Number of distinct sites visited by N random walkers

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We study the number of distinct sites visited by N random walkers after t steps $S_N(t)$ under the condition that all the walkers are initially at the origin. We derive asymptotic expressions for the mean number of distinct sites $\langle S_N(t)\rangle$ in one, two, and three dimensions. We find that $\langle S_N(t)\rangle$ passes through several growth regimes; at short times $\langle S_N(t)\rangle \sim t^d$ (regime I), for $t_\times \ll t \ll t'_\times$ we find that $\langle S_N(t)\rangle \sim (t \ln[N\, S_1(t)/t^{d/2}])^{d/2}$ (regime II), and for $t \gg t'_\times$, $\langle S_N(t)\rangle \sim N S_1(t)$ (regime III). The crossover times are $t_\times \sim \ln N$ for all dimensions, and $t'_\times \sim \infty$, $\exp N$, and N^2 for one, two, and three dimensions, respectively. We show that in regimes II and III $\langle S_N(t)\rangle$ satisfies a scaling relation of the form $\langle S_N(t)\rangle \sim t^{d/2} f(x)$, with $x \equiv N\langle S_1(t)\rangle/t^{d/2}$. We also obtain asymptotic results for the complete probability distribution of $S_N(t)$ for the one-dimensional case in the limit of large N and t.

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I. INTRODUCTION

One of the most important properties of a discrete-time lattice random walk is the number of distinct sites visited by a t-step walk [1–19]. This importance stems from the large number of models that are directly related to the first-passage events of the random walker. These events enter into the description of phenomena ranging from relaxation processes [3–6] and diffusion-limited reactions [3, 20, 21] such as defect annealing [2, 7] and exciton trapping [8, 9], to the spread of populations in ecology [22–25].

The analyses in the literature to date refer to the calculation of properties of $S_1(t)$, the distinct number of sites visited by a *single* random walker. For this case, the asymptotic forms for the mean number of sites visited $\langle S_1(t)\rangle$ in any number of dimensions d are well known. Knowledge of $\langle S_1(t)\rangle$ enables one to find a lowest-order approximation to the survival probability for the "trapping problem," in which a random walker moves in the presence of randomly distributed static traps [3]. This problem can be regarded as the simplest generalization of the Smoluchowski model for the rate of chemical reactions of the form $A + B \to B$, taking into account the possibility of a concentration of B's rather than the single B envisaged by Smoluchowski [3, 20, 21]. $\langle S_1(t)\rangle$ also appears in the solution of the "target" problem [3], in which a single target is static within a concentration of diffusing traps. The survival probability in this case is given exactly by $\exp[-k\langle S_1(t)\rangle]$ [3] (where k depends on the lattice and the concentration of traps), if the traps are initially Poisson distributed.

The properties of the number of distinct sites visited in the more general situation where there are N random walkers in the system are not related simply to the single-walker case. Here we calculate the asymptotic properties of $S_N(t)$, the number of distinct sites visited by $N \gg 1$ independent random walkers.

The quantity $S_N(t)$ has direct application to any situation that might be conceptualized as a "multiple-scavenger problem," in which mobile traps (or scavengers) react with stationary particles initially distributed homogeneously on the lattice, and a particle disappears when any one of the traps reaches it. It can also be useful for the analysis of the simultaneous survival of several particles diffusing in the presence of randomly distributed static traps, which is the generalization of the trapping problem described above (Fig. 1). The probability for the simultaneous survival of M of the initial N particles will be given by a combinatorial factor times $\langle \exp[-k\ S_M(t)] \rangle$, where k again depends on the concentration of traps and on the lattice. For short times or low concentration of traps, this survival probability can be approximated by $\exp[-k\ \langle S_M(t) \rangle]$.

We consider the case for which all N random walkers are assumed to be initially at the origin; generalizations are discussed in Sec. VII. The single-step displacements are characterized by a finite variance, so that the t-step transition probabilities (in the absence of boundaries) tend toward the Gaussian form predicted by the central-limit theorem. The random walkers move independently of one another with the consequence that multiple occupancy of a single site is allowed.

We find that the mean number of distinct sites $\langle S_N(t) \rangle$ passes through several distinct growth regimes in time. At very short times, we find the simple expression

$$\langle S_N(t) \rangle \sim At^d, \quad t \ll t_\times \quad \text{(regime I)} \tag{1.1}$$

where A depends on the lattice. Equation (1.1) simply states that every accessible site is occupied by a walker.

Regime I holds so long as there are many walkers at every accessible site, i.e., so long as $NP_{\min}(t) \gg 1$, where $P_{\min}$ is the smallest nonzero occupation probability on the lattice at time t. Then $P_{\min}(t) = z^{-t}$, where z is the number of nearest neighbors of a site, so regime I must terminate at a crossover time $t_\times$ which scales logarithmically with N,

$$t_\times \sim \ln N. \tag{1.2}$$

To discuss times greater than $t_\times$, we will calculate $\langle S_N(t) \rangle$ using generating function techniques. This analysis leads to a compact scaling expression for $\langle S_N(t) \rangle$,

$$\langle S_N(t) \rangle \sim t^{d/2} f(x), \quad t \gg t_\times \tag{1.3}$$

where the tilde denotes the fact that (1.3) holds for N and t both large. The scaled variable x is given by

$$x \equiv \begin{cases} N & (d=1) \\ N/\ln t & (d=2) \\ N/\sqrt{t} & (d=3), \end{cases} \tag{1.4}$$

and the scaling function $f(x)$ by

$$f(x) = \begin{cases} (\ln x)^{d/2}, & t_\times \ll t \ll t'_\times \quad \text{(regime II)} \\ x, & t \gg t'_\times \quad \text{(regime III)}. \end{cases} \tag{1.5}$$

Here the second crossover time $t'_\times$ is

$$t'_\times \sim \begin{cases} \infty & (d=1) \\ e^N & (d=2) \\ N^2 & (d=3). \end{cases} \tag{1.6}$$

The appearance of regime III (for $d \geq 2$) can be understood from the following heuristic argument. In regime II, all but an exponentially small fraction of the walkers are contained within a d-dimensional sphere of radius $\xi \sim t^{1/2}$. Hence $\langle S_N(t) \rangle$ must be bounded from above by the volume of this sphere $V(t) \sim t^{d/2}$. A second upper bound on $\langle S_N(t) \rangle$ is $N\langle S_1(t) \rangle$, where

$$\langle S_1(t) \rangle \sim \begin{cases} t^{1/2} & (d=1) \\ t/\ln t & (d=2) \\ t & (d=3) \end{cases} \tag{1.7}$$

is the number of distinct sites visited by one random walker. A crossover in $\langle S_N(t) \rangle$ will occur if the system passes from one constraint to the other. For $d = 1$, $V(t) < N\langle S_1(t) \rangle$ for all t, so no crossover occurs—regime II holds for arbitrarily large t, confirming the result (1.6a) above. For $d = 2$ and 3, we find $V(t) < N\langle S_1(t) \rangle$ initially, but for sufficiently large t, $V(t) > N\langle S_1(t) \rangle$. Thus $t'_\times$ is obtained from the condition

$$V(t'_\times) \sim N\langle S_1(t'_\times) \rangle. \tag{1.8}$$

For $d = 2$, (1.7) and (1.8) lead to $t'_\times \sim Nt'_\times/\ln t'_\times$, so that $t'_\times \sim e^N$; this confirms the result (1.6b) above. Similarly, for $d = 3$, $(t'_\times)^{3/2} \sim Nt'_\times$ implies $t'_\times \sim N^2$, confirming the result (1.6c). One can interpret $t'_\times$ as the time up to which the walkers visit the same places very frequently. For times longer than $t'_\times$, the walkers "almost" do not see each other, and can be treated independently. Thus one would expect the form $S_N(t) \sim NS_1(t)$ under these conditions.

