaling hypothesis is true for the Gibbs potential, and Eqs. (B1a), (B3a), and (B4a) are valid, then it is natural to define the singular part of the entropy as in (B6a). Alternatively,

$$C_{Hs}=\left(\frac{\partial\sigma}{\partial\tau}\right)_H=-\left(\frac{\partial^2G_s}{\partial\tau^2}\right)_H\ , \tag{B7}$$

where C_{Hs} is the singular part of the constant-field specific heat. The derivative of Eq. (B4a) with respect to T now gives

$$-S=-\sigma+\left(\frac{\partial G_{\rm reg}}{\partial T}\right)_H\ . \tag{B8}$$

If Eq. (B3b) is now assumed, and σ is identified as $(S-S_c)$, Eq. (B2a) follows from (B8) and the usual assumption that the magnetization function $M(T,H)$ has no regular part. Equations (B1)–(B4) and (B6) are therefore overcomplete, though fully self-consistent.

If the specific heat diverges at the critical point, then Eq. (B3b) is valid. Indeed Eq. (B3b) may be regarded as an approximate statement, which is better the closer one is to the critical point, and this is entirely in the spirit of the definition of the critical point exponents as representing the behavior of the leading terms of series at the critical point. Equations (B2a) and (B2b) may be similarly regarded as the first two terms of expansions, and therefore as approximations to the regular parts of the functions $G_{\rm reg}(T,H)$ and $E_{\rm reg}(S,H)$ at the critical point. With these provisos theorem 2*A* can be applied.

The particular class of cases for which Eq. (B3b) is definitely not valid is those models for which the specific heat tends to a finite value at the critical point with a cusp singularity. For these cases the specific heat may be written $C_H=C_{Hc}+A\tau^{-\alpha}$, where $\alpha<0$. The entropy is therefore approximately

$$S=S_c+C_{Hc}\tau+A\left|\tau\right|^{1-\alpha}/(1-\alpha). \tag{B9}$$

The singular part of the entropy is now a function of both entropy and temperature. It proves impossible to separate those potentials which depend on the entropy (enthalpy and internal energy) into a regular part and a singular part which depends in a GHF fashion on the singular part of the entropy. Theorem 2A cannot be proved and these potentials will not be GHFs although the scaling hypothesis is valid for the Gibbs and Helmholtz potentials. (Of course, all derivatives of G and A are GHFs.)

In summary, theorem 2A will be valid unless $\alpha < 0$ and $C_{Hc} \neq 0$. In systems for which this is the case we cannot relate the GHF properties of the Gibbs and Helmholtz potentials to those of the enthalpy and internal energy.

Correlation Function Scaling Implies Thermodynamic Scaling

If we postulate that the structure factor $S(\tau, H, q)$ is a GHF at small values of its arguments, then it automatically follows that the susceptibility

$$\chi(\tau, H) \equiv S(\tau, H, q=0) = G^{(0,2)}(\tau, H)$$

is a GHF. However, this does not immediately imply that $G(\tau, H)$ is a GHF, because to obtain G from χ a double integration is needed, and this introduces two functions of temperature as constants of integration $A(\tau)$ and $B(\tau)$, where

$$G(\tau, H) = \int^{H}\int^{H'} \chi(\tau, H'')\, dH'\, dH'' + A(\tau) H + B(\tau) \ . \qquad \text{(B10)}$$

Now $A(\tau)$ and $B(\tau)$ depend on the lower limit of integration that is chosen in (B10). Since $A(\tau)$ is introduced for $G^{(0,1)}(\tau, H) \equiv M(\tau, H)$, its value is determined by matching the magnetization and the first integral of χ. $B(\tau)$ contributes to the entropy and specific heat and can be determined by similar matching.

It is now possible to prove that $G(\tau, H)$ defined in (B10) has a singular part which is a GHF. The proof depends on the observation that the singularity structure of $G(\tau, H)$ is very simple in the Reτ, ReH plane. There is certainly a singularity at the critical point and discontinuities in derivatives on the τ axis for $T < T_c$, but there are no singularities at points for $H \neq 0$.

