Exact solution in an external magnetic field of Ising models with three-spin interactions*

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The three-spin interaction Hamiltonian $\mathcal{H}^{(3)} \equiv -J_3 \Sigma_{\langle ijk \rangle} s_i \sigma_j \sigma_k - H\Sigma_i s_i - H'\Sigma_j \sigma_j$ (the s and σ spins belong to different sublattices) is solved for some two-dimensional lattices by a generalized star-triangle transformation. The $H' = 0$ internal energy, specific heat, and magnetization are explicitly calculated, and the singularity structures of other functions (e.g.,χ) are studied. Although the actual critical-point exponents remain the same as for systems with two-spin interaction, the three-spin interaction gives rise to different amplitude and background terms. In particular, the $H' = 0$ magnetization is nonzero for *all* finite T, because of the lack of spin-reversal symmetry; the diameter is seen explicitly to have an energy singularity, with critical exponent $1-\alpha$.

Ising models with three-spin interactions have attracted considerable recent attention, in part because they represent an example of a many-body Ising spin system lacking the usual "up-down" spin-reversal symmetry.[1-5] Thus far the only results have been for the *zero-field* three-spin Ising model, and functions that might be expected to display highly unusual behavior (e.g., magnetic response functions such as the magnetization m) have not been obtained.

It seems desirable to study further three-spin interaction Hamiltonians and in particular to search for situations in which exact results can be obtained in nonzero external magnetic field.

Consider the three-spin Ising interaction Hamiltonian

$$\mathcal{H}^{(3)} \equiv -J_3 \sum_{\langle ijk \rangle} s_i \sigma_j \sigma_k - H \sum_i s_i - H' \sum_j \sigma_j , \qquad (1)$$

where the s spins belong to one sublattice [cf. Fig. 1(a)] and the σ spins to another ($s = \pm 1$, $\sigma = \pm 1$). The index i runs over all spin sites of the s sublattice, the indices j, k denote sites of the σ sublattice, and the notation $\langle ijk \rangle$ denotes that sites j, k are nearest neighbors of the site i. The magnetic fields H, H' act separately on the s and σ sublattices, respectively; the special cases $H = H'$ and $H = -H'$ correspond to direct and staggered magnetic fields. For many two-dimensional lattices (denoted Y), we can apply a generalized star-triangle (Y-Δ) transformation and relate the partition function $Z_Y^{(3)}(J_3, H, H')$ for the three-spin interaction Hamiltonian (1) to a partition function $Z_\Delta^{(2)}(J_2, H')$ for a two-spin interaction among the spins σ_i, σ_j on the σ sublattice. This relation permits study of the internal energy, specific heat, magnetization, and susceptibility for the three-spin Hamiltonian (1). Many interesting features are noted; in particular, the $H' = 0$ magnetization is found to be nonzero for *all* T, because of the lack of spin-reversal symmetry, and the diameter is seen explicitly to have an "energy singularity" with exponent $(1-\alpha)$.

Our solutions are based upon application of a generalized star-triangle (Y-Δ) transformation. We shall illustrate this approach by considering in detail the diced lattice[6] of Fig. 1(a), where the spins on the s sublattice are represented by closed circles and the spins on the σ sublattice by open circles. Figure 1(b) shows a "unit" consisting of an s spin and its three nearest-neighbor (nn) σ spins. The Hamiltonian (1) can be written in terms of "unit Hamiltonians"

$$\mathcal{H}^{(3)} = \sum_{i=1}^{2N/3} \tilde{\mathcal{H}}_i + H' \sum_{j=1}^{N/3} \sigma_j , \qquad (2a)$$

where

$$\tilde{\mathcal{H}}_i = -J_3 (s_i \sigma_{i1} \sigma_{i2} + s_i \sigma_{i2} \sigma_{i3} + s_i \sigma_{i3} \sigma_{i1}) - H s_i . \qquad (2b)$$

where in (2b) we denote the three σ spins in unit i by σ_{i1}, σ_{i2}, and σ_{i3}.