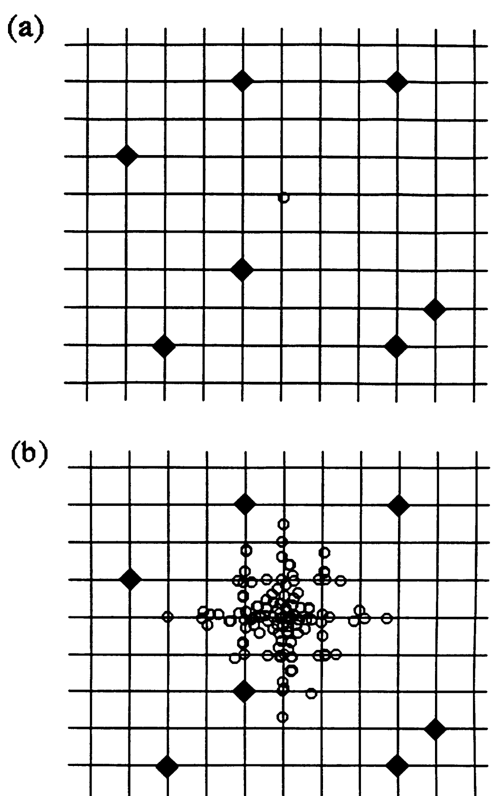

FIG. 1. Schematic illustration of the trapping problem in which the number of distinct sites enters explicitly into the expression for the survival probability P_{surv}. (a) A single walker, starting in the center of an infinite square lattice, can be trapped by fixed traps distributed randomly in the plane. (b) N walkers starting at the origin in the presence of the same set of randomly distributed traps. In both cases, $P_{\text{surv}} \sim \exp(-\langle S_N \rangle)$.

Following the same kind of reasoning, we can generalize the above argument to any spatial dimension d. The crossover time to the final regime will be given by

$$t'_\times \sim N^{2/(d-2)} \quad (d>2). \tag{1.9}$$

This result is a consequence of the fact that $\langle S_1(t)\rangle \sim t$ for any dimension larger than 2. Equation (1.9) shows the effect of the space dimension on $t'_\times$; it shows that when the dimension increases, the walkers become "independent" at shorter times $t'_\times$.

The above results will be derived in detail in this paper. The remarkable feature is the appearance of regime II. The behavior in regime I corresponds to the limit $[N \to \infty, t \text{ fixed}]$; the interface of the set of visited sites is smooth and $S_N(t)$ is easy to understand ($S_N \sim t^d$). The behavior in regime III corresponds to the opposite limit $[t \to \infty, N \text{ fixed}]$; the interface of the set of visited sites is extremely rough and the form of S_N is also easy to understand ($S_N \sim NS_1$). In regime II, the function S_N takes on an unexpected and nontrivial form. The walkers are largely confined to a sphere of radius $\sqrt{t}$ (in contrast to regime I, where they populate a sphere of radius t); the interface of the set of visited sites undergoes a progressive roughening which is readily apparent on visual inspection of the set of visited sites (Fig. 2).

We also carried out numerical calculations for $\langle S_N(t)\rangle$ using both the methods of Monte Carlo and exact enumeration. In particular, we confirmed the scaling form (1.3) [26].

The organization of this paper is as follows. In Sec. II we present the general formalism that we use to calculate the expressions for the mean number of sites visited by N random walkers and discuss regime I which is present for all dimensions. Sections III, IV, and V deal with the explicit calculations for the one-, two-, and three-dimensional cases, respectively. In Sec. VI we calculate the probability distribution for the number of distinct sites visited by N walkers in a one-dimensional system. The discussion and conclusions are in Sec. VII. We note that in this work, asymptotic results can mean one of two possibilities: (a) the step number tending to infinity with the number of random walkers held fixed, or (b) the number of random walkers tending to infinity while the step number is held fixed.

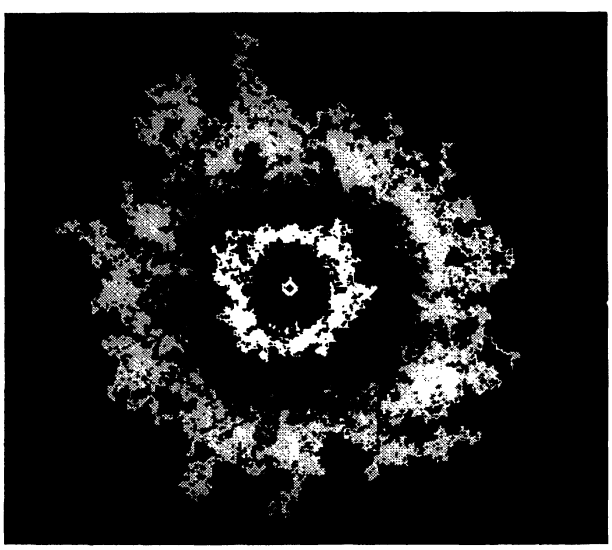

FIG. 2. Contours of the surface obtained from snapshots at successive times of the territory covered by N random walkers for the case $N = 500$ for a sequence of times in regime II. Note the roughening of the disc surface as time increases.

II. GENERAL FORMALISM AND REGIME I, ALL DIMENSIONS d

First we introduce some notation. The probability that a site $\mathbf{r}$ will be first visited at step t by a single random walker initially at the origin will be denoted by $f_t(\mathbf{r})$. We denote by $\Gamma_t(\mathbf{r})$ the probability that the site $\mathbf{r}$ has not been visited by a single random walker by step t. This function is related to the $f_{t'}(\mathbf{r})$ by

$$\Gamma_t(\mathbf{r}) = 1 - \sum_{t'=0}^{t} f_{t'}(\mathbf{r}). \tag{2.1}$$

The probability that $\mathbf{r}$ has been visited by at least one of the N random walkers in the course of t steps is $1-\Gamma_t^N(\mathbf{r})$. Thus, the expected number of distinct sites visited by the N random walkers by the tth step is $\langle S_N(t)\rangle$, where

$$\langle S_N(t)\rangle = \sum_{\mathbf{r}} [1 - \Gamma_t^N(\mathbf{r})], \tag{2.2}$$

where the sum is over all sites of the lattice.