This has simple consequences on the functions $A(\tau)$ and $B(\tau)$ since, being functions of τ only, a singularity in $A(\tau)$ or $B(\tau)$ would produce a line of singularities at constant τ in the (τ, H) plane. There are therefore two possibilities: Either the functions $A(\tau)$ and $B(\tau)$ are regular in τ or else the singularities in them are exactly cancelled at nonzero values of H by contributions from the double integral. (An example of each of these two types of behavior is given below.)

To be precise, we must choose the lower limits of the integral in (B10). Here we consider two cases.

Case (i): Lower limit is $H=0$ *and* $A(\tau)$, $B(\tau)$ *are singular*. Now the integral is exactly a GHF if χ is a GHF (converse to theorem 1), but the integral gives zero contribution on the temperature axis at $H=0$. Therefore, the functions $A(\tau)$ and $d^2B/d\tau^2$ are the zero-field magnetization and specific heat, respectively. These are singular at the critical point and $A(\tau)$ and $B(\tau)$ both give a line of singularities in the (τ, H) plane at $\tau = 0$ [with singular $M(0, H)$, $C_H(0, H)$ all along this line]. This line of singularities must therefore be exactly cancelled by contributions from the integral. This is only possible if the scaling powers of the integral exactly match the behavior of the zero-field magnetization and of the specific heat. This implies that the whole singular part of $G(\tau, H)$, as defined in (B10), is a GHF.

Case (ii): Lower limit is $H = -\infty$ (and $H, H' < 0$) *and* $A(\tau)$, $B(\tau)$ *are regular*. Now $A(\tau)$ and $(\partial^2 B/\partial\tau^2)$ are the infinite-field magnetization and specific heat, respectively, and these are regular. The integral can be broken into two parts $I(\tau, H) - I(\tau, -\infty)$, where

$$I(\tau, H) \equiv + \int_{H}^{0} \int_{H'} \chi(\tau, H'')\, dH'\, dH'' \ . \qquad \text{(B11)}$$

Now $I(\tau, H)$ is a GHF and the analogous singularities to those discussed in case (i) are contained in $I(\tau - \infty)$, the same arguments apply, and we conclude that despite the arbitrary functions $A(\tau)$, $B(\tau)$, the singular part of $G(\tau, H)$ is a GHF and the GHF hypothesis for $\chi(\tau, H)$ implies a GHF hypothesis for $G(\tau, H)$.

APPENDIX C: INEQUALITIES OBEYED BY SCALING POWERS a_i

The scaling powers a_i of the intensive variables x_i obey some simple inequalities. These are required because of the physical condition that the extensive variables remain finite at the critical point. The extensive variables are first-order derivatives of the Gibbs function; these obey the equations

$$\frac{\partial G}{\partial x_i} \propto |x_i|^{(1-a_i)/a_i} \qquad \text{(C1)}$$

close to the critical point. If the extensive variable does not diverge, then neither can its scaling part $(\partial G/\partial x_i)$. Therefore, this must go to zero at the critical point and its exponent must be positive:

$$(1 - a_i)/a_i > 0 \qquad \text{(C2)}$$

which may be rewritten

$$0 < a_i < 1 \qquad \text{(C3)}$$

for every scaling power a_i corresponding to an independent variable x_i.

The scaling powers a_s corresponding to a strongly[42] fluctuating quantity $(\partial G/\partial x_s)$ have a stronger condition. This arises because the corresponding susceptibility $\chi_s \equiv (\partial^2 G/\partial x_s^2)$ diverges as the critical point is approached. The scaling power a_χ is therefore necessarily negative. But

$$a_X = (1 - 2a_s) \tag{C4}$$

and so (C3) becomes

$$\tfrac{1}{2} < a_s < 1 \tag{C5}$$

for strong directions of approach to the critical point.

No similar inequality holds for weakly fluctuating variables. But considerations of Appendix B show that theorem 2 on Legendre transforms is only useful when Eq. (C5) holds for the scaling powers corresponding to the weak variables, since we need $a_C = (1 - 2a_w) < 0$.

APPENDIX D: SYSTEMS WHICH SATISFY SCALING HYPOTHESIS AND SYSTEMS WHICH DO NOT

There are relatively few exactly soluble model systems on which we can test the scaling hypothesis of Sec. III A 1—indeed, even the well-known $d=2$ Ising model is solved only for zero field.

