Early warning of large volatilities based on recurrence interval analysis in Chinese stock markets

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Forecasting extreme volatility is a central issue in financial risk management. We present a large volatility predicting method based on the distribution of recurrence intervals between successive volatilities exceeding a certain threshold \(Q\), which has a one-to-one correspondence with the expected recurrence time \(\tau_Q\). We find that the recurrence intervals with large \(\tau_Q\) are well approximated by the stretched exponential distribution for all stocks. Thus, an analytical formula for determining the hazard probability \(W(\Delta t|t)\) that a volatility above \(Q\) will occur within a short interval \(\Delta t\) if the last volatility exceeding \(Q\) happened \(t\) periods ago can be directly derived from the stretched exponential distribution, which is found to be in good agreement with the empirical hazard probability from real stock data. Using these results, we adopt a decision-making algorithm for triggering the alarm of the occurrence of the next volatility above \(Q\) based on the hazard probability. Using the 'receiver operator characteristic' analysis, we find that this prediction method efficiently forecasts the occurrence of large volatility events in real stock data. Our analysis may help us better understand reoccurring large volatilities and quantify more accurately financial risks in stock markets.

Keywords: Extreme volatility; Risk estimation; Recurrence interval; Large volatility forecasting; Distribution; Hazard probability

JEL Classification: G130, G110

1. Introduction

Predicting extreme volatility events in financial markets is essential in risk estimation. A standard approach to extreme event prediction is to find the precursory patterns prior to an extreme event or to quantify the probability that a given pattern is a precursor to an extreme event (Hallerberg et al. 2007, Hallerberg and Kantz 2008). Bogachev and Bunde (2009a, 2011) propose a new method based on the statistics of the recurrence intervals between events exceeding a threshold to determine the risk probability \(W(\Delta t|t)\) that an extreme event will occur within the next \(\Delta t\) intervals when the last extreme event occurred \(t\) periods ago. They find that when examining real market data and model data with a low level of noise, the predicting method based on recurrence interval analysis produces better forecasts than the method based on precursor pattern recognition (Bogachev and Bunde 2009a, 2011).

Understanding the recurrence interval, defined as the waiting time between consecutive events with values greater than a predefined threshold \(Q\), is essential in uncovering the underlying laws governing extreme events in many fields. Recurrence interval analysis has been carried out on many kinds of time series in predicting the probability that an extreme event will occur, including records of climate (Bunde et al. 2004, 2005), seismic activities (Corral 2003, Saichev and Sornette 2006), energy dissipation rates of three-dimensional turbulence (Liu et al. 2009), heartbeat intervals in medical science (Bogachev et al. 2009), precipitation and river run-off (Bogachev and Bunde 2012), Internet traffic (Bogachev and Bunde 2009b, Cai et al. 2009), financial volatilities...
Anumber of studies ranging from daily to high-frequency data and from developed to emerging markets (Wang et al. 2006, 2007, Jung et al. 2008, Qiu et al. 2008, Ren et al. 2009a, 2009b, Jeon et al. 2010, Wang and Wang 2012, Xie et al. 2014), have reported that the distribution of the recurrence intervals of financial volatility is a stretched exponential. The stretched exponential distribution is also observed in the recurrence time between returns above a given positive threshold or below a negative threshold for the index spot and futures in the Chinese future markets (Suo et al. 2015), which is in contrast to the $q$-exponential distribution observed in the recurrence intervals between the losses in financial returns (Ludescher et al. 2011, Ludescher and Bunde 2014).

In this paper, we describe the materials and methods in section 2, determine the distribution of the recurrence intervals between large volatilities in section 3 and report the hazard probability results and predicting algorithm performance in section 4. In section 5, we summarize our findings.

2. Materials and methods

2.1. Data description

To carry out a detailed recurrence interval analysis of the Chinese stock markets, we include as many Chinese stocks in our analysing sample as possible. The minute-by-minute price data of all stocks in the Chinese markets are extracted from the RESSET financial database. The extracting period is from 26 July 1999 to 30 December 2011, which is the maximum spanning period allowed in the RESSET database. To ensure that the recurrent interval results between the top 1% volatilities will have more than 1000 data points, we select only those stocks that have a minimum of two years of trading records. The large sample size lowers the error rate when we use a maximum likelihood estimation (MLE) to fit the distributions. Finally, we have 1820 stocks in our sample, including 799 A-shares, 54 B-shares and 51 ChiNext shares in the Shenzhen market, and 862 A-shares and 54 B-shares in the Shanghai market.