The analysis for regimes II and III requires the introduction of a generating function $S(u;t)$ defined with respect to the number of random walkers N as

$$S(u;t) \equiv \sum_{N=0}^{\infty} \langle S_N(t)\rangle u^N. \tag{2.3}$$

Since N occurs as a power in Eq. (2.2) we can immediately find an expression for $S(u;t)$ in the form

$$\begin{aligned} S(u;t) &= \sum_{\mathbf{r}} \left(\frac{1}{1-u} - \frac{1}{1-u\Gamma_t(\mathbf{r})} \right) \\ &= \frac{u}{1-u} \sum_{\mathbf{r}} \frac{1-\Gamma_t(\mathbf{r})}{1-u\Gamma_t(\mathbf{r})}. \end{aligned} \tag{2.4}$$

From Eq. (2.2), we are in a position to derive a general result for the limit defined by t fixed and $N \to \infty$, in the case that the transition probabilities of the random walk have compact support, e.g., when the walker is allowed to make steps to nearest neighbors only. After t steps, the set of all accessible sites for a single random walker Ω_t consists of a finite number of elements $N(\Omega_t)$. Furthermore, $\Gamma_t(\mathbf{r}) < 1$ whenever $\mathbf{r} \in \Omega$, or $\lim_{N\to\infty} \Gamma_t^N(\mathbf{r}) = 0$. It therefore follows that $\lim_{N\to\infty} \langle S_N(t)\rangle = N(\Omega_t)$. That is to say, when there are N independent random walkers they will tend to visit all possible visitable sites when the limit $N \to \infty$ is taken. For example, in the case of a nearest-neighbor random walk in d dimensions, the maximum number of sites that can be visited in t steps is $\sim t^d$ which leads to the result [27]

$$\lim_{N\to\infty} \langle S_N(t)\rangle \sim t^d \quad \text{(regime I)}. \tag{2.5}$$

For finite but large N, this initial growth regime, regime I, holds as long as $N \gg z^t$, where z is the coordination number of the lattice [26].

III. ONE DIMENSION, REGIME II

Next we are interested in finding, for $d = 1$, the form of $\langle S_N(t)\rangle$ in the limit of a large number of steps (regime II). We use the expression for $f_t(x)$, where the position x is now a continuous variable, based on the use of a continuum approximation for the probability distribution $p_t(x)$, the probability that a walker initially at the origin will be at position x at time t. The result is [28]

$$f_t(x) \sim \frac{|x|}{2\sigma\sqrt{\pi t^3}} \exp\left(-\frac{x^2}{4\sigma^2 t}\right) \tag{3.1}$$

for $t \gg 1$.

To be consistent in the use of a continuum approximation, we replace the summation in Eq. (2.4) by an integral. For this purpose we need the approximate form for $\Gamma_t(x)$ which is derived from Eq. (3.1). This is

$$\Gamma_t(x) \sim \int_t^\infty f_{t'}(x)dt' = \text{erf}\left(\frac{|x|}{2\sigma\sqrt{t}}\right). \tag{3.2}$$

On substituting Eq. (3.2) into (2.4), we find that the generating function $S(u;t)$ can be approximated by

$$\begin{aligned} S(u;t) &\sim \frac{2\sigma\sqrt{t}\,u}{1-u}\int_0^\infty \frac{1-\text{erf}(v)}{1-u\,\text{erf}(v)}dv \\ &= \frac{2\sigma\sqrt{t}u}{1-u} I(u). \end{aligned} \tag{3.3}$$

Equation (3.3) defines the integral $I(u)$. Using standard Tauberian methods for power series [29] we relate the behavior of $S(u;t)$ in the limit $u = 1$ to the large-N limit of $\langle S_N(t)\rangle$. If we simply set $u = 1$ in the integrand of $I(u)$, we see that the integrand is identically equal to 1, with the result that the value of the integral must be infinite due to the behavior of the integrand at large v. Hence, in determining the singular behavior of $I(u)$ in the limit $u \to 1$, we can replace erf(v) by its value at large v,

$$1 - \text{erf}(v) \sim \frac{e^{-v^2}}{v\sqrt{\pi}}. \tag{3.4}$$

It then follows that

$$I(u) \sim \int_0^\infty \frac{1}{1+\sqrt{\pi}ve^{v^2}(1-u)}dv. \tag{3.5}$$

On defining the small parameter

$$\epsilon \equiv \sqrt{\pi}(1-u) \tag{3.6a}$$

as well as a function

$$\phi(v) \equiv ve^{v^2} \tag{3.6b}$$

and introducing an integral representation of the integrand of Eq. (3.5) we can rewrite (3.5) as a Laplace transform

$$\begin{aligned} I(u) &\sim \frac{1}{\epsilon}\int_0^\infty \frac{dv}{(1/\epsilon)+\phi(v)} \\ &= \frac{1}{\epsilon}\int_0^\infty \exp\left(-\frac{\xi}{\epsilon}\right)d\xi \int_0^\infty \exp[-\xi\phi(v)]dv. \end{aligned} \tag{3.7a}$$

Since we are interested in the evaluation of this integral in the limit $\epsilon \to 0$ we use a combination of Abelian and Tauberian theorems to find an approximation to the value of $I(u)$ at the singular point. The crucial contribution to the Laplace transform in ξ space results from the region $\xi \sim 0$; hence it is in this limit that we must evaluate the integral with respect to v. However, because of the restriction to small ξ, we need only consider contributions to the integral from the region in which $\phi(v)$ is large, which we can identify as $v \gg 1$. We may estimate this contribution by converting the v integral to a Laplace transform by introducing a new variable of integration, $\rho \equiv \phi(v)$. With this substitution we have

$$\int_0^\infty e^{-\xi\phi(v)}dv = \int_0^\infty e^{-\xi\rho}f(\rho)d\rho, \tag{3.7b}$$

where $f(\rho) = 1/\phi'(v)$, it being understood that the value of v to be substituted into this expression is to be written in terms of ρ.

The right-hand side of Eq. (3.7a) is a Laplace transform, and in order to approximate its value for small ξ we must estimate the behavior of $f(\rho)$ for large ρ. An approximate solution to the transcendental equation $\phi(v) = \rho$, in the region $\rho \gg 1$, or $v \gg 1$, is obtained by taking logarithms of both sides of the equation, and retaining only the lowest order term. This procedure yields

$$v \sim \sqrt{\ln(\rho)}, \tag{3.8}$$

or

$$f(\rho) \sim \frac{1}{2\rho\sqrt{\ln(\rho)}}, \quad \rho \to \infty. \tag{3.9}$$

Our approximation to $f(\rho)$ is only valid in the limit of large ρ. Hence we may change the lower limit on the ρ integral in Eq. (3.7b) to some value that is $O(1)$ in order to get around any difficulties due to the logarithm going negative (we will see later that the precise value is irrelevant), thereby limiting the validity of our results to the region of small ξ. We find that the transformation in Eq. (3.7b) can be approximated by

$$\begin{aligned} \int_0^\infty e^{-\xi\rho}f(\rho)d\rho &\sim \tfrac{1}{2}\int_1^\infty \frac{e^{-\xi\rho}}{\rho\sqrt{\ln(\rho)}}d\rho \\ &= \tfrac{1}{2}\int_\xi^\infty \frac{e^{-\lambda}}{\lambda\sqrt{\ln(\lambda/\xi)}}d\lambda \\ &\sim \frac{1}{2\sqrt{\ln(1/\xi)}}\int_\xi^\infty \frac{e^{-\lambda}}{\lambda}d\lambda \sim \tfrac{1}{2}\sqrt{\ln(1/\xi)}. \end{aligned} \tag{3.10}$$