Perhaps the simplest exactly soluble system is the mean field theory (mft) which actually corresponds to a magnetic system in which each moment interacts with every other moment in the system with an equal interaction energy. The Helmholtz potential for the mft is[83]

$$A^{\mathrm{mft}}(T,M) = NkT\left[-\ln 2 + \tfrac{1}{2}\ln(1-\bar{M}^2) + \tfrac{1}{2}\bar{M}\ln\left(\frac{1+\bar{M}}{1-\bar{M}}\right) - \frac{\bar{M}^2 T_c}{2T}\right], \tag{D1}$$

where $\bar{M}$ denotes the relative magnetization $M(T,H)/M(0,0)$. Now it is evident from (D1) that at the critical point $(T=T_c, M=0)$, $A_c = A(T_c, 0) = -NkT_c \ln 2$. We form $A_{\mathrm{reg}} \equiv A_c - S_c\tau$ (as in Appendix B) and $A_s = A - A_{\mathrm{reg}}$. Expanding the logarithms in (D1) about $\bar{M}=0$, we see that

$$A_s^{\mathrm{mft}}(\tau, M) = Nk(\tfrac{1}{2}\bar{M}^2\tau + \tfrac{1}{12}\bar{M}^4 T) + \cdots . \tag{D2}$$

From inspection of (D2) we see that close to the critical point

$$A_s^{\mathrm{mft}}(\lambda^{1/2}\tau, \lambda^{14}M) = \lambda A_s^{\mathrm{mft}}(\tau, M), \tag{D3}$$

so that $a_\tau = \frac{1}{2}$ and $a_M = \frac{1}{4}$ (thus $a_H^{\mathrm{mft}} = \frac{3}{4}$).[84] Hence the scaling hypothesis would appear to be valid for the mft.

One can similarly show that the three-dimensional ($d=3$) spherical model[85] satisfies the static scaling hypothesis,[86] with $a_\tau = \frac{1}{3}$ and $a_H = \frac{5}{6}$.[87]

However, not all model systems satisfy the scaling hypothesis, as one can see by examining the Gibbs potential for the $d=1$ Ising model[88]

$$G(T,H) = -NkT\ln[e^{\mathcal{J}}\cosh h + (e^{2\mathcal{J}}\sinh^2 h + e^{-2\mathcal{J}})^{1/2}], \tag{D4}$$

where h is the magnetic field in dimensionless units and $\mathcal{J} \equiv J/kT$, with J the nearest-neighbor exchange energy and k the Boltzmann constant. Indeed, the zero-field susceptibility $\chi_T(T,H=0) = G^{(0,2)}(T,H=0)$ for the $d=1$ Ising model varies as $T^{-1}e^{2J/kT}$, and this essential singularity is not a power law singularity.[88] Hence by theorem 1 and the corollary to theorem 3, the Gibbs potential cannot be a GHF.

Similarly, the two-dimensional six-vertex or "KDP" (potassium dihydrogen phosphate) models treated by Lieb and others[89] in recent years also involve essential singularities and do not satisfy the scaling hypothesis.

Most systems in nature are not described well by any of the exactly soluble models described above (the mft, the $d=3$ spherical model, the $d=1$ Ising model, the $d=2$ KDP models). However, the critical behavior of a wide class of real systems is well described by the hierarchy of classical spin Hamiltonians[90]

TABLE VII. Numerical values of the scaling powers a_τ, a_H for thermodynamic functions, and b_τ, b_H, and b_r for static correlation functions, for some model systems thought to obey the scaling hypotheses. The notation $\simeq$ indicates that the number is based upon numerical approximation methods. Only three of the four scaling powers b_x are independent—e.g., b_M is related to b_H, b_r by Eq. (4.26). Note that $2b_M$ is *almost* unity for D-dimensional spins situated on a d-dimensional lattice (D finite), so that the "two-exponent relations" (4.24a) and (4.24b) are *almost* valid.

