Stochastic Process with Ultraslow Convergence to a Gaussian: The Truncated Lévy Flight

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We introduce a class of stochastic process, the *truncated* Lévy flight (TLF), in which the arbitrarily large steps of a Lévy flight are eliminated. We find that the convergence of the sum of n independent TLFs to a Gaussian process can require a remarkably large value of n—typically $n \approx 10^4$ in contrast to $n \approx 10$ for common distributions. We find a well-defined crossover between a Lévy and a Gaussian regime, and that the crossover carries information about the relevant parameters of the underlying stochastic process.

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In physical systems, the variance of any stationary processes is *finite,* so that a Gaussian behavior is expected in the absence of long range correlations. Gaussian behavior arises from the central limit theorem (CLT), which is fundamental to statistical mechanics [1], and states that the sum

$$z_n \equiv \sum_{i=1}^{n} x_i \tag{1}$$

of n stochastic variables $\{x\}$ that are statistically independent, identically distributed, and with a finite variance converges when $n \longrightarrow \infty$ to a normal (Gaussian) stochastic process [2]. On the other hand, Lévy flights [3–10] which have *infinite* variance—or a related process called Lévy walks [11,12]—have been observed experimentally in fluid dynamics [13] and polymers [14] and have been used to describe subrecoil laser cooling [15], turbulent fluids [11,16], very stiff polymers [17], and the spectral random walk of a single molecule embedded in a solid [18,19].

In these physical systems, an unavoidable cutoff is always present. For example, in the case of a single molecule embedded in a solid [18,19], due to the minimal length between the molecule and the nearest two-level systems, a cutoff is present in the distribution of the jumps of the resonance frequency. The jump of the frequency is induced by the thermal transitions occurring in the surrounding two-level systems. A second example is turbulence: In numerical simulations performed using Lévy walks in a Boltzmann lattice gas, the jumps of the particles are limited by the finite size of the simulated system [16].

Lévy flights have mathematical properties that discourage a physical approach: They have (i) infinite variance, and (ii) an analytical form is known only for few special cases. Here we show that the paradox of infinite variance may be resolved by introducing a variant of the Lévy flight (not requiring the hypothesis of a spatiotemporal coupling [12]), which we term the truncated Lévy flight (TLF). A TLF has finite variance. We show that the convergence of a sum of TLFs to a normal process can be so slow that a huge number ($n \gtrsim 10^4$) of independent events (or time intervals) may be necessary to ensure convergence to a Gaussian stochastic process. Thus, in systems with finite variance, we can observe a sum of a huge number of independent stochastic variables which converges to a stochastic process characterized by a probability distribution that differs from a Lévy stable distribution only in the very far wings. From the point of view of an experimental study, we find an apparent failure of the CLT in that a stochastic process with finite variance is apparently not converging to the expected Gaussian behavior (Fig. 1).

We define a TLF to be a stochastic process $\{x\}$ characterized by the following probability distribution:

$$T(x) \equiv \begin{cases} 0, & x > l, \\ c_1 L(x), & -l \le x \le l, \\ 0, & x < -l, \end{cases} \tag{2}$$

where

$$L(x) \equiv \frac{1}{\pi} \int_0^{+\infty} \exp(-\gamma q^{\alpha}) \cos(qx)\, dq \tag{3}$$

is the symmetrical Lévy stable distribution of index α $(0 < \alpha \le 2)$ and scale factor γ $(\gamma > 0)$, c_1 is a normalizing constant, and l is the cutoff length [20]. For the sake of simplicity, we set $\gamma = 1$.

We investigate the probability distribution $P(z_n)$ of the stochastic process of Eq. (1) when $\{x\}$ is a TLF, i.e., a stochastic process with probability distribution given by Eq. (2). Of particular interest is the probability of return $P(z_n = 0)$, which we study as a function of n, l, and α.

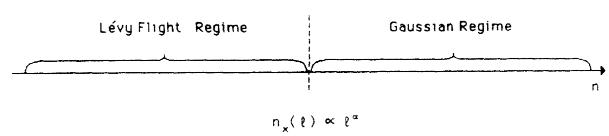

FIG. 1. Schematic illustration of our results for the TLF. Shown is the crossover found between Lévy flight behavior for small n and Gaussian behavior for large n. The crossover value $n_\times$ increases rapidly with the cutoff length l.

For small values of n, $P(z_n = 0)$ takes a value very close to the one expected for a Lévy stable process,

$$P(z_n = 0) \simeq L(z_n = 0) = \frac{\Gamma(1/\alpha)}{\pi \alpha n^{1/\alpha}}. \quad (4)$$

For large values of n, $P(z_n = 0)$ assumes the value predicted for a normal process,

$$P(z_n = 0) \simeq N(z_n = 0) = \frac{1}{\sqrt{2\pi}\,\sigma_0(\alpha, l) n^{1/2}}, \quad (5)$$

where $\sigma_0(\alpha, l)$ is the standard deviation of the TLF stochastic process $\{x\}$.