As mentioned, the lower limit in the integral on the right-hand side of this equation is not important since its effect shows up only as a logarithmic term, which does not contribute to the lowest-order term in the limit $\xi \to 0$. Finally, we may evaluate the integral with respect to ξ in Eq. (3.7a),

$$I(u) \sim \frac{1}{2\epsilon}\int_0^\infty \exp\left(-\frac{\xi}{\epsilon}\right)\sqrt{\ln(1/\xi)}d\xi \sim \tfrac{1}{2}\sqrt{\ln(1/\epsilon)}, \quad (3.11)$$

which, when combined with Eq. (3.3), implies that as $u \to 1$ the singular behavior of $S(u;t)$ is

$$S(u;t) \sim \frac{\sigma\sqrt{t}}{1-u}\sqrt{\ln\left(\frac{1}{1-u}\right)}. \quad (3.12)$$

It therefore follows, by a Tauberian theorem for power series, that for long times the mean number of distinct sites visited by N random walkers in one dimension is

$$\langle S_N(t)\rangle \sim \sigma\sqrt{t\ln N} \quad \text{(regime II)}. \quad (3.13)$$

Thus the $\sqrt{t}$ dependence on the number of steps remains unchanged from the single-walker result, and the entire effect of having multiple random walkers appears in the factor $\sqrt{\ln(N)}$. The logarithmic dependence on N can be interpreted as a result of a "screening effect" accounting for the overlap of the contributions of different walkers, albeit not as pronounced as in regime I where $\langle S_N(t)\rangle$ does not depend explicitly on N.

We emphasize that the expression in Eq. (3.13) is only true for $N \gg 1$ and $t \to \infty$. If t is fixed and N is increased so $t \ll \ln N$, we have shown at the end of Sec. I that $\langle S_N(t)\rangle$ is proportional to t rather than $\sqrt{t}$. The step number $t_\times$ at which the crossover between the two growth regimes takes place is

$$t_\times \sim \ln(N). \quad (3.14)$$

IV. TWO DIMENSIONS, REGIMES II AND III

To derive an approximation to $\langle S_N(t)\rangle$ for long times in two dimensions it is necessary to use the asymptotic form of $\Gamma_t(\mathbf{r})$ in the limit of large r^2 and large t,

$$\Gamma_t(\mathbf{r}) \sim 1 - \frac{2\sigma^2 t}{r^2\ln(t)}\exp\left(-\frac{r^2}{4\sigma^2 t}\right), \quad (4.1)$$

in order to evaluate the sum in Eq. (2.4), in the limit $r \to \infty$ (see Appendix A).

This approximate form for $\Gamma_t(\mathbf{r})$ allows us to derive a corresponding approximation for the function $S(u;t)$, defined in Eq. (2.4). Then, after a replacement of the sum over $\mathbf{r}$ by an integral with respect to r, we find the integral representation, valid in the limit $u \to 1$,

$$S(u;t) \sim \frac{2\pi}{(1-u)}\int_0^\infty \frac{r}{[r^2\ln(t)/2\sigma^2 t](1-u)\exp(r^2/4\sigma^2 t)+1}dr = \frac{4\pi\sigma^2 t}{1-u}\int_0^\infty \frac{1}{\epsilon v e^v + 1}dv, \quad (4.2)$$

in which the parameter ϵ is now defined by $\epsilon \equiv 2(1-u)\ln(t)$. This form of the integrand is quite different from that required for the analysis in one dimension, since in the present case the parameter ϵ contains both $1-u$ and $\ln(t)$, rather than just $1-u$. This leads to the appearance of an additional regime in the behavior of $\langle S_N(t)\rangle$ regarded as a function of the two variables N and t.

Accordingly, we will consider the behavior of $S(u;t)$ in the two limits $\epsilon \to \infty$ and $\epsilon \to 0$, which corresponds, respectively, to $t \to \infty$ for fixed N, and $N \to \infty$ for fixed but large t. In the first case we can neglect the factor of 1 in comparison to $\epsilon v e^v$. This appears to introduce a singularity into the integral because of the term $1/v$, but the singularity is apparent rather than real, as can be seen from the rigorously correct version of $S(u;t)$ given in Eq. (2.4), and the consideration that $\Gamma_t(0) = 0$. It therefore follows that $S(u;t)$ has the scaling form

$$S(u;t) \sim C\frac{t}{(1-u)^2\ln(t)}, \quad (4.3a)$$

where C is a constant that appears difficult to calculate accurately. The singularity in u in (4.3a) leads to

$$\langle S_N(t)\rangle \sim Nt/\ln(t). \quad (4.3b)$$

Thus, for times such that $t \gg t'_\times$, where $t'_\times \sim e^N$, the expected number of distinct sites visited by N random walkers is proportional to N times the expected number of distinct sites visited by a single random walker [14, 15].

Let us now consider the limit $\epsilon \to 0$, for which the neighborhood of $v = 0$ gives the most significant contribution to the integral in Eq. (4.2). When the term ϵ is factored out of the denominator of the integral in Eq. (4.2) we find

$$S(u;t) \sim \frac{4\pi\sigma^2 t}{(1-u)\epsilon}\int_0^\infty \frac{1}{ve^v + (1/\epsilon)}dv$$

$$= \frac{4\pi\sigma^2 t}{(1-u)\epsilon}\int_0^\infty \exp\left(-\frac{\xi}{\epsilon}\right)d\xi \times \int_0^\infty \exp(-\xi v e^v)dv. \quad (4.4)$$

We see that the integral with respect to ξ can be regarded as a Laplace transform of the function

$$F(\xi) \equiv \int_0^\infty \exp(-\xi v e^v) dv. \quad (4.5)$$

Because of our assumption that u is close to 1 the only values of ξ that give a significant contribution to the transform are those for which $\xi \sim 0$. But because of the form of the function $F(\xi)$, we infer that this contribution is controlled by the behavior of the integrand for very large v. To ascertain the nature of this behavior let us set $\rho = ve^v$. At large v one has the approximation $v \sim \ln \rho$ and $dv = d\rho/\rho$. If we were to substitute these formulas into Eq. (4.5), then we would find an infinity that arises from the behavior of the integrand at $\rho = 0$. However, it is also evident from the original form of the integral that no such infinity exists. To eliminate this artificial infinity we introduce a cutoff in the lower limit of the integral at $\rho = 1$, in which case we have

$$F(\xi) \sim \int_1^\infty \frac{\exp(-\xi\rho)}{\rho} d\rho = \int_\xi^\infty \frac{e^{-\beta}}{\beta} d\beta, \quad (4.6)$$

which, it is to be understood, is valid only in the limit $\xi \to 0$. Thus we see that $F(\xi)$ is an exponential integral, and from the properties of such integrals we find that

$$F(\xi) \sim \ln\left(\frac{1}{\xi}\right). \quad (4.7)$$

Thus

$$\begin{aligned} S(u;t) &\sim \frac{2\pi\sigma^2 t}{(1-u)^2 \ln(t)} \\ &\quad \times \int_0^\infty \exp\left(\frac{-\xi}{(1-u)^2\ln(t)}\right) \ln\left(\frac{1}{\xi}\right) \\ &\sim \frac{4\pi\sigma^2 t}{(1-u)} \ln\left(\frac{1}{(1-u)^2 \ln(t)}\right), \end{aligned} \quad (4.8)$$

which implies that there is a time regime $\ln N \ll t \ll e^N$ for which the number of distinct sites visited by N walkers in two dimensions is

$$\langle S_N(t)\rangle \sim 4\pi\sigma^2 t \ln\left(\frac{N}{\ln(t)}\right) \quad \text{(regime II)}. \quad (4.9)$$