Model	a_τ	a_H	b_τ	b_H	b_r	b_M
Mean field theory (mft)	$\frac{1}{2}$	$\frac{3}{4}$	...	...	...	...
$d=2$ Ising model ($D=1$)	$\frac{1}{2}$	$\frac{15}{16}$	4	$\frac{15}{2}$	-4	$\frac{1}{2}$
$d=3$ Ising model ($D=1$) (taking $\nu=0.638$ and $\eta=0.041$)	$\simeq\frac{8}{15}$	$\simeq\frac{5}{6}$	$\simeq 1.506$	$\simeq 2.3526$	$\simeq -0.9606$	$\simeq 0.4708$
$d=3$ plane rotator model ($D=2$) (taking $\nu=0.680$, $\eta=0.040$)	$\simeq\frac{1}{2}$	$\simeq\frac{5}{6}$	$\simeq 1.414$	$\simeq 2.3567$	$\simeq -0.9615$	$\simeq 0.4721$
$d=3$ classical Heisenberg model ($D=3$) ($\nu=0.717$, $\eta=0.040$, $\gamma=1.405$, $\Delta=1.77$)	$\simeq\frac{10}{21}$	$\simeq\frac{5}{6}$	$\simeq 1.341$	$\simeq 2.3736$	$\simeq -0.9615$	$\simeq 0.4891$
$d=3$ spherical model ($D=\infty$)	$\frac{1}{3}$	$\frac{5}{6}$	1	$\frac{5}{2}$	-1	$\frac{1}{2}$

TABLE VIII. Estimates for critical-point exponents for a three-dimensional lattice ($d=3$) for different values of spin dimensionality D. Shown in the last column is the prediction of the bilinear form hypothesis (Ref. 92).

Exponent	$D=1$ (Ising)	$D=2$ (planar)	$D=3$ (Heisenberg)	$D=\infty$ (spherical)	Bilinear hypothesis
a_T	$\simeq \frac{8}{15}$	$\simeq \frac{1}{2}$	$\simeq \frac{10}{21}\cdots$	$\frac{1}{3}$	$\frac{1}{3}\left(\frac{7+D}{4+D}\right)$
a_H	$\simeq \frac{1}{8}$	$\simeq \frac{5}{6}$	$\simeq \frac{5}{6}\cdots$	$\frac{5}{6}$	$\frac{5}{6}\,(1)$
α	$\simeq \frac{5}{6}$	$\simeq 0$	$\simeq -\frac{1}{10}\cdots$	-1	$-1\left(\frac{-2+D}{7+D}\right)$
β	$\simeq \frac{5}{16}$	$\simeq \frac{1}{3}$	$\simeq \frac{7}{20}\cdots$	$\frac{1}{2}$	$\frac{1}{2}\;\frac{4+D}{7+D}$
γ	$\simeq \frac{5}{4}$	$\simeq \frac{4}{3}$	$\simeq \frac{7}{5}\cdots$	2	$2\left(\frac{4+D}{7+D}\right)$
δ	$\simeq 5$	$\simeq 5$	$\simeq 5\cdots$	5	$5\,(1)$
Δ	$\simeq \frac{25}{16}$	$\simeq \frac{5}{3}$	$\simeq \frac{7}{4}\cdots$	$\frac{5}{2}$	$\frac{5}{2}\left(\frac{4+D}{7+D}\right)$

$$\mathcal{H}^{(D)} = -J\sum_i \sum_\delta \vec{S}_i^{(D)} \cdot \vec{S}_{i+\delta}^{(D)}, \tag{D5}$$

where J is the nearest-neighbor exchange potential and the spins $\vec{S}_i^{(D)}$ and $\vec{S}_{i+\delta}^{(D)}$ are isotropically interaction D-dimensional unit vectors situated on neighboring sites i and $i+\delta$ of a lattice. (Note that here D denotes the dimensionality of the spin space, while d is the lattice dimensionality.) The hierarchy of models described by the Hamiltonian (D5) cannot be solved exactly except in the limit $D\to\infty$, in which case the solution becomes essentially identical to that for the spherical model.[91] However, for finite D, very realistic approximation procedures have been derived,[92] and it has recently been verified numerically that the scaling hypothesis (3.3a) is valid for both $D=1$ and $D=3$, corresponding to the Ising and classical Heisenberg models, respectively.[22,39] Hence it is likely that scaling hypothesis is valid for a large class of real experimental systems, and indeed the detailed predictions of Eq. (3.3a) are borne out by a wide variety of experimental measurements.

The scaling parameters a_T and a_H for systems thought to obey the scaling hypothesis are given in Table VII and a bilinear form hypothesis[93] describing the variation of scaling powers and exponents with D is displayed in Table VIII.