In the interval $1 \le \alpha < 2$, we can calculate the crossover between the two regimes by equating Eqs. (4) and (5), and writing explicitly the dependence of σ_0 on α and l. The analytical form of the variance of a TLF is known only for $\alpha = 1$. An approximate relation for any value of $1 < \alpha < 2$ and l can be found if one uses a series expansion [21] of the symmetrical Lévy distribution valid in the interval $1 < \alpha < 2$

$$L(z) \simeq -\frac{1}{\pi} \sum_{k=1}^{m} \frac{(-1)^k}{k!} \frac{\Gamma(\alpha k + 1)}{z^{\alpha k + 1}} \sin\left(\frac{k \pi \alpha}{2}\right) + R(z),$$

where (6a)

$$R(z) = O(z^{-\alpha(m+1)-1}). \quad (6b)$$

By using the leading term of the series, we find that the standard deviation of a TLF is approximately given by

$$\sigma_0(\alpha, l) \simeq \left[\frac{2\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi(2-\alpha)}\right]^{1/2} l^{(2-\alpha)/2}. \quad (7)$$

Using Eqs. (7), (5), and (4) we find

$$n_\times \simeq A\, l^\alpha, \quad (8a)$$

where

$$A = \left[\frac{\pi\alpha}{2\Gamma(1/\alpha)[\Gamma(1+\alpha)\sin(\pi\alpha/2)/(2-\alpha)]^{1/2}}\right]^{2\alpha/(\alpha-2)}. \quad (8b)$$

From Eq. (8a), we see that, for a given value of l, the number of variables required to see the crossover increases with the value of α.

To quantify the "distance" between the TLF and the Gaussian stochastic processes, we introduce a quantity

$$\Delta \equiv \log_{10} \frac{T(0)}{N_s(0)}, \quad (9)$$

where $T(0)$ is the TLF probability of return, and $N_s(z_n = 0) \equiv N(z_n = 0) n^{1/2}$ is the scaled probability of return for