In regime II, $S_N(t)$ increases only logarithmically with the number of walkers, in contrast to regime III discussed earlier for which $S_N(t)$ increased linearly with N. The form of $\langle S_N(t)\rangle$ in Eq. (4.9) suggests the same screening effect that we found in one dimension, i.e., the parameter N, which would appear if the contribution from each random walker were to be counted separately, is essentially replaced by $\ln(N)$. The degree of overlap ("screening") is thus much less in two dimensions than in one, as one might guess. Notice that there are now two crossovers to be taken into account. The first is that from the behavior $\langle S_N(t)\rangle \sim t^2$ appropriate at short times, to the form given in Eq. (4.9). The second crossover occurs from the "screened" value of $\langle S_N(t)\rangle$ to the unscreened value, which occurs for $t'_\times \sim \exp(N)$; that is, when the logarithm in Eq. (4.9) becomes $O(1)$. Because of the exponential dependence of $t'_\times$ on N, the effect of screening will generally persist for a very large number of steps.

V. THREE DIMENSIONS, REGIMES II AND III

The calculation of the asymptotic behavior of $\langle S_N(t)\rangle$ in three dimensions is quite similar to that for two dimensions. The appropriate expression for $\Gamma_t(r)$ is (see Appendix B)

$$\Gamma_t(\mathbf{r}) \sim 1 - \frac{1}{2\pi\sigma^2 p(\mathbf{0};1) r} \operatorname{erfc}\left(\frac{r}{\sigma\sqrt{2t}}\right). \quad (5.1)$$

With this formula we can convert the sum in Eq. (2.4) into an integral, taking the spherical symmetry into account. In this way we find

$$S(u;t) \sim \frac{2}{\sigma^2 p(\mathbf{0};1)(1-u)} \int_0^\infty \frac{r^2 \operatorname{erfc}(r/\sigma\sqrt{2t})}{r(1-u) + [1/2\pi\sigma^2 p(\mathbf{0};1)]\operatorname{erfc}(r/\sigma\sqrt{2t})} dr = \frac{4\pi\sigma^3 (2t)^{3/2}}{1-u} \int_0^\infty \frac{v^2 \operatorname{erfc}(v)}{\epsilon\sqrt{t}v + \operatorname{erfc}(v)} dv, \quad (5.2)$$

in which the parameter ϵ is defined by $\epsilon \equiv 2^{3/2}\pi\sigma^2 p(\mathbf{0};1)(1-u)$.

Again, the result of setting $u = 1$ is an integral divergent due to its behavior at the upper limit. Hence we can introduce the asymptotic form of erfc(v) to capture the divergent behavior of the integral as $u \to 1$. Using this type of approximation we find that the integral in Eq. (5.2) can be transformed into one that is very similar to that given in Eq. (4.2), and whose properties can also be found by using essentially the same analysis. The starting point—as for $d = 2$—is the representation

$$\begin{aligned} S(u;t) &\sim \frac{4\pi\sigma^3(2t)^{3/2}}{(1-u)} \int_0^\infty \frac{v^2}{\epsilon\sqrt{t}\psi(v) + 1} dv \\ &= \frac{4\pi\sigma^3(2t)^{3/2}}{(1-u)} I(u,t), \end{aligned} \quad (5.3)$$

in which the function $\psi(v)$ is

$$\psi(v) = \sqrt{\pi} v^2 e^{v^2}. \quad (5.4)$$

As was the case in two dimensions, the coefficient of $\psi(v)$ in the denominator of Eq. (5.3) contains both $1-u$ and a

t-dependent factor, which again leads to the appearance of two regimes in the behavior of $\langle S_N(t)\rangle$.

Since the function $\psi(v)$ in the denominator is proportional to $\epsilon\sqrt{t}$, we expect that results in the two limits $\epsilon\sqrt{t} \ll 1$ and $\epsilon\sqrt{t} \gg 1$ differ from each other just as we found in $d = 2$. The first of these cases corresponds to keeping the time fixed and allowing N to increase indefinitely, and the second to reversing the order in which these limits are taken.

We first consider the case $\epsilon\sqrt{t} \ll 1$ (regime II), and apply the transformation in Eqs. (3.7a) and (4.5). In analogy with Eqs. (3.7b)–(3.10) we conclude that

$$\begin{aligned} I(u,t) &\sim \frac{1}{\epsilon\sqrt{t}} \int_0^\infty \frac{1}{(1/\epsilon) + \psi(v)} dv \\ &= \frac{1}{\epsilon\sqrt{t}} \int_0^\infty \exp\left(-\frac{\xi}{\epsilon\sqrt{t}}\right) d\xi \\ &\quad \times \int_0^\infty v^2 \exp[-\xi\psi(v)] dv. \end{aligned} \tag{5.5}$$

As a matter of detail in analyzing the integral over v for small ξ, we can make the transformation suggested in Eq. (3.7a), remembering that there is now an extra v^2 in the integrand in Eq. (5.5). On taking this into account in the calculation, one finds that the analog of Eq. (3.11) is

$$I(u,t) \sim \tfrac{1}{2} \ln^{3/2}\left(\frac{1}{(1-u)\sqrt{t}}\right). \tag{5.6}$$

On substituting this approximation into the representation of $S(u;t)$ in Eq. (5.3) we find

$$S(u;t) \sim \frac{2\pi\sigma^3(2t)^{3/2}}{1-u} \ln^{3/2}\left(\frac{1}{(1-u)\sqrt{t}}\right). \tag{5.7}$$

A Tauberian theorem then allows us to conclude that

$$\langle S_N(t)\rangle \sim 2\pi\sigma^3(2t)^{3/2} \ln^{3/2}\left(\frac{N}{\sqrt{t}}\right) \quad \text{(regime II)}, \tag{5.8}$$

which is valid when $N \gg \sqrt{t}$. Again we see that there exists a "screening" growth regime of $\langle S_N(t)\rangle$ after the initial regime for which $\langle S_N(t)\rangle \sim t^3$.

The elaborate analysis leading to Eq. (5.8) is again unnecessary in the opposite limit in which $\epsilon\sqrt{t} \gg 1$, since, as in two dimensions, we can simply omit the factor 1 that appears in the denominator of Eq. (5.3) and evaluate the resulting integral. In this way we find

$$S(u;t) \sim \frac{2\sigma t}{\sqrt{\pi}p(\mathbf{0};1)(1-u)^2}, \tag{5.9}$$

which implies that in regime III

$$\langle S_N(t)\rangle \sim \frac{2\sigma}{\sqrt{\pi}p(\mathbf{0};1)} Nt \quad \text{(regime III)}. \tag{5.10}$$

Equation (5.10) is valid in the limit $N \ll t^{1/2}$, corresponding to the "unscreened" regime in which the overlap of the contributions of each walker becomes negligible. The crossover from screened to unscreened behavior occurs at a step number $t'_\times \sim N^2$. The order of magnitude of $t'_\times$ differs considerably from the result found in two dimensions, for large numbers of random walkers.