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[32] E. Riedel and F. Wegner, Phys. Rev. Letters 24, 730 (1970); E. Riedel, J. Appl. Phys. 42, 1383 (1971).

[33] Good background reading in functional equations, with a brief discussion of generalized homogeneous functions, is to be found in J. Aczél, *Lectures on Functional Equations and their Applications* (Academic, New York, 1966).

[34] This was not always appreciated—see, e.g., the recorded discussion of the 1968 International Statistical Mechanics Conference [J. Phys. Soc. Japan 26S (1969)].

[35] We can always arrange that one of the scaling powers in a scale-invariant function is unity, by choosing $p=1/a_1$ in Eq. (2.2).

[36] See also T. Chang, A. Hankey, and H. E. Stanley, Bull. Am. Phys. Soc. 17, 277 (1972); and unpublished.

[37] We are indebted to R. Krasnow for discussions that helped clarify these matters.

[38] See, e.g., Refs. 1 and 5.

[39] S. Milošević and H. E. Stanley, Phys. Rev. B 5, 2526 (1972); 6, 986 (1972); 6, 1002 (1972).

[40] S. Milošević, D. Karo, R. Krasnow, and H. E. Stanley, Thirteenth International Low-Temperature Physics Conference, Boulder, Colo., 1972 (unpublished).

[41] We use the terminology scaling law to mean only the exponent relations, such as $\alpha+2\beta+\gamma=2$, that are predicted by the scaling hypothesis. This terminology has been used by others—cf. Ref. 31.

[42] R. B. Griffiths and J. C. Wheeler, Phys. Rev. A 2, 1047 (1970).

[43] Specifically, one uses variables h_1', h_2', where h_2' lies in the coexistence surface and h_1' does not (and both h_1', h_2' are not in the critical surface). The extension to multicomponent systems is straightforward. See Ref. 36.

[44] An example of critical-point exponents that are *not* linearly independent are the $\tau\rightarrow 0$ exponents for the entropy $S(\tau, H)\propto G^{(1,0)}(T, H)\sim|\tau|^{1-\alpha}$ and the constant-field specific heat $C_H(\tau,H)\propto G^{(2,0)}(\tau, H)\sim|\tau|^{-\alpha}$.

[45] See, e.g., H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford U.P., New York, 1971), Sec. 3.5.

[46] This example is of more than mere academic interest, as extensive data yet to be analyzed concern constant-entropy measurements. [A. Arrott (private communication). We wish to thank Professor Arrott for helpful discussions on this section of the manuscript.]

[47] For Ni, see S. Kouvel and J. B. Comly, Phys. Rev. Letters 20, 1237 (1968); for $CrBr_3$, see J. T. Ho and J. D. Litster, Phys. Rev. Letters 22, 603 (1969); Phys. Rev. B 2, 4523 (1970); for EuO, see N. Menyuk, K. Dwight, and T. B. Reed, Phys. Rev. B 3, 1689 (1971).

[48] A detailed comparison of scaling functions for numerous experimental systems with the Heisenberg model scaling function appears in Refs. 39 and 40.

[49] The idea behind "data collapsing" is the mathematical fact that if a function $f(x,y)$ is a GHF, then one can determine its value everywhere in the x-y plane by specifying it on *any closed contour surrounding the origin.* Similarly, for $f(x,y,z)$, a GHF of three variables, one can determine its value everywhere providing one specifies $f(x,y,z)$ on *any surface containing the origin.* Thus "scaling functions" are one-dimensional curves for GHFs of two independent variables, two-dimensional surfaces for GHFs of three independent variables, and, in general, $(n-1)$-dimensional hypersurfaces for GHFs of n independent variables (cf. Figs. 1, 3, and 4).

[50] The independent variables τ, H are appropriate to the T-H ensemble commonly used for the description of magnetic systems; for a different ensemble the independent variables would of course be different. See, e.g., the discussion by G. H. Wannier [*Statistical Physics* (Wiley, New York, 1966)].

[51] Note that we are treating r as a continuous variable rather than as a variable with meaning only when it joins a pair of lattice sites. Accordingly, we shall find integrations rather than summations in relations that follow [cf. Eqs. (4.4) and (4.10)]. If one chooses not to adopt the present convention, one obtains still a self-consistent theory, but the relation (4.9) between the a_x and the b_x is altered accordingly.