Taking the trace over the s spin in unit i, we have

$$\sum_{s_i = \pm 1} \exp\{-\beta \tilde{\mathcal{H}}_i\} = I(J_3, H) \times \exp\{-\beta J_2 [\sigma_{i1}\sigma_{i2} + \sigma_{i3}\sigma_{i2} + \sigma_{i3}\sigma_{i1}]\} , \qquad (3)$$

in analogy to the conventional Y-Δ transformation; here $\beta \equiv 1/kT$. The quantities $I(J_3, H)$ and $J_2(J_3, H)$ may be determined by substituting particular values of the σ_{ij} in (3); thus

$$I(J_3, H) = 2e^{R(K,h)} (\cosh K \cosh h - \sinh K \sinh h), \qquad (4a)$$

where $K \equiv \beta J_3$, $R \equiv \beta J_2$, $h \equiv \beta H$, and J_2 is given in terms of J_3 and H by means of the following expression for $R \equiv R(K, h)$,

$$e^{4R} = \frac{\cosh 3K + \sinh 3K \tanh h}{\cosh K + \sinh K \tanh h} . \qquad (4b)$$

Since the two-spin interaction parameter J_2 couples only nn spins on the σ sublattice (a triangular lattice), it follows from (2) and (3) that the parti-

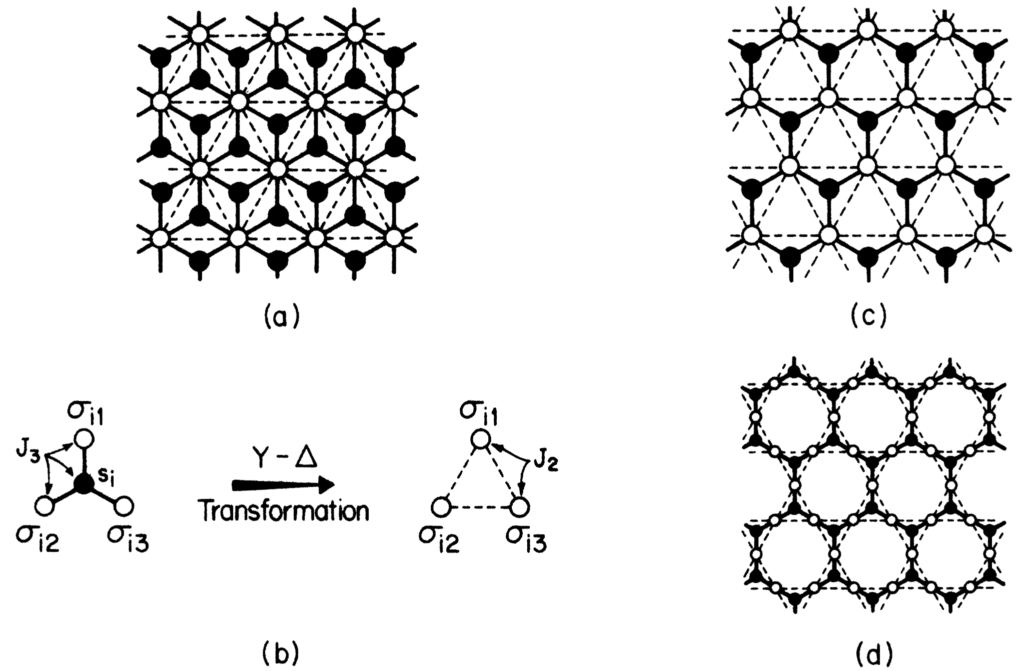

FIG. 1. (a) Diced lattice. (b) The star-triangle ($Y-\Delta$) transformation which changes a three-spin interaction involving spins belonging to both the s (black circles) and σ (open circles) sublattices into a two-spin involving spins belonging only to the σ sublattice. The (c) honeycomb and (d) decorated honeycomb lattices; the partition functions are given by Eqs. (11) and (12) respectively. In the case of the decorated honeycomb lattice, one *can* consider the sublattice to be the open circles and the σ sublattice to be the black circles; by applying the decoration-iteration transformation, the partition function for this altered Hamiltonian is given in terms of the two-spin honeycomb lattice [Eq. (13)].