A combination of the results in Eqs. (5.8) and (5.10) suggests that the function $\langle S_N(t)\rangle$ can be written in scaling form

$$\langle S_N(t)\rangle \sim Ct^{3/2} f\left(\frac{N}{\sqrt{t}}\right). \tag{5.11}$$

Here C is a constant and the function $f(x)$ has the properties

$$f(x) \sim \begin{cases} \ln^{3/2}(x), & x \gg 1 \\ x, & x \ll 1. \end{cases} \tag{5.12}$$

The scaling form of Eq. (5.11) can also be deduced from the integral representation in Eq. (5.3). To see this divide out a factor of $(1-z)^{-1}$ from the denominator of Eq. (5.3). This allows us to express $S(u;t)$ as

$$S(u;t) \sim \frac{4\pi\sigma^3(2t)^{3/2}}{(1-u)^2} \int_0^\infty \frac{v^2}{\rho(v)\sqrt{t} + [1/(1-u)]} dv, \tag{5.13}$$

where the form of the function $\rho(v)$ is determined from Eq. (5.3). One can now verify that the integral in Eq. (5.13) is a slowly varying function [30] when regarded as a function of $(1-u)^{-1}$. To demonstrate this, let the value of the integrand be denoted by $J[(1-u)^{-1}; v, t]$. This function is slowly varying in the limit $u \to 1$ provided that

$$\lim_{u\to 1} \frac{J[c(1-u)^{-1}; v, t]}{J[(1-u)^{-1}; v, t]} = 1 \tag{5.14}$$

for any positive value of c. That this property holds is easily verified from the explicit form of the function J, that is shown in Eq. (5.13). A Tauberian theorem for power series then allows us to conclude that

$$\langle S_N(t)\rangle \sim 4\pi\sigma^3(2t)^{3/2} \int_0^\infty v^2 \left(\frac{2\rho(v)(\sqrt{t}/N) + 1}{[\rho(v)(\sqrt{t}/N) + 1]^2}\right) dv. \tag{5.15}$$

Our contention is now proved since the integral is manifestly equal to a function of $\sqrt{t}/N$, which is equivalent to the scaling form found in Eq. (5.11). As mentioned in Sec. II, we performed tests of (5.11) using exact enumeration procedures, and found good agreement with the scaling form.

VI. DISTRIBUTION OF $S_N(t)$ FOR ONE DIMENSION

In addition to the mean number of distinct sites $\langle S_1(t)\rangle$, asymptotic results for the distribution of S_1 are known for the case of a single random walker for $d = 1$ and for $d \geq 3$ [13]. For the N walker case, in $d = 1$, we can derive the probability density of the overall span of the N diffusing particles, that is, the distribution of end-to-end distances between the rightmost and the leftmost positions reached by any of the walkers in the continuum limit. The span, in one dimension, can be regarded as the continuous analog of the number of distinct sites visited by a discrete random walk. The span for this system can be decomposed into the sum of the furthest displacement in the positive x direction, and the corresponding furthest displacement in the direction of negative x. Hence, the first problem to be considered is that of calculating at time t the probability density for the maximum displacement of a diffusing particle, initially at $x = 0$. Let D be the diffusion constant and let $\mathcal{B}_1(t)$ be the maximum displacement in the positive x direction for a single particle at time t. The probability that $\mathcal{B}_1(t) \leq b$ is easily calculated to be

$$P(b,t) = \operatorname{erf}\left(\frac{b}{\sqrt{4Dt}}\right). \tag{6.1}$$

We are interested in the distribution of the maximum displacement of the largest of the N values of $\mathcal{B}_1(t)$, and from this we wish to calculate the probability density of the global maximum displacement, a random variable which we denote by $\mathcal{B}_N(t)$. In a formal sense, this is an easy problem since the probability that none of the N particles has ever moved beyond b is just $[P(b,t)]^N$, and the probability density is found by differentiating this formula. Following this route would require the evaluation of very complicated functions, and we therefore use the theory of extreme-value statistics [31], which will yield results for $N \gg 1$ and $t \to \infty$, but in which the results can be expressed in terms of universal functions that are rather simple.

One result required for the following calculations is the limiting form of $P(b,t)$ as b increases indefinitely. This is found from Eq. (6.1) to be

$$P(b,t) \sim 1 - \frac{\sqrt{2Dt}}{b\sqrt{\pi}} \exp\left(-\frac{b^2}{2Dt}\right). \tag{6.2}$$

We next make use of the following result from the theory of extreme-value statistics [30]:

Define a function $R(\xi)$ *by*

$$R(\xi) = \frac{\int_\xi^\infty [1 - P(b,t)]db}{1 - P(\xi,t)} \tag{6.3}$$

and assume that as $\xi \to \infty$

$$\lim_{\xi\to\infty} \frac{1 - P(\xi + xR(\xi), t)}{1 - P(\xi,t)} = e^{-x}, \tag{6.4}$$

then there exist sequences, a_N *and* $c_N > 0$, $N = 1, 2, \ldots,$ *such that as* $N \to \infty$

$$\lim_{N\to\infty} \operatorname{Prob}\{\mathcal{B}_N(t) < a_N + c_N x\} = \exp(-e^{-x}). \tag{6.5}$$

The constants a_N *and* c_N *can be chosen as the solutions to*

$$a_N = \inf\left\{b;\ 1 - P(b,t) \leq \frac{1}{N}\right\}, \quad c_N = R(a_N). \tag{6.6}$$

Making use of the asymptotic form of $P(b,t)$ in Eq. (6.2), we find the asymptotic form of $R(\xi)$ to be

$$R(\xi) \sim \xi \exp\left(\frac{\xi^2}{4Dt}\right) \int_\xi^\infty \frac{\exp(-\rho^2/4Dt)}{\rho} d\rho \sim \frac{2Dt}{\xi} \tag{6.7}$$

from which one can verify that Eq. (6.4) is indeed satisfied. The constants a_N and c_N that appear in Eq. (6.5) can be found from Eq. (6.6) to be

$$a_N \sim \sqrt{4Dt \ln(N)}, \quad c_N \sim \sqrt{Dt/\ln(N)}, \tag{6.8}$$

so that, according to Eq. (6.5),

$$\lim_{N\to\infty} \operatorname{Prob}\left\{\frac{\mathcal{B}_N(t) - \sqrt{4Dt\ln(N)}}{\sqrt{Dt/\ln(N)}} < x\right\} = \exp(-e^{-x}). \tag{6.9}$$

This explicit expression for the cumulative probability allows us to calculate all of the moments of the random variable $\mathcal{B}_N(t)$. For example, the first moment and the variance of $\mathcal{B}_N(t)$ in the limit of large N are

$$\begin{aligned} \langle \mathcal{B}_N(t) \rangle &= \sqrt{4Dt\ln(N)} + \gamma\sqrt{\ln(N)/Dt}, \\ \sigma^2[\mathcal{B}_N(t)] &= \left(\frac{\pi^2}{6} + \gamma\right)\frac{Dt}{\ln(N)}, \end{aligned} \tag{6.10}$$

where γ is Euler's constant equal to $0.577\ldots$.

