To What Class of Fractals Does the Alexander-Orbach Conjecture Apply?

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Alexander and Orbach have recently made the remarkable numerical discovery that for the incipient infinite cluster in percolation the ratio of d_f (the fractal dimension of the aggregate) to d_w (the fractal dimension of a random walk on the aggregate) is approximately "superuniversal"—independent of d for $d > 1$. Does this discovery also hold for aggregates other than percolation? A plausibility argument (rigorous for the Cayley tree) is presented that it should hold, exactly, for "homogeneous" fractals, but need not for nonhomogeneous fractals such as the percolation backbone, the Sierpinski gasket, and the Havlin carpet.

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Highly ramified fractal aggregates have recently attracted considerable interest, in part because of their potential to describe a wide range of nonregular structures ranging from colloids and polymer gels to galactic structures. These may be quantitatively characterized by the fractal dimension d_f which relates the dependence of the diameter ξ_f on the mass N_f $[N_f \sim \xi_f^{d_f}]$. To describe the dynamics of, e.g., electrical transport, interest has also focused on the behavior of a random walk on a fractal aggregate (the de Gennes "ant").[1-3] The fractal dimension d_w of the walk relates the rms displacement ξ_w to the number of steps N_w $[N_w \sim \xi_w^{d_w}]$.

At the percolation threshold p_c both d_f and d_w depend strongly on d, the spatial dimension.[2-6] Therefore considerable interest arose when Alexander and Orbach (AO)[5] recently made the remarkable numerical discovery that the ratio d_f/d_w takes on a value that is roughly independent of d; for this reason they made the rather bold conjecture that $d_f/d_w = \frac{2}{3}$ (the Cayley-tree value) for all d. The AO conjecture has been confirmed by very precise MD calculations for $d = 2$ and 3.[3,4] It has the startling implication that *dynamic* critical exponents are related to *static* exponents; e.g., the electrical conductivity exponent t is given by $t/\nu = d - 2 + \frac{1}{2}d_f$.[2-6]

Is the AO conjecture exact, or only approximate? Equally important, how "general" is the AO discovery—to what class of fractals does it apply? Very recent calculations on Witten-Sander aggregates[7,8] and lattice animals[9] for $d = 2$ and 3 suggest that $d_f/d_w = \frac{2}{3}$ for these fractals. On the other hand, for other fractals d_f/d_w may depend strongly on the nature of the fractal as well as on the space dimension d. Examples include deterministic fractals such as the Sierpinski gasket and the Havlin carpet (Fig. 1 of Ref. 4), as well as nondeterministic fractals such as the percolation backbone.[10] Clearly it is important to understand the general features of an aggregate that determine whether or not the AO conjecture should hold. We present a compelling plausibility argument to support the numerical result $d_f/d_w = \frac{2}{3}$ (exactly) for percolation clusters in any dimension. Further, we conjecture that $d_f/d_w = \frac{2}{3}$, either exactly or quite accurately, for arbitrary homogeneous fractals of dimension greater than one. In general, we may expect other, nonuniversal values for inhomogeneous fractals.[4] This argument, which builds on the basic picture of Rammal and Toulouse,[6] is rigorous for the Cayley tree—thus providing a simple and exact picture for that important case.

By *homogeneous* we mean that there are no bottlenecks that hinder the diffusion of the de Gennes ant. More precisely, let C be a set that totally surrounds a fractal of characteristic linear dimension ξ. For a homogeneous fractal, the number of elements of such a surrounding set should always scale as a normal boundary on a fractal —i.e., as ξ^{d_f-1}. In contrast, *nonhomogeneous* fractals such as the Sierpinski gasket and Havlin carpet (Fig. 1) have boundary sets with exceptionally dense boundaries (hence the term nonhomogeneous).

Our argument is that $S \sim N^{d_f/d_w} \sim N^{2/3}$, where S is the number of sites visited by the ant in $N = N_w$ steps. It is most convincingly presented for the case of percolation on a Cayley tree with z nearest neighbors (nn). We focus on that portion of the incipient infinite cluster visited by the ant (Fig. 2), and we shall create this set *while* the ant is walking. Imagine that the ant is put free at the origin at time $t = 1$; we call the origin an S site and the z nn G sites ("growth" sites into which the ant could move). At $t = 1$, the ant randomly chooses one of these G sites and assigns a random number (RN). If RN > p_c $[= 1/(z-1)]$,