2.2. Definition of volatilities and recurrence intervals

Following the review of volatility estimator in Bollen and Inder (2002), for a given minute-by-minute price series $I(t)$, the minute-by-minute volatility $\omega(t)$ can be estimated using

$$\omega(t) = [\ln I(t) - \ln I(t - 1)].$$

Such definition is also widely used in the reference on recurrence interval analysis of financial volatilities (Yamasaki et al. 2005, Wang et al. 2006, 2008, 2009, Xie et al. 2014). Using absolute return as financial volatility not only has the advantage of considering the extreme negative and positive returns at the same time, but also allows us to directly compare our results with the results reported in the literature.

In order to eliminate the influence of the daily periodic patterns, we remove the intraday patterns from the volatility series $\omega(t)$ on each trading day,

$$\omega'(s) = \omega(s)/G(s),$$

where $G(s) = \sum_{i=1}^{N} \omega(i, s)/N$. Here, $\omega(i, s)$ represents the volatility at time $s$ on day $i$. The normalized volatility series $v(t)$ is then obtained by dividing $\omega'(t)$ by its standard deviation,

$$v(t) = \frac{\omega'(t)}{\sqrt{\langle (\omega'(t))^2 \rangle - \langle (\omega'(t))^2 \rangle^2}},$$

where $(\cdot)$ means the average value of $(\cdot)$.

The focus of our study is the recurrence interval between the normalized volatilities exceeding a predefined threshold $Q$. To compare the results between different stocks, we quantify $Q$ by its mean recurrence time $\tau_Q$. There is a one-to-one correspondence between $Q$ and $\tau_Q$, such that (Podobnik et al. 2009)

$$\frac{1}{\tau_Q} = \int_{Q}^{\infty} p_{\nu}(v)dv,$$

where $p_{\nu}(v)$ is the probability distribution of the volatility. Using copula, Chicheportiche and Chakraborti (2014) also prove this equation and state that this equation is universal no matter what the underlying dependence structure in the analysing series is. Here, we restrict $\tau_Q$ to a range of [20, 100]. This range corresponds to the extreme volatilities from a top value of 5–1%, which is often considered in risk estimation.
3. Distribution of recurrence intervals between large volatilities

The distributions of recurrence intervals are found to exhibit a scaling behaviour for the large volatilities filtered by different thresholds (Yamasaki et al. 2005, Wang et al. 2006), which means that the interval distribution is independent of the thresholds and thus the distribution of recurrence intervals between extreme events can be inferred from the distribution of recurrence intervals between non-extreme events. However, such scaling behaviours are rejected by follow-up analysis (Wang et al. 2008, 2009, Ren and Zhou 2008), which states that the extreme event filtering threshold should have an influence on the recurrence interval distribution. This behaviour is corroborated by the fact that the estimated distributional parameters are found to have a strong dependence on the thresholds when the recurrence intervals are fitted by given distribution functions, such as stretched exponential distribution (Xie et al. 2014, Suo et al. 2015) and q-exponential distribution (Ludescher et al. 2011, Chicheportiche and Chakraborti 2014). More interestingly, Ludescher et al. (2011) and Ludescher and Bunde (2014) argue that the distribution of recurrence intervals depends only on the mean recurrence interval \( \tau_Q \), and not on a specific asset or on the time resolution of the data. According to their analysis, the distribution of recurrence intervals from different financial series should overlap on the same curve for the same threshold. The above discussions motivate us to understand the following three problems for the recurrence interval between large volatilities in the Chinese stock markets: (1) whether there is a scaling behaviour for the recurrence intervals between the large volatilities with different thresholds for individual stocks; (2) whether the recurrence intervals between large volatilities with the same threshold exhibit the same distribution for different stocks; and (3) whether the market dynamics have any influence on the distribution of recurrence intervals.

In order to have an overview of the distribution of the recurrence intervals between large volatilities, we firstly plot the recurrence interval distribution of randomly chosen stocks in log–log scale in figure 1. As shown in figure 1(a), the distributions of the recurrence interval between large volatilities obtained from different thresholds are scaled by their \( \tau_Q \). The distribution of different stocks is shifted vertically for better visibility. One can see that the scaled recurrence interval distributions corresponding to different thresholds are overlapping on the same curve for the five stocks. Obviously, the appearance of the collapsing behaviours indicates that the recurrence interval distributions possess scaling behaviours for the five randomly chosen stocks. Figure 1(b) illustrates the collapsing behaviours of the recurrence interval distributions obtained from the same \( \tau_Q \) for different stocks. Ten stocks are randomly chosen to present the distribution overlapping behaviour. In order to increase visibility, the distribution curves of different \( \tau_Q \) values are transformed by a factor. It is found that for the same \( \tau_Q \) value, the recurrence time distributions of different stocks are overlapping on the same curve. This means that the return intervals between volatilities that exceed a threshold \( Q \) may exhibit a universal distribution for different stocks when \( \tau_Q \) is fixed. The results in figure 1 provide an amazing picture that the scaled recurrence interval distribution is independent of the \( \tau_Q \) for individual stocks and the recurrence intervals with the same \( \tau_Q \) have the same distribution across different stocks, which leads to the hypothesis that the distribution of scaled recurrence interval between large volatilities is universal, depends neither on the value of \( \tau_Q \) nor on specific stocks.