These expressions show that the maximum displacement is proportional to the value $\sqrt{4Dt\ln(N)}$ with a variance that vanishes as $N \to \infty$, i.e., the probability density for $\mathcal{B}_N(t)$ tends towards a δ function in that limit. Since the underlying diffusion processes are symmetric, we may expect the same dependence on the parameters t and N for the furthest displacement in the x direction. The results in Eq. (6.10) indicate that to a very good approximation the span is proportional to $\sqrt{Dt\ln(N)}$, which exhibits the same dependence on time and the number of random walkers as that given in Eq. (3.13). Because the N-dependent term in the expression for $\sigma^2[\mathcal{B}_N(t)]$ is inversely proportional to $\ln(N)$ we expect the δ-function approximation to the probability density of $\mathcal{B}_N(t)$ to be accurate only for enormously large values of N. Otherwise, the distribution is rather wide.

A complete solution in the same sense as is implied in Eq. (6.9) is not available for random walks for $d > 1$ because the number of distinct sites visited cannot be identified in terms of the span.

VII. DISCUSSION AND EXTENSIONS

We have restricted our analysis to the case of random walkers, all starting from the *same* initial position; this is responsible for the the "screening effects" that appear in the first and second growth regimes. Yet, depending on the degree of initial localization of the walkers, one may reduce the overlap of the different contributions thus reducing the time in which the expressions for $\langle S_N(t)\rangle$ for the first and second regime are valid. If the initial localization of the walkers is sparse enough, one might skip the initial-growth regimes altogether.

TABLE I. Summary of the expressions for the number of distinct sites $\langle S_N(t)\rangle$ (denoted S for compactness) for the different growth regimes, as well as the crossover times separating these regimes. The expressions for general dimension $d > 3$ in regimes II and III are conjectured from the results in one, two, and three dimensions.

d	Regime I	$t_\times$	Regime II	$t'_\times$	Regime III
1	$S \sim t$	$\ln N$	$S \sim \sqrt{t \ln N}$	∞	
2	$S \sim t^2$	$\ln N$	$S \sim t\ln(N/\ln t)$	e^N	$S \sim Nt/\ln t$
3	$S \sim t^3$	$\ln N$	$S \sim [t\ln(N/\sqrt{t})]^{3/2}$	N^2	$S \sim Nt$
d	$S \sim t^d$	$\ln N$	$S \sim [t\ln(Nt^{1-d/2})]^{d/2}$	$N^{2/(d-2)}$	$S \sim Nt$

The effects of the initial localization can be estimated by considering that the walkers are initially distributed within a region of linear size l, and comparing l with the characteristic width $\xi(t) \sim t^{1/2}$ of the distribution of positions of the N walkers that are initially at the origin. Then we can distinguish three types of behavior for $S_N(t)$:

(i) If $l \ll \xi(t_\times) \sim \sqrt{ln(N)}$, we expect to observe all three growth regimes of $S_N(t)$.

(ii) If $\xi(t_\times) \ll l \ll \xi(t'_\times)$, then the system would still pass through regime II and regime III.

(iii) If $l \gg \xi(t'_\times)$, then regime III will hold for all times.

In summary, we find that the number of distinct sites visited by N random walkers passes through several growth regimes, depending on the degree of overlap of the contributions of each walker. We also obtain the asymptotic expressions for $\langle S_N(t)\rangle$ in each regime in $d = 1, 2, 3$, as well as the crossover times. The distinct growth regimes of $\langle S_N(t)\rangle$ and the crossover times are summarized in Table I. Making use of the theory of extreme order statistics [30], we also found the asymptotic form of the distribution of the $S_N(t)$ for $d = 1$.

There is considerable recent interest in the modifications in basic physical laws that are required when the underlying substrate is a fractal object instead of a Euclidean space [32, 33]. An analysis of the generalization requiring us to find the expected number of distinct sites visited on a fractal relies much more heavily on computer simulations and will be investigated in a work to follow, as well as the addition of a biasing field, which will also fundamentally modify the interference effects and the properties of the number of distinct sites.

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APPENDIX A: DERIVATION OF EQ. (4.1)

The generating function for $f_t(\mathbf{r})$ as a function of t is readily calculated in terms of that for the probability that a walker initially at the origin will be at position $\mathbf{r}$ at time t, $p_t(\mathbf{r})$. Let these two generating functions be denoted by $f(\mathbf{r}; z)$ and $p(\mathbf{r}; z)$, respectively. Then it is well known that when $\mathbf{r} \neq 0$, $f(\mathbf{r}; z)$ and $p(\mathbf{r}; z)$ are related by $f(\mathbf{r}; z) = p(\mathbf{r}; z)/p(0; z)$. This relation will be exploited in the analysis to follow. Since one can always express $p(\mathbf{r}; z)$ in terms of a d-dimensional integral containing the characteristic function for a single step for random walks in which the steps are independent random variables $P(\mathbf{r})$, the function $f(\mathbf{r}; z)$ can be represented by a ratio of d-dimensional integrals for these walks.

To derive expression (4.1), we start by defining the generating function of the $\Gamma_t(\mathbf{r})$ with respect to t will be denoted by $\Gamma(\mathbf{r}; z)$, where [14, 15]

$$\Gamma(\mathbf{r}; z) = \frac{1}{1-z}[1 - f(\mathbf{r}; z)] = \frac{1}{1-z}\left(1 - \frac{p(\mathbf{r}; z)}{p(\mathbf{0}; z)}\right), \tag{A1}$$

in which $p(\mathbf{r}; z)$ is the generating function of the t step transition probabilities. The two-dimensional form of $p(\mathbf{r}; z)$ can be expressed in terms of the generating function for the characteristic function $P(\mathbf{r})$, $\hat{p}(\boldsymbol{\theta}) = \sum_{\mathbf{r}} p(\mathbf{r}) \exp(i\mathbf{r} \cdot \boldsymbol{\theta})$ as

$$p(\mathbf{r}; z) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{-i\mathbf{r}\cdot\boldsymbol{\theta}}}{1 - z\hat{p}(\boldsymbol{\theta})} d^2\boldsymbol{\theta}. \tag{A2}$$

We are only interested in the large-t behavior of $\Gamma_t(\mathbf{r})$, which by an argument familiar in the literature of random walks [28] is found from the asymptotic dependence of $p(\mathbf{r}; z)$ on z in the neighborhood of $z = 1$. For simplicity we restrict ourselves to the case in which steps along the x and y coordinates are uncorrelated and the variance of step length is a constant σ^2. When the limit $u \to 1$ is taken, it is known that $p(\mathbf{0}; u)$ has the singular behavior [14, 15]

$$p(\mathbf{0}; z) \sim \frac{1}{2\pi\sigma^2} \ln\left(\frac{1}{1-z}\right). \tag{A3}$$

When z is set equal to 1 in Eq. (A2) the double integral is singular because the denominator vanishes at $\theta = \mathbf{0}$. Because of this we can find the asymptotic behavior as $z \to 1$ by approximating to $\hat{p}(\theta)$ in the neighborhood of the origin, and extending the limits of integration to $\pm\infty$. This leads to the approximation

$$p(\mathbf{r}; z) \sim \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\mathbf{r}\cdot\boldsymbol{\theta}}}{1 - z + (\sigma^2/2)\theta^2} d^2\boldsymbol{\theta}. \tag{A4}$$