tion function Z_d of the diced lattice with three-spin interactions J_3 and fields H and H' is related directly to the partition function Z_t of the triangular lattice with two-spin nn interactions J_2 and field H',

$$Z_d^{(3)}[J_3, H, H', N] = [I(J_3, H)]^{2N/3} Z_t^{(2)}[2R(K,h), H', \tfrac{1}{3}N)] . \quad (5)$$

Here the factor $2R$ arises from the fact that each bond of the triangular lattice is summed over twice, while the factors $\frac{2}{3}N$ and $\frac{1}{3}N$ reflect the fact that $\frac{2}{3}$ of the spins are on the s sublattice while $\frac{1}{3}$ of the spins are on the σ sublattice [cf. Fig. 1(a)].

From (5) it follows that the Gibbs potential per spin for the three-spin interaction,

$$g_d^{(3)}[J_3, H, H'] = -\tfrac{2}{3}kT\ln I - \tfrac{1}{3}g_t^{(2)}[2R(K,h), H'] , \quad (6)$$

has a singularity at $H=H'=0$ at a temperature T_c given by

$$K_c = J_3/kT_c = \mathrm{arc\,cosh}[\tfrac{1}{2}(3+\sqrt{3})^{1/2}] . \quad (7)$$

By differentiation of (6) we have obtained expressions for various thermodynamic functions for the *three-spin* Hamiltonian (1) in terms of the thermodynamic functions of the *two-spin* Hamiltonian. In zero field, some of these expressions are known exactly. Thus for $H=H'=0$, the specific heat per spin for the diced lattice is

$$c_d^{(3)}(K) = \frac{1}{3}\left(\frac{K}{R}\frac{\partial R}{\partial K}\right)^2 c_t(2R) - K^2 k \frac{\partial^2 R}{\partial K^2}[(RkT)^{-1} e_t(2R) - 2] + 2K^2 k\,\mathrm{sech}^2 K . \quad (8)$$

The dominant singularity is determined by the (logarithmic) singularity in the two-spin specific

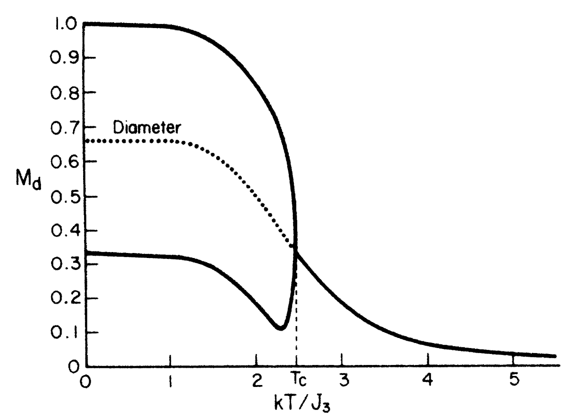

FIG. 2. Sketch of the temperature dependence of the zero-field magnetization of the diced lattice, $m_d^{(3)}(K)$, from Eq. (9).

heat $c_t(2R)$, but the *amplitude is altered and there exists an additional singularity due to the energy* $e_t(2R)$. At high T, $c_d^{(3)}(K) \sim 2/T^2$, which is consistent with the fact that $\frac{2}{3}$ of the spins (the s sublattice) have three nn and $\frac{1}{3}$ of the spins (the σ sublattice) have six nn, so that the "average coordination number" is 4.