[52] See Appendix A of Ref. 45 for a derivation of (4.4a). We have omitted for the sake of notational convenience the proportionality factor $g^2\mu_B^2/kT$ that should appear on the right-hand side of (4.4a), where g denotes the gyromagnetic ratio, μ_B is the Bohr magneton, k is the Boltzmann constant, and T is the temperature. Note that had we, in Sec. III, chosen the scaling power of $\chi_T(\tau,H)$ to be $1-db_q$, then the a's would necessarily equal the b's.

[53] The generalized correlation length $\xi_\Phi(\tau, H)$ has been defined by M. E. Fisher [Phys. Rev. 180, 594 (1969)].

The corresponding critical-point exponents ν_ϕ and μ_ϕ [cf. Eqs. (4.15a) and (4.15b)] were found to satisfy exponent inequalities by L. L. Liu, R. I. Joseph, and H. E. Stanley [Phys. Rev. B 6, 1963 (1972)].

[54]Note from (4.12), (4.14a), and (4.15a) that the zero-field correlation function of (4.2) is a homogeneous function of r and correlation length $\xi_\phi \equiv |\tau|^{b_r/b_\tau}$, $C_2(\lambda\xi_\phi, 0, \lambda r) = \lambda^{1/b_r} C_2(\xi_\phi, r)$ or, alternatively, the zero-field structure factor of (4.3a) is a homogeneous function of q and $\kappa_\phi \equiv \xi_\phi^{-1} = |\tau|^{b_q/b_\tau}$, $S(\lambda\kappa_\phi, 0, \lambda q) = \lambda^{b_S/b_q} S(\kappa_\phi, 0, q)$. These homogeneity relations are physically plausible in that they say that contributions to the correlation function are determined by the ratio r to the correlation length or, for the structure factor, by the ratio of the inverse correlation length (cf. Ref. 31).

[55]This definition of η_ϕ was made by Liu, Joseph, and Stanley (Ref. 53) in their derivation of some exponent inequalities involving η_ϕ. When $\phi = 0$, $\eta_\phi = \eta$, as pointed out by Fisher (Ref. 53), this definition includes, but is rather more general than, the usual definition of η in terms of the decay of the correlation function at the critical point: $C_2(0, 0, r) \sim r^{-(d-2+\eta)}$ for large r. By substituting $\lambda = (1/r)^{1/b_r}$ into Eq. (4.2) the reader can verify that $(d-2+\eta) = -1/b_r$, so that the simpler definition also results in Eq. (4.19). A third definition, fully equivalent to the second, is that $S(\tau = 0, H = 0, q) \sim q^{-2+\eta}$, and the reader can verify from (4.3a), that this predicts $-2+\eta = (1-db_q)/b_q = -d - 1/b_r$ which again results in (4.19).

[56]That Eqs. (4.24) do not follow from homogeneity alone (cf. Ref. 54) has been appreciated by many—though certainly not all—workers in the field of critical phenomena; see, e.g., the work of Fisher (Ref. 11), Stell (Refs. 16, 20, 21, and 24), and Snider (Ref. 23). The approach described here shows explicitly that the correction terms to (4.24) are such that the relations are turned into thermodynamic-function scaling laws (though even this point may be obtained from Ref. 20).

[57]G. Stell [Phys. Rev. Letters 20, 533 (1968)] was perhaps the first to suggest the occurrence of log terms. M. A. Moore [Phys. Rev. B 1, 2238 (1970)] argued that their presence was not consistent with numerical results when $d = 3$ but was when $d = 4$.

[58]See the extensive work of Stell (Refs. 20, 21, and 24).

[59]B. D. Josephson, Proc. Phys. Soc. (London) 92, 269 (1967); 92, 276 (1967). See also a very recent proof [G. Stell (report of work prior to publication)] which proceeds along rather different lines than those of Josephson.

[60]M. J. Buckingham and J. D. Gunton, Phys. Rev. 178, 848 (1969). The Buckingham-Gunton inequalities were first proposed—but not proved rigorously—by G. Stell [Phys. Rev. Letters 20, 533 (1968)] and by J. D. Gunton and M. J. Buckingham [*ibid.* 20, 143 (1968)]. Fisher (Ref. 53) presents a nice discussion of these inequalities.