By exponentiating the denominator using the formula $u^{-1} = \int_0^\infty \exp(-ut)dt$, we transform this last equation into

$$p(\mathbf{r};z) \sim \frac{1}{4\pi^2}\int_0^\infty e^{-(1-z)t}dt \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\mathbf{r}\cdot\boldsymbol{\theta}-(\sigma^2\theta^2 t/2)}d^2\theta = \frac{1}{2\pi\sigma^2}\int_0^\infty e^{-(1-z)t-r^2/2\sigma^2 t}\frac{dt}{t} = \frac{1}{\pi\sigma^2}K_0\left(\frac{r}{\sigma}\sqrt{2(1-z)}\right), \tag{A5}$$

where $r^2 = \mathbf{r}\cdot\mathbf{r}$, and where $K_0(x)$ is a Bessel function. Hence it follows that in the limit $u \to 1$ the form of $\Gamma(\mathbf{r};u)$ is approximately equal to

$$\Gamma(\mathbf{r};u) \sim \frac{1}{1-u}\left[1 - \frac{2}{\ln[1/(1-u)]}K_0\left(\frac{r}{\sigma}\sqrt{2(1-u)}\right)\right]. \tag{A6}$$

From this formula we will deduce the form of $\Gamma_t(\mathbf{r})$ for large r, which, by Eq. (2.4) governs the behavior of $S(u;t)$ for large t.

To do so we must essentially determine the inverse from the second term in Eq. (A6). For this purpose we regard the generating function as a Laplace transform by setting $u = \exp(-s)$. The limit $u \to 1$ is then equivalent to $s \to 0$. Notice that the Bessel function that appears in Eq. (A6) is singular at $r = 0$. This singularity is an artificial one since $\Gamma_t(\mathbf{0}) = 0$ in the original lattice model, which allows us to finesse any difficulties due to it in our later evaluations. The inverse of the term $(1-u)^{-1}$ is equal to 1. We therefore seek to find the inverse transform of the function

$$\hat{\Gamma}(s) \equiv \mathcal{L}_t\{\Gamma_t(s)\} = \frac{1}{s\ln(1/s)}K_0\left(\frac{r\sqrt{2s}}{\sigma}\right), \tag{A7}$$

where $\mathcal{L}_t$ denotes a Laplace transform. The desired result need only be valid in the limit $t \to \infty$, which is equivalent to $s \to 0$ in the Laplace transform domain. In this limit $\ln(1/s)$ is a slowly varying function [29] and contributes a term $1/\ln(t)$ to the inverse. The inverse transform of the remaining terms is

$$\mathcal{L}^{-1}\left\{\frac{K_0(r\sqrt{2s}/\sigma)}{s}\right\} = \tfrac{1}{2}E_1\left(\frac{r^2}{4\sigma^2 t}\right), \tag{A8}$$

where $E_1(x)$ is the exponential integral [19]. The important regime in this calculation is $r^2 \gg 4\sigma^2 t$, in which we can make the approximation

$$\tfrac{1}{2}E_1\left(\frac{r^2}{4\sigma^2 t}\right) \sim \frac{\sigma^2 t}{r^2}\exp\left(-\frac{r^2}{4\sigma^2 t}\right), \tag{A9}$$

with the result that $\Gamma_t(\mathbf{r})$ behaves as

$$\Gamma_t(\mathbf{r}) \sim 1 - \frac{2\sigma^2 t}{r^2\ln(t)}\exp\left(-\frac{r^2}{4\sigma^2 t}\right), \tag{A10}$$

in the limit $r \to \infty$.

APPENDIX B: DERIVATION OF EQ. (5.1)

Following the analysis of Appendix A, we first observe that in three dimensions $p(\mathbf{0};1)$ is finite. If we again restrict ourselves to completely symmetric random walks with uncorrelated displacements along the three coordinate axes as well as equal variances, then a calculation similar to that given in Eq. (A5) yields the approximation

$$p(\mathbf{r};z) \sim \frac{1}{2\pi\sigma^2 r}\exp\left(-\frac{r}{\sigma}\sqrt{2(1-z)}\right). \tag{B1}$$

This, in turn, implies that $\Gamma(\mathbf{r};z)$ behaves as

$$\Gamma(\mathbf{r};z) \sim \frac{1}{1-z}\left[1 - \frac{1}{2\pi\sigma^2 p(\mathbf{0};1)r} \times \exp\left(-\frac{r}{\sigma}\sqrt{2(1-z)}\right)\right]. \tag{B2}$$

While it is possible to calculate an asymptotic approximation to $\Gamma_t(\mathbf{r})$ by expanding the exponential in Eq. (B2) and using a Tauberian theorem for power series [28] to find the contribution from each individual term, it is somewhat simpler to use the fact that t is large, and replace the power series that defines $\Gamma(\mathbf{r};z)$ by a Laplace transform. This is equivalent to replacing u by e^{-s}, where for present purposes $s \sim 0$. In this regime $\Gamma(\mathbf{r};z)$ is to be regarded as a Laplace transform which we denote by $\hat{\Gamma}(\mathbf{r};s)$. This function is given, in the region $s \sim 0$, by

$$\hat{\Gamma}(\mathbf{r};s) \sim \frac{1}{s}\left[1 - \frac{1}{2\pi\sigma^2 p(\mathbf{0};1)r}\exp\left(-\frac{r}{\sigma}\sqrt{2s}\right)\right]. \tag{B3}$$

Since the inverse transforms of the functions appearing in this equation are known, we can write

$$\Gamma_t(\mathbf{r}) \sim 1 - \frac{1}{2\pi\sigma^2 p(\mathbf{0};1)r}\operatorname{erfc}\left(\frac{r}{\sigma\sqrt{2t}}\right). \tag{B4}$$

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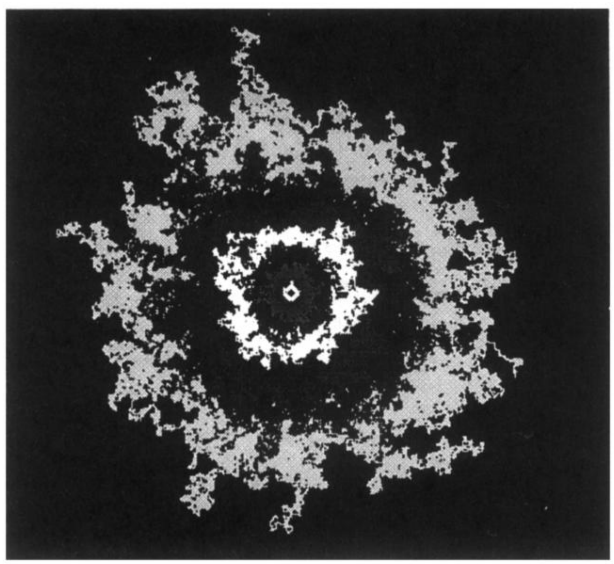

FIG. 2. Contours of the surface obtained from snapshots at successive times of the territory covered by N random walkers for the case $N = 500$ for a sequence of times in regime II. Note the roughening of the disc surface as time increases.