To quantitatively test the hypothesis, the best strategy is to find a suitable distribution function to fit the recurrence interval distribution. This allows us to verify the above hypothesis by comparing with the estimated distributional parameters corresponding to different stocks and thresholds. It is well known that the memory behaviour in the underlying process plays a very important role in the distribution form of the recurrence intervals. The memory-less process always results in an exponential long distribution of recurrence time. For the process with long memory, the situation is much more complicated. If only linear long memory is incorporated, the stretched exponential and Weibull distribution of recurrence intervals are supported both in theoretical derivations (Santhanam and Kantz 2008, Olla 2007) and numerical experiments (Altmann and Kantz 2005, Penttinen 2006, Eichner et al. 2007). Furthermore, power-law distributions of recurrence time are uncovered in synthetic processes with non-linear long memory behaviours (multifractal processes) (Bogachev et al. 2007, 2008a, Bogachev and Bunde 2008). As with the synthetic processes, the recurrence interval analysis on the realistic processes, for example the empirical analysis in financial markets, also provides inconsistent results, such that the recurrence time is found to follow a stretched exponential distribution (Wang et al. 2006, 2007, Jung et al. 2008, Qiu et al. 2008, Jeon et al. 2010, Wang and Wang 2012, Suo et al. 2015), a power-law distribution with an exponential cut-off (Kaizoji and Kaizoji 2004, Yamasaki et al. 2005, Lee et al. 2006, Greco et al. 2008), and a q-exponential distribution (Ludescher et al. 2011, Ludescher and Bunde 2014). The q-exponential distribution is also reported to nicely fit the inter-trade time, which is the waiting time between consecutive trades (Politi and Scalas 2008, Scalas et al. 2004, Jiang et al. 2008). Here, we propose to use four candidate distributions to fit the recurrence intervals between large volatilities in the Chinese stock markets. Our purpose is to find the best fits out of the four distributions for the following predicting analysis. The following are four candidate distributions: the stretched exponential distribution,

\[
p_r(\tau) = a \exp[-(b \tau)^\mu],
\]

the power law distribution with an exponential cut-off,

\[
p_p(\tau) = c \tau^{-\gamma -1} \exp(-k \tau),
\]

the q-exponential distribution,

\[
p_q(\tau) = (2 - q) \lambda [1 + (q - 1) \lambda \tau]^{-1/\theta},
\]

and the Weibull distribution,

\[
p_w(\tau) = \frac{\tau}{d} \left( \frac{\tau}{d} \right)^{\gamma - 1} \exp \left[ - \left( \frac{\tau}{d} \right)^{\gamma} \right].
\]

We have the following two equations \( \int_0^\infty p(\tau) d\tau = 1 \) and \( \int_0^{\infty} \tau p(\tau) d\tau = \tau_Q \) for any distribution formula of \( p(\tau) \), which allow us to reduce the number of estimated parameters for the four candidate distributions. The second equation is universal, independent of any memory structure of the underlying process (Chicheportiche and Chakraborti 2014, 2013). With the two equations, we have

\[
a = \frac{\gamma (\gamma + 1)}{\Gamma(\gamma)} \tau_Q \quad \text{and} \quad b = \frac{\Gamma(\gamma)}{\Gamma(\gamma + 1)} \tau_Q.
\]
To determine which distribution has the best performance, we better fits the distribution for small intervals for two stocks, 000001 and 900956. One can see that results of the four candidate distributions of the recurrence solution for the estimated parameter. Obviously, we have only one parameter to estimate for the four candidate distributions. We use the MLE method to estimate the parameters, we discretize the estimating parameters in their equation analytically by taking a derivative of the loga-
distribution and \( d = \frac{\tau_0}{(1 + \gamma)} \) for the Weibull distribution. Obviously, we have only one parameter to estimate for the four candidate distributions. We use the MLE method to estimate the distributional parameters. Because it is very hard to obtain the equation analytically by taking a derivative of the logarithmic likelihood function with the corresponding estimated parameters, we discretize the estimating parameters in their fitting range with a step of \( 10^{-6} \) (\( \mu \in (0, 1), \gamma \in (-1, 0), q \in (1, 1.5), \) and \( \zeta \in (0, 1) \)) and evaluate the logarithmic likelihood function \( \ln L \) of these discrete values. The discrete value associated with the maximum \( \ln L \) is took as the final solution for the estimated parameter.