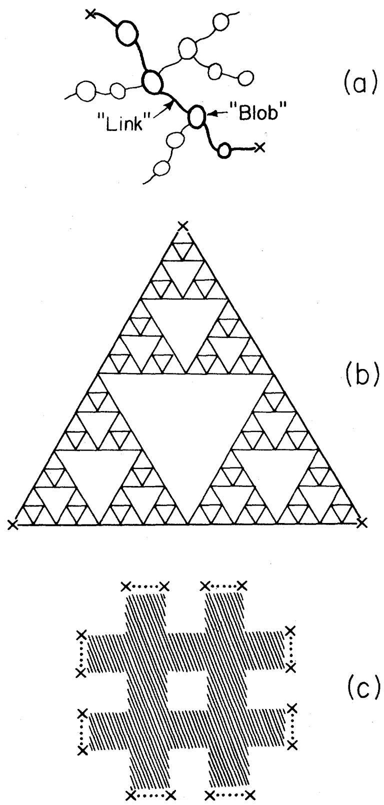

FIG. 1. Examples of three nonhomogeneous fractals: (a) percolation backbone, (b) Sierpinski gasket, and (c) Havlin carpet (Ref. 4). The crosses indicate sets of "blocking points" that could serve to confine a random walker. The Havlin carpet is constituted by replacing every elementary square in the figure shown by the figure itself, suitably scaled (lengths by 5, mass by 16), so that $d_f = \ln 16/\ln 5 \cong 1.72$. Note that the set of "blocking sites" for the Havlin carpet forms a Cantor set with $d_f = \ln 2/\ln 5 \cong 0.42$, which is less than $1.72 - 1$.

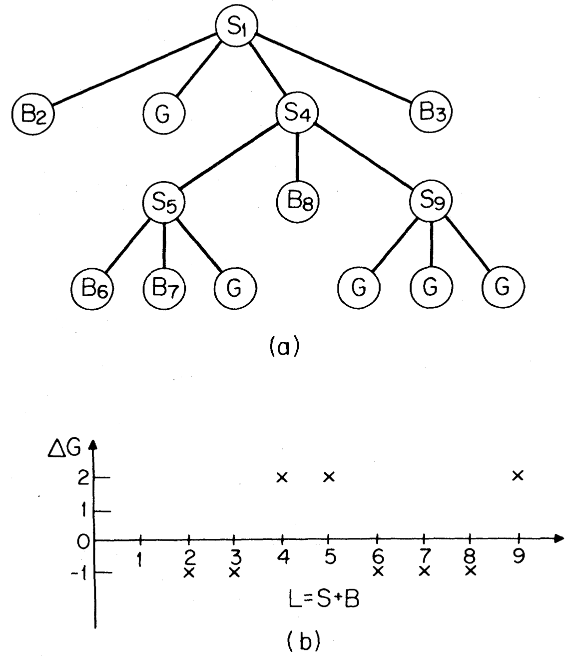

FIG. 2. (a) Schematic illustration of a few steps taken by the de Gennes ant on a Cayley tree with $z = 4$. The letters S, B, and G denote sites visited by the ant, blocked sites, and growth sites, respectively. The subscripts denote successful values of $L \equiv S + B$ $[L = 2, 3, \ldots, 9]$. (b) Values of $\Delta G(L)$ corresponding to the trajectory of (a). Note that $+2$ occurs roughly twice as often as -1, and that ΔG is an independent random variable with zero mean.

then the site chosen is renamed a B ("blocked") site and the ant does not move, while if RN $< p_c$ it is renamed an S site and the ant moves there.[11] At $t = 2$, the ant again randomly selects one of the not blocked nn of the site it is on, and the process is continued [Fig. 2(a)].

At each time step, we can calculate the quantity G, the number of growth sites. Now G can only change when either S or B grows by one; i.e., if we introduce the variable $L = S + B$, then

$$G(L) = \sum_{L'=2}^{L} \Delta G(L'), \tag{1}$$

where $\Delta G(L') = G(L') - G(L' - 1)$ and $G(1) = z$. It is clear that the ΔG are independent random variables [Fig. 2(b)], so that for large L, $G(L)$ is normally distributed with a mean of $L\,\mathrm{mean}(\Delta G)$ and a variance of $L^{1/2}\mathrm{var}(\Delta G)$. If follows that

$$G(L) = L\,\mathrm{mean}(\Delta G) + L^{1/2}\,\mathrm{var}(\Delta G). \tag{2}$$

Now $\mathrm{mean}(\Delta G) = 0$ for the particular choice p_c of the occupying probability. However since $L \sim S[1 + (1 - p_c)/p_c]$, the rate of growth of S is clearly given by the ratio of G sites to S sites,

$$\Delta S/\Delta N \sim G/S \sim S^{-1/2}, \tag{3}$$

from which we obtain $S \sim N^{d_f/d_w}$ with[12]

$$d_f/d_w = \tfrac{2}{3}. \tag{4}$$

This reasoning cannot be carried over in its present form to d-dimensional percolation, since $\Delta G(L)$ can take a *range of different values* if the site is occupied, while *it is always -1* if the site is blocked. While the "decisions" whether to block or occupy are still independent, the ΔG values for each site depend on the previous history of the walk.