[61]Relations analogous to (4.28) have been obtained by Stell (Ref. 20) [who writes $d\nu = (2-\alpha) + q\nu$] and by Snider (Ref. 23) [who writes $dw = (2-\alpha)$]. Clearly, $q = X/\nu$ and $w = d\nu/(d\nu - X)$. Snider somewhat misleadingly treats w as a new exponent, and he implies that w is related to both of Stell's new exponents (θ and q); in fact, $w = d/(d-q)$.

[62]Relation (4.30a) follows only on making certain additional assumptions. See, e.g., the discussion by Liu, Joseph, and Stanley (Ref. 53).

[63]Of course, in practice, data may be available only for a fixed value of one variable—e.g., structure factor data may be available only for $H = 0$, $S(\tau, 0, \vec{q})$.

[64]I. W. Smith, M. Giglio, and G. B. Benedek, Phys. Rev. Letters 27, 1556 (1971). We wish to thank I. Smith for providing us with his data in tabular form.

[65]G. Tuthill (private communication); and unpublished.

[66]G. Tuthill, F. Harbus, and D. Karo (private communication); and unpublished.

[67]M. P. Schulhof, P. Heller, R. Nathans, and A. Linz, Phys. Rev. Letters 24, 1184 (1970); Phys. Rev. B 1, 2304 (1970).

[68]In Sec. IV, b_r was related to b_H in (4.26) by means of a general assumption that only appears to be valid for two particular models (the $d = 2$ Ising and $d = 3$ spherical models). In contrast, the relations stated in this section are based on more specific microscopic considerations (Ref. 31).

[69]Equation (5.19) appears in H. Wagner, Phys. Letters 33A, 58 (1970). Wagner assumes that the magnetic field scales for both ferromagnets and antiferromagnets, which is in contradiction to the principles formulated by Griffiths and Wheeler (Ref. 42) who emphasize the effects of the fact that H is not the ordering field for an antiferromagnet. Wagner next assumes $b_\chi = 0$, when in fact only $b_\chi \geq 0$ is required for χ to be finite at T_N. Wagner also makes the second assumption of (4.25b) that $2b_M = 1$. By this reasoning, he obtains $z = \frac{3}{2}$ for antiferromagnets. A detailed discussion of scaling in antiferromagnets (and multicomponent systems in general) using the GHF approach appears in T. S. Chang, A. Hankey, and H. E. Stanley, Phys. Rev. (to be published).

[70]L. P. Kadanoff and J. Swift, Phys. Rev. 166, 89 (1968); K. Kawasaki, Ann. Phys. (N.Y.) 61, 1 (1970). We wish to thank Professor Kadanoff for emphasizing to us that the relation $\gamma - a = \nu$ is for $d = 3$ only.

[71]Riedel and Wegner, Ref. 18; also, see F. Wegner, J. Phys. (Paris) 32S, 519 (1971).

[72]M. Suzuki, Phys. Letters 35A, 23 (1971); Progr. Theoret. Phys. (Kyoto) 46, 1054 (1971).

[73]R. Abe, Progr. Theoret. Phys. (Kyoto) 44, 339 (1970).

[74]L. P. Kadanoff (unpublished lectures); see also Ref. 28.

[75]A. Coniglio, Phys. Letters 38A, 105 (1972); Physica 58, 489 (1972).

[76]G. Paul and H. E. Stanley [Phys. Letters 37A, 347 (1971); Phys. Rev. B 5, 2578 (1972)] considered (6.1) for both sc and fcc lattices, and calculated the spin correlation function, the second moment, the susceptibility, and the specific heat. The Hamiltonian (6.1) was considered independently, for only the case of the sc lattice, and for only the susceptibility function, by J. Oitmaa and I. G. Enting [J. Phys. C 5, 231 (1972), and references contained therein]. These workers incorrectly interpreted their expansions as indicating a continuous variation of γ with anisotropy parameter R, and they failed to notice that the region of three-dimensional behavior for small R, while small, is nevertheless present [cf. Fig. 2(c)].

[77]R. B. Griffiths, Phys. Rev. Letters 24, 1479 (1970).