Figure 2 shows the empirical distributions and the fitting results of the four candidate distributions of the recurrence intervals for two stocks, 000001 and 900956. One can see that in the central regions of the distributions, all four candidates agree with the empirical data. The \( q \)-exponential distribution better fits the distribution for small \( \tau_Q \) and the stretched exponential distribution gives better fits in the tail for large \( \tau_Q \). To determine which distribution has the best performance, we utilize likelihoods and KS statistics to assess the validity of the estimates. As we known, KS statistic measures the distance between the fitting cumulative distribution and the empirical cumulative distribution. Here, we have four candidate distributions to fit the recurrence intervals. If one distribution gives the minimum KS statistic, this distribution can be regarded as the closest to the empirical distribution, which indicates the best fit. This is applied in finding the best truncated boundary when fitting to the left-truncated distributions (Clauset et al. 2009, Jiang et al. 2013). Therefore, the candidate distribution giving the maximum likelihood or the minimum KS statistic is considered to fit the recurrence intervals best.

Taking stock 000001 as an example, figure 3(a) highlights the candidate distribution which has the maximum likelihood when it is fitted to the recurrence intervals for each \( \tau_Q \). Figure 3(b) is the same as (a), except that we highlight the distributions on the basis of the minimum KS statistics. The symbols ‘s’, ‘p’, ‘q’ and ‘w’ correspond to stretched exponential distribution, power-law distribution with an exponential cut-off, \( q \)-exponential distribution and Weibull distribution, respectively. Both panels indicate that the \( q \)-exponential distribution gives the best fits when \( \tau_Q \) is not large and the stretched exponential distribution performs the best for large values of \( \tau_Q \). However, the transition point is different for the two validity measurements. The transition point from \( q \)-exponential distribution to stretched exponential distribution is close to \( \tau_Q = 90 \) for KS statistics, which is much larger than that for likelihoods, \( \tau_Q = 60 \). Such results allow us to further classify the stocks into one group for each \( \tau_Q \) if the same distribution fits their recurrence intervals best.

We count the number of stocks in each group for each \( \tau_Q \) and each validity measurement and report the results in table 1. One can observe that more than 90% of stocks are in \( q \)-exponential and stretched exponential groups for all values of \( \tau_Q \) and both validity measurements favour the \( q \)-exponential distribution (respectively, stretched exponential distribution) when \( \tau_Q \) approaches 20 (100). However, the classification of stocks based on likelihoods and KS statistics does not tell us anything about the significance of the fits. Following the methods (Jiang et al. 2013), we employ both KS tests and CvM tests to check the goodness of the fits in each group for each \( \tau_Q \). The null hypothesis \( H_0^\alpha \) for both tests is that the data are drawn from a testing distribution. The testing distribution for a given group corresponds to the distribution that fits the recurrence intervals best for the stocks in that group. The stocks whose recurrence intervals cannot reject the hypothesis \( H_0^\alpha \) at the significant level of 0.01 for either of the two tests are labelled as passing the statistical tests.

We also list the number of stocks passing the statistical tests in table 1. We find that only a small number of stocks are found to be in the group of power-law distribution with an exponential cut-off and Weibull distribution. However, most of them do not fail the statistical tests. It is also observed that only a small number of stocks in \( q \)-exponential group pass the statistical tests for all \( \tau_Q \). This means that \( q \)-exponential distribution is not suitable to fit the recurrence intervals between extreme volatilities in the Chinese stock markets, although the validity measurements favour it for small \( \tau_Q \). In contrast, the number of stocks passing the tests in stretched exponential group keeps increasing with the increment of \( \tau_Q \). The percentage of stocks out of the Chinese markets whose recurrence intervals cannot reject the stretched exponential distribution is more than 60%
(respectively, 56%) for the validity measurements of likelihoods (respectively, KS statistics) when $\tau_Q = 100$. The results indicate that the stretched exponential distribution is a good candidate to fit the recurrence intervals with large $\tau_Q$ in the Chinese stock markets. Another intriguing observation in table 1 is that very few stocks pass the statistical tests for small $\tau_Q$. It could be attributed to that the parameter estimating methods are based on the continuous distribution formula. Low $\tau_Q$ will lead to a sample of recurrence intervals containing many small discrete values, which makes the estimator biased.