The following approach, while certainly not rigorous, does support the validity of the AO conjecture for d-dimensional percolation. First partition all of space into cells of edge M where $1 \ll M \ll \xi_w$. From the time the ant enters a cell until he leaves the cell, we treat all consecutive changes in G as if they were a single event. This coarse graining has no effect on the asymptotic behavior of G, S, or N. Correlations within the cell have a finite range (since M is finite), and therefore are of no consequence for the asymptotic behavior of the random walks. Correlations within the cell have a finite range (since M is finite), and therefore are of no consequence for the asymptotic behavior of the random walks. Correlations between different cells will decrease as the ratio of surface to volume approaches zero.

For diffusion on an *arbitrary* fractal, the argument is more hypothetical. Let N and S be defined as before, but redefine G sites as those nonvisited sites *belonging to the aggregate* that are nn of the already visited sites. We do not create the fractal as the ant diffuses, so that we can dispose of the need for B sites. Assume that (i) the fractal is infinite, but diffusion on it is anomalous in the sense that $S/N \to 0$ as $N \to \infty$ at least as fast as some power of N; (ii) the fractal is not topologically one dimensional (i.e., the perimeter of a cluster grows with its size); and (iii) the fractal is homogeneous in the sense described above.

Since $S \ll N$ for large clusters, it is clear that the walk will only rarely hit a G site, thereby increasing S. Such events are well separated in time by assumption (i), and hence are presumably uncorrelated. Hence it is plausible that

$$\Delta S/\Delta N \sim G/S, \qquad (5)$$

where G/S is the probability of hitting a G site; by (iii) all G sites should be equally probable for sufficiently long times. Since G only changes when S does, we have

$$G(S) = \sum_{S'=1}^{S} \Delta G(S'). \qquad (6)$$

The desired result, (4), follows if it can be determined that $G \sim S^{1/2}$. This would be true if the $\Delta G(S')$ were without long-range correlations. In general, this is not the case, since there are explicit counterexamples. However, under assumptions (i)–(iii) it may be a reasonable approximation to make. It is to be noted, in any case, that all known counterexamples to the AO conjecture are strongly inhomogeneous. We cannot tell whether the homogeneity definition we have given is indeed the appropriate one, nor why it should be so crucial in the argument apart from the following: On an inhomogeneous fractal the random walker will alternatively be "bogged down" in a bottleneck or, on the contrary, just have passed one. This will occur on all scales. This may lead to strong correlations in the ΔG's and hence to a breakdown of the argument. From the above follows Eq. (2) with L replaced by S. We can argue again that mean$(\Delta G) = 0$, by assuming the contrary: Were mean$(\Delta G) > 0$, then $G \sim S$, implying that $\Delta S/\Delta N = O(1)$ contradicting the hypothesis of anomalous diffusion. Were mean $(\Delta G) < 0$, then the walk would have to terminate, contradicting the hypothesis of an infinite fractal. Hence (2) implies $G \sim S^{1/2}$ and the argument proceeds exactly as in the case of the Cayley tree.

In summary, then, we have taken seriously the AO numerical discovery that $d_f/d_w \cong \frac{2}{3}$ for percolation fractals. We have presented an argument that justifies this result for the Cayley tree and somewhat improves upon the Rammal-Toulouse argument for general-d percolation clusters. We define a class of homogeneous fractals, and argue that similar reasoning should apply. Nonhomogeneous fractals are seen to be quite different, and we predict that the AO conjecture does not generally hold for these. Our prediction is borne out by calculations on the percolation backbone, Sierpinski gasket, and Havlin carpet.

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[11]Thus a fraction p_c of the time steps result in the ant moving while a fraction $1-p_c$ result in no movement. In many simulations the ant moves at each time step; to make contact with this work, we only need rescale time by a factor of p_c.

[12]The quantity $2d_f/d_w = d_s$ is termed the spectral or "fracton" dimension. No rationale for notation has emerged in this field. For example, d_f is denoted $\bar{d}$ in Refs. 4–6, D in Refs. 2 and 7; d_w is denoted D in Ref. 4, $2+\delta$ in Ref. 5, $1/\nu$ in Ref. 6; finally, d_s is denoted $\bar{\bar{d}}$ in Refs. 4 and 5 and $\tilde{d}$ in Ref. 6. By placing a subscript on all three quantities d_f, d_w, and d_s we let the reader know immediately which fractal dimension is being denoted.