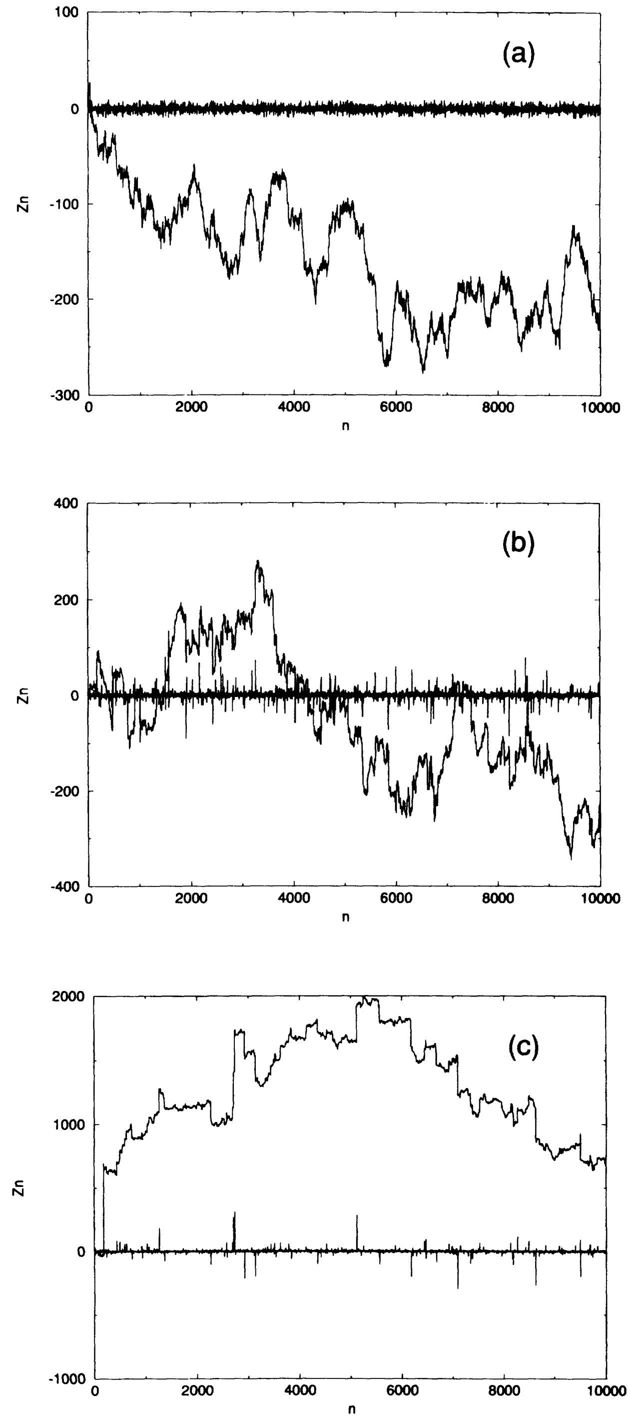

FIG. 2. Numerical simulations of the stochastic variable z_n defined by Eq. (1). The stochastic variables $\{x\}$ are also shown. The x_i variables are TLFs of index $\alpha = 1.2$ and scale factor $\gamma = 1$. The cutoff length l is given by (a) $l = 10$, (b) $l = 100$, and (c) $l = 1000$. For values of l closer to γ [e.g., the case $l = 10$, of (a)], large jumps of size greater than 10 are completely forbidden and the profile of the z_n random walk resembles Brownian motion. By increasing l, large jumps occur [(b)]. Finally, if $l \gg \gamma$ [e.g., the case $l = 1000$, of (c)], then the profile strongly resembles the profile of a Lévy flight of the same index α.

the associated normal process. We find

$$\Delta \simeq \log_{10} \frac{2\Gamma(1/\alpha)}{\pi\alpha} + \frac{1}{2}\log_{10}\left(\frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{2-\alpha}\right) + \frac{2-\alpha}{2}\log_{10} l. \tag{10}$$

For a selected value of l, Δ is maximal for $\alpha = 1$ and is equal to 0 for $\alpha = 2$; this means that the distance between the two asymptotic regimes decreases as α increases.

Next we test our theoretical predictions by performing numerical simulations of the TLF stochastic process [22]. To generate a Lévy stable stochastic process of index α and scale factor $\gamma = 1$, we use the algorithm of Ref. [23].

In Figs. 2(a)–2(c), we show three different realizations of the stochastic process z_n obtained by generating TLF characterized by $\alpha = 1.2$ and $l = 10, 100, 1000$, respectively. We also plot the values of the x variables used to obtain each realization. From Eq. (2), it follows that these values are bounded between $-l$ and $+l$. By increasing l, the features of the random walk performed by z_n range from those of Brownian motion [Fig. 2(a)] to those of a Lévy flight of index α [Fig. 2(c)].

In Fig. 3, we show the probability of return obtained by simulating the z_n process when $\alpha = 1.2$ and $l = 10$, 100, 1000. We also show the asymptotic behaviors predicted by using Eq. (4) (solid line) and Eq. (5) (dotted lines). We clearly see the crossover between the two regimes, as predicted by Eqs. (4) and (5).

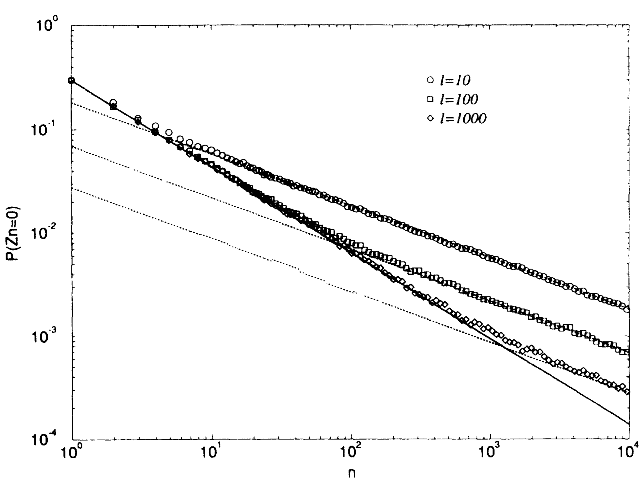

FIG. 3. Numerical simulation of the probability of return $P(z_n = 0)$, where z_n is a sum of TLFs of index $\alpha = 1.2$. Three different values of the cutoff length l ($l = 10, 100, 1000$) are shown. The small-n regime of Eq. (4) is shown as a solid line, while the large-n asymptotic regime of (5) is shown as three dotted lines, one for each value of l. The values of the standard deviation of the TLF, $\sigma_0(\alpha, l)$, used to draw the dotted lines have been calculated numerically. The crossover $n_\times$ between the two asymptotic regimes increases dramatically when l increases. For the case $l = 1000$, a behavior very close to the one predicted for a Lévy flight is observed up to $n \simeq 1000$.

We systematically studied the dependence of $n_\times$ and Δ on the parameters α and l [22]. The simulations are compared with the theoretical predictions of Eqs. (8a) and (10) in Figs. 4(a) and 4(b), respectively; we find striking agreement.