Despite the statistical tests for goodness of fits do not give affirmative results of accepting the null hypothesis $H_0^i$ for all distributional fits, the four candidate distributions are still a very good approximation to the empirical recurrence interval distributions, as evidenced in figure 2. Therefore, it is feasible to discuss the scaling behaviour for individual stocks with resort to the dependence between the distribution parameters and $\tau_Q$. As stated above, if the recurrence intervals exhibit a scaling behaviour, the estimated parameters of the four distributions should be independent of $\tau_Q$, which means that the slope should be zero if the estimated parameters are regressed against $\tau_Q$. We plot the estimate parameters as a function of $\tau_Q$ in figure 4 for stock 000001 (a) and stock 900956 (b). It is observed that the four parameters present an asymptotic horizontal line with respect to $\tau_Q$ for both stocks. The least linear square fits of $\mu$, $\gamma$, $q$ and $\xi$ with respect to $\tau_Q$ give the four slopes of $\kappa_\mu = -8.84 \times 10^{-4}$, $\kappa_\gamma = 1.1 \times 10^{-3}$, $\kappa_q = 3.43 \times 10^{-4}$, and $\kappa_\xi = -6.79 \times 10^{-4}$ (respectively). We further propose a strict statistical test in the spirit of bootstrapping to check whether there exists a scaling behaviour in recurrence intervals for individual stocks. The null hypothesis $H_0^i$ is that the distribution parameters ($\mu$, $\gamma$, $q$ and $\xi$) are independent of $\tau_Q$. If the null hypothesis $H_0^i$ cannot be rejected, the recurrence intervals can be argued to possess a scaling behaviour for a given stock. For a given distribution, we estimate the slope $\kappa$ by linearly regressing its parameter against $\tau_Q$. We define the statistics as the slope $\kappa$.

For a given stock, we perform the statistical test on each stock and find that 1586 are independent of $\tau_Q$. Therefore, it is feasible to discuss the scaling behaviour for individual stocks with resort to the dependence between the distribution parameters and $\tau_Q$. As stated above, if the recurrence intervals exhibit a scaling behaviour, the estimated parameters of the four distributions should be independent of $\tau_Q$, which means that the slope should be zero if the estimated parameters are regressed against $\tau_Q$. We plot the estimate parameters as a function of $\tau_Q$ in figure 4 for stock 000001 (a) and stock 900956 (b). It is observed that the four parameters present an asymptotic horizontal line with respect to $\tau_Q$ for both stocks. The least linear square fits of $\mu$, $\gamma$, $q$ and $\xi$ with respect to $\tau_Q$ give the four slopes of $\kappa_\mu = -8.84 \times 10^{-4}$, $\kappa_\gamma = 1.1 \times 10^{-3}$, $\kappa_q = 3.43 \times 10^{-4}$, and $\kappa_\xi = -6.79 \times 10^{-4}$ (respectively).
For the same $\tau_Q$ and a given distribution, we regard the corresponding estimated parameters of different stocks as one sample and estimate the mean, standard deviation, skewness and kurtosis. The results are reported in table 2. It is observed that the mean value of the distribution parameters exhibit a weak trend against $\tau_Q$, which is in agreement with the above analysis on individual stocks. Another intriguing finding is that the standard deviation is stable with the increment of $\tau_Q$ for each candidate distribution. Such narrow standard deviation implies that the recurrence intervals of different stocks roughly conform to the same distribution when $\tau_Q$ has the same value (Ludescher et al. 2011, Ludescher and Bunde 2014). We also observe that the skewness does not equal to 0 and the kurtosis is greater than 3, which indicate that the distributions of estimated parameters are non-normal. More interesting, the skewness of $\mu$ and $\zeta$ decreases from positive to negative with the increase in $\tau_Q$, which indicates that the distribution shifts from right-skewed to left-skewed, while for the skewness of $\gamma$ and $q$, the situation is the opposite.

In order to determine whether the distribution parameters are influenced by market states, i.e. bull or bear, we use a moving window analysis to track the evolution of estimating parameters ($\mu$, $\gamma$, $q$, and $\beta$). We fix the window size at 48 months and exclude stocks with trading periods shorter than 97 months. We are left with 948 stocks as subject for our rolling window analysis. The window (ending time) varies from 2003 to 2011 with a step of one year. We also discard first-month trading data for the remaining because first-month records tend to be partial (i.e. do not span an entire month) and the volatilities for new IPOs are excessively large (Guo et al. 2013), which is caused by the existence of information asymmetry for IPO firms and the incomplete knowledge about IPO firms for investors (Hussein and Zhou 2014).