The zero-field magnetization per spin for the three-spin interaction is

$$m_d^{(3)}(K) = \frac{1}{3}\left[m_t^{(2)}(2R) - R^{-1}\left(\frac{\partial R}{\partial H}\right)_{H=0} e_t(2R) + 2\left(\frac{\partial R}{\partial H}\right)_{H=0} - 2\tanh K \right], \tag{9}$$

and is plotted in Fig. 2. Notice that the first term is $\frac{1}{3}$ the magnetization of the triangular lattice, as one might expect. The remaining terms contribute a value $\frac{2}{3}$ at $T=0$ (so that the phase boundary extends from $\frac{2}{3}-\frac{1}{3}$ to $\frac{2}{3}+\frac{1}{3}$) and give rise to a singularity in the diameter which is linear in the energy of the two-spin system and hence has critical exponent $(1-\alpha)$; they also contribute *above* the critical temperature, and the magnetization remains nonzero for all finite T, with $m_d^{(3)} \to 0$ *only* when $T\to\infty$ [since spin-reversal symmetry is lacking in (1) and hence $g_d(-H) \neq g_d(H)$]. In fact, the spin-reversal symmetry is only "partially destroyed," since (1) is still invariant if under reversal of all the spins on the σ sublattice and hence for $T > T_c$, $\langle\sigma\rangle = 0$ and the zero-field magnetization arises completely from the s spins. This point is made quite clearly if one notes that even for the single unit [Fig. 1(b)],

$$\langle s\rangle = \sum_{s_i,\sigma_{1i}\sigma_{2i}\sigma_{3i}} s_i e^{-\beta\mathcal{H}_i} \Big/ \sum_{s_i\sigma_{1i}\sigma_{2i}\sigma_{3i}} e^{-\beta\mathcal{H}_i}, \tag{10}$$

is finite for all temperature.

The susceptibility may also be related directly to thermodynamic functions of the two-spin interaction model. Again the leading singularity is determined by the susceptibility for the two-spin model, but for this function the "background" terms are also singular at the same temperature, so that in addition to the γ $(=\frac{7}{4})$ divergence of the two-spin interaction susceptibility, we find terms that vary like the specific-heat and the temperature derivative of the magnetization and hence contribute α (logarithmic) and $1-\beta$ $(=\frac{7}{8})$ divergences.

The above results may be generalized in several directions. Firstly, they are not limited to the "$S=\frac{1}{2}$ Ising model," but can be carried through without effort for any value of the spin quantum number.[7] Secondly, they apply to a large number of lattices related by the Y-Δ transformation such that the interaction after the transformation connects only nearest neighbors. Thus, for example, the honeycomb (h) lattice becomes the triangular (t) lattice, and

$$Z_h^{(3)}(J_3, H, H', N) = I^{N/2} Z_t^{(2)}[R(K,h), H', \tfrac{1}{2}N]\ . \tag{11}$$

Similarly, the decorated honeycomb (dh) lattice becomes the Kagome (K) under the Y-Δ transformation, and

$$Z_{dh}^{(3)}(J_3, H, H', N) = I^{N/2} Z_K^{(2)}[R(K,h), H', \tfrac{1}{2}N]\ . \tag{12}$$

Note that the above statements about the diced lattice carry through the honeycomb and the decorated honeycomb lattices, and the analysis may be carried out by inspection.

Finally, we note that we may also obtain analogous results for situations in which the *decoration-iteration* transformation is applied.[8] For example, consider the dh lattice again, but *interchange* the s and the σ sublattices. Then

$$Z_{dh}^{(3)}(K, H, H', N) = I^{N/2} Z_h^{(2)}[R(K,h), H', \tfrac{1}{2}N]\ , \tag{13}$$

where now

$$I = I(J_3, H) = \cosh R(K,h)\ , \tag{14a}$$

and

$$\tanh R(K,h) = \tanh K \tanh h\ . \tag{14b}$$

Note that in the limit of $H\to 0$, there is no coupling among the spins on the σ sublattice.

Thus, in summary, we have seen that some of the effects of breaking of spin-reversal symmetry can be studied by studying the behavior of three-spin interaction Ising models whose field-dependent partition function can be related to the partition function of a two-spin interaction model on a different lattice. (Although the actual critical-point exponents remain the same as for systems with two-spin interaction,[9] the three-spin interaction gives rise to different amplitude and background terms.) The latter clearly give rise to the angular behavior in the diameter and to a zero-field magnetization function that does not vanish except when $T\to\infty$.

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