In conclusion, theoretical predictions and numerical simulations demonstrate the existence of values of the control parameters α and l for which the sum of TLFs requires a huge number of independent variables to converge to a normal process [24]. This implies that the

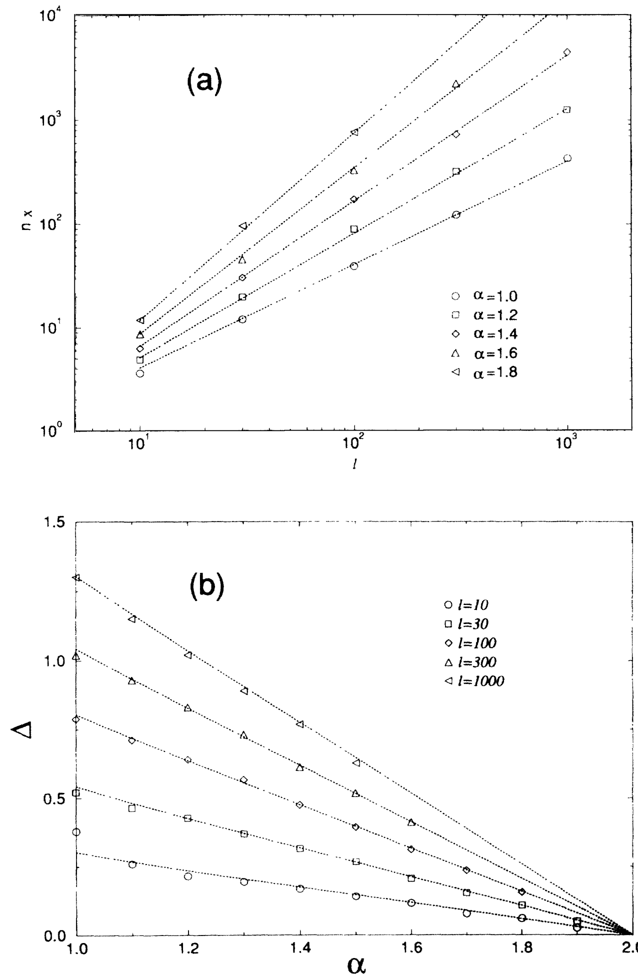

FIG. 4. (a) Minimum value of n, $n_\times$, required to observe a crossover between the Lévy and Gaussian regimes. Symbols refer to the values of $n_\times$ we find in numerical simulations of $P(z_n = 0)$ for a range of indices α and cutoff lengths l. Shown as dotted lines are the theoretical predictions of Eq. (8a) for $\alpha = 1.0, 1.2, 1.4, 1.6,$ and 1.8 from bottom to top. (b) "Distance" Δ between $T(0)$ and $N_s(0)$, the scaled probability of return of the associated normal process [see (9)]. Symbols are the values of $n_\times$ found in numerical simulations of z_n characterized by different values of the control parameters α and l. The theoretical predictions of Eq. (10) are also shown as dotted lines, each corresponding to a value of l: from bottom to top, $l = 10$, 30, 100, 300, and 1000.

CLT correctly predicts the final distribution only if one considers the sum of an incredibly large number of TLFs. For $n \ll n_\times$, a violation of the CLT is observed. When this violation occurs, the sum of TLFs maintains statistical properties for a large value of n that are effectively indistinguishable from the statistical properties of Lévy flights (except for the most rare events). Because of this "long-time" violation of the CLT, we expect that a number of physical phenomena may show statistical properties very close to those expected for Lévy flights for long intervals of the control variables, even in the absence of a spatiotemporal memory.

Our study also shows that a clear crossover between Lévy and Gaussian regimes is observable by investigating the probability of return of a stochastic process with finite variance, which shows a Lévy-like probability distribution for a long (but finite) interval of independent variables n. Since the crossover is a function of the index α and cutoff length l, our method allows the parameters α and l to be determined from experimental data: e.g., by measuring the index α (inverse of the absolute value of the slope fitting the Lévy regime in Fig. 3) and the crossover $n_\times$, one can determine l by using Eq. (8a). The knowledge of α and l in physical systems could be useful to fully characterize the analyzed process.

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[2] In a dynamical system, Eq. (1) defines a random walk if the variable x is the jump size performed after a time interval Δt and n is the number of time intervals. In this Letter, the "number of variables" n and the "time" $t = n\Delta t$ can be interchanged everywhere.

[3] A Lévy flight is a random walk whose step length is chosen from the distribution (3), so arbitrarily large steps occur with power-law frequency, in contrast to a conventional random walk for which large steps are exponentially rare.

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