For each stock, we perform the same analysis in each window as for in the entire period. The recurrence intervals associated with different values of $\tau_Q$ are fitted by the four candidate distributions, which give the corresponding distribution parameters ($\mu$, $\gamma$, $q$ and $\zeta$). As stated above, the distribution parameters are universal across different stocks when $\tau_Q$ has the same value. We estimate the average and standard deviations of the estimated parameters of different stocks for the same $\tau_Q$ in each window. The results are listed in table 3. Generally speaking, the distribution parameters estimated from the moving windows and the entire period share the same
4. Predicting large volatilities

We use the hazard probability \( W(\Delta t|t) \) to forecast the occurrence of large volatility events. The \( W(\Delta t|t) \) is the probability that there will be additional waiting time \( \Delta t \) before another large volatility event occurs when the previous large volatility event occurred \( t \) time ago, which can be formulated as (Sornette and Knopoff 1997, Bogachev et al. 2007),

\[
W(\Delta t|t) = \int_{t}^{t+\Delta t} p(t) dt / \int_{0}^{\infty} p(t) dt, \tag{9}
\]

where \( p(t) \) is the probability distribution of the recurrence intervals between extreme events. This probability is the key early warning measurement for the occurrence of extreme volatilities. The early warning is triggered when the probability \( W(\Delta t|t) \) is greater than a predefined hazard threshold. We can theoretically derive this hazard probability if we have the distribution of the recurrence intervals between consecutive extreme volatilities.

If we designate the top 1% of volatility values (corresponding to the mean recurrence time \( \tau_Q = 100 \)) to be extreme events, we can estimate the hazard probability \( W_q(\Delta t, t) \) in \( t \) when fixing \( \Delta t \). The recurrence intervals between volatilities with \( \tau_Q = 100 \) are well approximated by the stretched exponential distribution for most stocks, as evidenced in table 1. This allows us to approach the theoretical hazard probability \( W_{SE}(\Delta t, t) \) in terms of the stretched exponential distribution. By substituting equation (5) into equation (9), we obtain

\[
W_{SE}(\Delta t|t) = \frac{b \Gamma_s \left( \frac{1}{\mu}, (bt)^\mu \right) - \Gamma_u \left( \frac{1}{\mu}, (bt + \Delta t)^\mu \right)}{\Gamma_s \left( \frac{1}{\mu}, (bt)^\mu \right)}, \tag{10}
\]

where \( \Gamma_s(s, x) \) and \( \Gamma_u(s, x) \) are lower and upper incomplete gamma functions and \( a \) and \( b \) are dependent on \( \mu \). On the other hand, the empirical hazard function \( W_{emp} \) can be evaluated as follows,

\[
W_{emp}(\Delta t|t) = \frac{\#(t < \tau < t + \Delta t)}{\#(\tau > t)}, \tag{11}
\]

where the denominator \( \#(\tau > t) \) is the number of recurrence intervals whose values are greater than \( t \), and the numerator \( \#(t < \tau < t + \Delta t) \) is the number of recurrence intervals which locates in the range of \( (t, t + \Delta t) \).

Figure 5 shows the hazard probability for two stocks when \( \Delta t = 1, 5 \) and 10. The solid curves stand for the analytical solution \( W_{SE} \) in equation (10) and the markers represent the empirical hazard probability in equation (11). One can see that in each panel, the curve and the markers decrease slowly and are in good agreement. The decreasing trend of \( W(\Delta t|t) \) means that the probability of observing another following large volatility decreases with the elapsing \( t \) if an extreme volatility occurs. This implies the existence of potential dependent structures in triggering the large volatilities, which is inconsistent with the clustering behaviour in the volatility series. The oscillations of \( W_{emp} \) in panels (c) and (f) are attributed to the poor statistics of recurrence time in the sampling interval of \( (t, t + \Delta t) \). Besides stock volatilities, the general formula of hazard probability

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Table 3. Results of the rolling window analysis. \(\mu\), \(\gamma\), \(q\) and \(\zeta\) correspond to the parameters of stretched exponential distribution, power-law distribution with an exponential cut-off, \(q\)-exponential distribution and Weibull distribution. We list the mean for the sample of the parameters from different stocks in the same window with the same \(\tau\), as well as the standard deviation in parentheses.