[78]Thus, for the Ising-model example of (6.1), $a_\tau = \frac{8}{15}$, $a_H = \frac{5}{6}$, $\bar{a}_\tau = \frac{1}{2}$, and $\bar{a}_H = \frac{15}{16}$.

[79]F. Harbus, R. Krasnow, and L. Liu (unpublished). See also Figs. 6(a), 7, 8, and 11 of G. Paul and H. E. Stanley, Phys. Rev. B 5, 2578 (1972).

[80]R. J. Birgeneau (private communication). See, e.g.,

R. J. Birgeneau, H. J. Guggenheim, and G. Shirane, Phys. Rev. Letters 22, 720 (1969); Phys. Rev. B 1, 2211 (1970), and references contained therein. This point [item (ii)] was independently made by M. E. Fisher (unpublished).

[81]The reader will note that the authors of Ref. 71 make two separate hypotheses, one for the Gibbs potential and one for the correlation length. This is not necessary, since (6.24) implies homogeneity of both these functions (see discussion in Sec. III and Appendix B).

[82]The detailed application of the GHF approach to tricritical behavior is treated elsewhere [cf. A. Hankey, H. E. Stanley, and T. S. Chang, Phys. Rev. Letters 29, 278 (1972)] and to be published. The first application of scaling to tricritical behavior appears in E. K. Riedel, Phys. Rev. Letters 28, 675 (1972); his approach also uses essentially the GHF formalism.

[83]See, e.g., Ref. 45, Eq. (6.29).

[84]There are systems for which the thermodynamic potential is analytic, yet exponents other than these of conventional mft are found. J. Rowlinson [*Liquids and Liquid Mixtures* (Butterworths, London, 1969), 2nd ed., p. 86] discusses the general structure of such a system. A lattice-model realization of such a system is given by J. F. Nagle [Phys. Rev. A 2, 2124 (1970)] and a fluid realization by P. C. Hemmer and G. Stell [Phys. Rev. Letters 24, 1284 (1970); J. Chem. Phys. 56, 4274 (1971)]. In such models $\alpha' \neq \alpha$, for example.

[85]T. H. Berlin and M. Kac, Phys. Rev. 186, 821 (1952).

[86]A. good review of the scaling properties of the spherical model will appear in G. S. Joyce, Phase Transitions, edited by C. Domb and M. S. Green (unpublished). The equivalence of the spherical model to the $D \to \infty$ limit of (D5) was pointed out first in Ref. 90:

[87]For $d>4$, the spherical model does not satisfy the two-exponent relations [e.g., (4.24)] but does satisfy the three-exponent relations [e.g., (4.22)]. The spherical model was probably first discussed in this connection by G. Stell [Phys. Rev. 184, 135 (1969)]. For $d=4$ the presence of log terms destroys thermodynamic homogeneity altogether for the spherical model; this is also discussed by Stell.

[88]See, e.g., Ref. 45, Chap. 8, Eq. (8.42).

[89]E. H. Lieb, Phys. Rev. Letters 18, 692 (1967); 19, 108 (1967); 19, 1046 (1967); Phys. Rev. 162, 162 (1967); B. Sutherland, Phys. Rev. Letters 19, 103 (1967); C. P. Yang, *ibid.* 19, 586 (1967); B. Sutherland, C. N. Yang, and C. P. Yang, *ibid.* 19, 588 (1967).

[90]The Hamiltonian of Eq. (D5) was first proposed in H. E. Stanley, Phys. Rev. Letters 20, 589 (1968). For a discussion of scaling for the case $D=2$ (which possibly represents $SrTiO_3$) see H. E. Stanley, in *Structural Phase Transitions and Soft Modes*, edited by E. J. Samuelsen, E. Andersen, and J. Feder (Universitetsforlaget, Oslo, 1971), p. 271.

[91]H. E. Stanley, Phys. Rev. 176, 718 (1968); 179, 570 (1969); J. Phys. Soc. Japan 26S, 102 (1969). See Ref. 45, Sec. 8.4, and Ref. 90.

[92]For reviews, see C. Domb, Advan. Phys. 19, 339 (1970); 9, 149 (1960); Ref. 45, Chap. 9.

[93]H. E. Stanley and D. D. Betts, Phys. Rev. (to be published).