<table>
<thead>
<tr>
<th>Moving window</th>
<th>Panel A: Stretched exponential distribution ((\mu))</th>
<th>Panel B: Power-law distribution with an exponential cutoff ((\gamma))</th>
<th>Panel C: (q)-exponential distribution ((q))</th>
<th>Panel D: Weibull distribution ((\zeta))</th>
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<tr>
<td>20</td>
<td>(0.70(0.07))</td>
<td>(-0.87(0.07))</td>
<td>(1.20(0.07))</td>
<td>(0.88(0.07))</td>
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<tr>
<td>30</td>
<td>(0.66(0.07))</td>
<td>(-0.81(0.07))</td>
<td>(1.22(0.07))</td>
<td>(0.85(0.07))</td>
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<tr>
<td>40</td>
<td>(0.64(0.07))</td>
<td>(-0.79(0.07))</td>
<td>(1.22(0.07))</td>
<td>(0.84(0.07))</td>
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<tr>
<td>50</td>
<td>(0.62(0.08))</td>
<td>(-0.77(0.08))</td>
<td>(1.23(0.08))</td>
<td>(0.83(0.07))</td>
</tr>
<tr>
<td>60</td>
<td>(0.62(0.08))</td>
<td>(-0.76(0.08))</td>
<td>(1.23(0.08))</td>
<td>(0.82(0.07))</td>
</tr>
<tr>
<td>70</td>
<td>(0.61(0.09))</td>
<td>(-0.75(0.09))</td>
<td>(1.22(0.09))</td>
<td>(0.82(0.07))</td>
</tr>
<tr>
<td>80</td>
<td>(0.61(0.09))</td>
<td>(-0.74(0.09))</td>
<td>(1.22(0.09))</td>
<td>(0.81(0.07))</td>
</tr>
<tr>
<td>90</td>
<td>(0.61(0.10))</td>
<td>(-0.74(0.09))</td>
<td>(1.21(0.09))</td>
<td>(0.81(0.07))</td>
</tr>
<tr>
<td>100</td>
<td>(0.62(0.10))</td>
<td>(-0.74(0.10))</td>
<td>(1.21(0.10))</td>
<td>(0.81(0.07))</td>
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</table>
\( W(\Delta t | t) \) in equation (9) can be also applied to estimate the risk probability of coming extreme events in climate, seismic activities, stock returns, floods and so on.

Based on equation (10), we can estimate the risk probability of an incoming extreme event in the following period of \( \Delta t \). If one sets \( \Delta t = 1 \), we will immediately have the information of hazard probability in next unit of time (minute here). Using a decision-making algorithm, we are allowed to predict whether there is a large volatility in the next minute (Bogachev and Bunde 2011). Specifically, we need to set a threshold \( Q_p \) for the hazard probability to trigger early warnings that a large volatility is about to occur. When the hazard probability exceeds \( Q_p \), an alarm that a large volatility will occur during the next time point is activated. By comparing with actual extreme events, we can estimate the correct prediction and false alarm rates to evaluate the predicting performance. Instead of choosing a specific value of the hazard threshold, \( Q_p \) is varied in the range of \([0, 1] \). Such a strategy could avoid the situation of considering only one hazard threshold value: that a large value will increase the number of missing events and a small value will increase the number of false alarms. The plots of correct prediction rates with respect to false alarm rates from all possible hazard thresholds \( Q_p \) will give rise to the famous 'receiver operator characteristic' (ROC) curve (Bogachev and Bunde 2009a, 2009b, 2011, Bogachev et al. 2009), which graphically displays the forecasting efficiency.

To estimate the correct prediction and false alarm rates, we generate for a given \( Q_p \) two forecasting signals—alarms and non-alarms—at each time point. By comparing the forecasting signals with the real data, we obtain one of the four outcomes at each time point (Bogachev and Bunde 2011), (1) a correct prediction of a large volatility event, (2) a correct prediction of a non-large volatility event, (3) a missed event and (4) a false alarm. By recording in our testing records how many times each outcome occurs, we can estimate the correct prediction rate \( D \) and the false alarm rate \( A \) using

\[
D = \frac{n_{11}}{n_0 + n_{11}}, \quad A = \frac{n_{10}}{n_0 + n_{10}},
\]

where \( n_{11} \) is the number of large volatility events that are correctly predicted, \( n_{00} \) the number of non-large volatility events that are correctly predicted, \( n_{01} \) the number of missed events and \( n_{10} \) the number of false alarms. All possible pairs of \((D, A)\) will be obtained if we vary the \( Q_p \) range from 0 to 1.

By definition, the ROC curve will be \( D = A = 1 \) if \( Q_p = 0 \) and \( D = A = 0 \) if \( Q_p = 1 \). The ROC curve joins the point (0, 0) in the left bottom corner to the point (1, 1) in the right top corner. For the random guess outcome, \( D = A \), a straight line is observed between the two corners. This occurs when there is no memory in the data. The area under the ROC curve is a measure of predicting performance. In practical application, people are more interested in the predicting model with less false alarms. Hence, we define the following performance statistics to evaluate the predicting power,

\[
AUC_m = \int_0^{0.3} D(A) dA,
\]

where \( AUC_m \) is nothing but the area below the ROC curve in the range of \( 0 \leq A \leq 0.3 \). Focusing \( A \) in this range is that the predictions are commonly useless if the false alarms are large. From this definition, the statistic \( AUC_m \) locates in the range of \([0.045, 0.3]\). The bottom limit corresponds to the random guess and the up limit is associated with the perfect prediction.

We estimate the volatility series \( v(t) \) for each stock and regard the last year as an out-of-sample predicting period. The data not in the last year are considered as the in-sample training set, which is used to calibrate the model parameters. The forecasting of large volatilities is implemented as follows:

- The stretched exponential distribution parameter \( \mu \) is estimated through fitting the recurrence intervals between large volatilities above a threshold \( Q \) associated with \( r_0 = 100 \) for the data in the in-sample training set.
- The events of large volatilities are identified in the out-of-sample predicting period based on the threshold \( Q \) in the in-sample calibrating period.
- The hazard probability \( W(\Delta t | t) \) in the out-of-sample predicting period is determined based on the in-sample distribution parameter \( \mu \) and the out-of-sample large volatilities.
- The correct prediction rates \( D \) and the false alarm rates \( A \) are calculated by comparing the predicting large volatilities with the actual out-of-sample large volatilities. The predicting large volatilities are obtained through making comparisons of the hazard probability \( W(\Delta t | t) \) and the hazard threshold \( Q_p \), which is varying from 0 to 1.
- The ROC curve is obtained through plotting \( D \) with respect \( A \). The performance statistic \( AUC_m \) is estimated based on equation (13).

Figure 6(a) plots a subseries of the volatility in the out-of-sample predicting period and highlights events above threshold \( Q \) in the top panel. The risk probability \( W(1 | t) \) is shown in the bottom panel. Note that \( W(1 | t) \) decreases as time \( t \) elapsed from the last large volatility event increases. Threshold \( Q_p \) is plotted as a horizontal line to show the activating alarm process. By varying \( Q_p \) within a \([0, 1]\) range, we obtain all pairs of \((A, D)\). Figure 6(b) shows the ROC curves for two stocks. The two curves are above the dashed line \( D = A \), indicating that our prediction is not random. The two curves do not overlap, indicating that the accuracy of this prediction algorithm varies for different stocks. We can observe that the \( AUC_m \) of stock 000001 is greater than that of stock 900956, which means that this model offers a relative better forecasting performance for stock 000001. We also calculate \( AUC_m \) for all stocks. Figure 6(c) shows the frequency plots of \( AUC_m \). Note that the peak is centred at \( \approx 0.08 \), about two times of a random prediction. We also find there are 22 stocks with \( AUC_m > 0.15 \). The top three values are 0.2130, 0.2088 and 0.1993 for stocks 0000557, 00050529 and 000628, indicating that our forecasting algorithm can accurately predict the large volatilities of these three stocks. Previous research has indicated that the efficiency of the algorithm is primarily influenced by the linear and non-linear memory behaviour in the original volatilities (Bogachev and Bunde 2011). Our results could be explained by that the behaviour of stocks with stronger memory behaviours, such as volatility clustering and multifacality, could be more accurately predicted using our algorithm. Our algorithm only takes into consideration the probability distribution of recurrence intervals; if the memory behaviour of recurrence intervals is also included, we believe that its predictive accuracy would be greatly enhanced.
5. Conclusion

In this work, we have utilized a decision-making algorithm to forecast the occurrence of large volatilities in the Chinese stock markets based on the hazard probability, which is derived from the distribution of recurrence intervals between the volatilities exceeding a threshold $Q$. By fitting the volatility recurrence intervals by means of four candidate distributions and assessing the fitting validity by means of likelihoods and KS statistics, we have found that the volatility recurrence intervals are well approximated by a stretched exponential distribution. All the four distribution parameters ($\mu$, $\gamma$, $q$ and $\zeta$) are found to be dependent of the mean recurrence time $\tau_Q$, which has a one-to-one correspondence with the threshold $Q$, for all the stocks in our sample. Using a moving window analysis, we have found that the parameters are influenced by market status and exhibit a monotonic trend with the evolution of time, that $\mu$ and $\zeta$ decrease and $\gamma$ and $q$ increase. These behaviours are consistent with the inefficiency of the Chinese stock markets from 2007 to 2011.

Using the stretched exponential distribution formula, we have derived an analytical solution of the hazard probability $W(\Delta t| t)$ of the next large volatility event above the threshold $Q$ within a short time interval $\Delta t$ after an elapsed time $t$ from the last large volatility above $Q$. This analytical solution $W(\Delta t| t)$ is in good agreement with the empirical risk probability estimated from real stock data. Based on a decision-marking algorithm, we have used the hazard probability to forecast large volatilities in the out-of-sample predicting period. We have found that the average statistic $AUC_m$, which corresponds to the area below the ROC curve in the range of $0 \leq A \leq 0.3$, is 0.091 for all stocks. We have also found that there are two stocks with $AUC_m > 0.2$, indicating that our predicting algorithm is accurate in forecasting the large volatilities. Our findings may shed new light on our understanding of extreme volatility behaviour and may have potential applications in managing stock market risk.

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Disclosure statement

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Early warning of large volatilities based on recurrence interval analysis in Chinese stock markets

We refer to the references provided in the document and include the necessary details.

**References**


