Nonlinear Solutions of Renormalization-Group Equations*

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We give the first nonlinear solution of renormalization-group equations. This solution, based on the differential generator of Wegner and Houghton, exhibits an explicit mixing of (or crossover between) critical and mean-field behavior. The solution is given for all values of the spin dimension n and to first order in $\epsilon \equiv 4-d$, where d is the lattice dimensionality.

Recently[1,2] much work has been devoted to the renormalization-group equations linearized around various fixed points. Each fixed-point Hamiltonian governs a particular class of critical phenomena. The linearized equations about a fixed point have solutions which represent scaling equations of state, with critical-point exponents which are simply calculated from the eigenvalues of the linearized renormalization-group equations. The physically measurable exponents are those of the dominant fixed point. The analysis of a single fixed point is therefore sufficient to discuss the critical behavior *asymptotically close to* the critical point. However, *at finite distances from the critical point*, the competing influences of the many other fixed points may become important. This competition between fixed points is loosely described as "crossover"; the physical system passes from the domination of one fixed point to the domination of another.

Riedel and Wegner,[3] using a semimicroscopic model which simulates renormalization-group crossover, have discussed the competition between tricritical and critical behavior. Here we present the first cross over solution based *directly* on the nonlinear renormalization-group equations. The solution given describes the transition from true critical behavior near the critical point to mean-field–like behavior at higher temperatures.[4,5]

To preface the discussion of the *nonlinear* solution itself, we will first give a general abstract description of the solution of a *linear* renormalization-group equation. This will also serve to establish our notation. Generally a renoralization-group representation near a fixed point can be written as a set of linear differential equations. For example, a model Hamiltonian parametrized by variables p and q might be described by the equations.

$$\dot{p} = 2p, \tag{1a}$$
$$\dot{q} = \epsilon q, \tag{1b}$$

where the dot denotes the derivative with respect to the renormalization parameter l and $\epsilon \equiv 4-d$, where d is the lattice dimension. The fundamental equation defining the renormalization parameter itself is given by the renormalization trajectory for the correlation length, $\xi(p, q)$,

$$\dot{\xi} = -\xi. \tag{2}$$

The solutions of Eq. (1) are

$$p = \text{const}\, e^{2l}, \tag{3a}$$

$$q = \text{const}\, e^{\epsilon l}. \tag{3b}$$

The solution of Eq. (2) is a generalized homogeneous function,

$$\xi(\lambda^2 p, \lambda^\epsilon q) = \lambda^{-1}\xi(p, q). \tag{4}$$

The correlation-length solution is more usually written as

$$\xi(p, q) = x^{-1/2}P(p^\epsilon/q^2), \tag{5}$$

where P is any arbitrary function which, however, is assumed to be regular and nonzero at $p=0$. We call p and q scaling fields. They play the same role in Eq. (4) as the scaling variables of the usual scaling theory. In this case, the critical-point exponent $\nu = \frac{1}{2}$.

More generally, Eqs. (1) will have nonlinear terms as well as linear ones. However, there will still be functions of p and q (not simply equal to p and q) which have a simple exponential dependence on the renormalization parameter. We will call these functions the nonlinear scaling fields.[6] The correlation length is again a generalized homogeneous function, not of p and q, but of the corresponding nonlinear scaling fields.

Wegner and Houghton[1] have suggested a differential generator for the renormalization group which reproduces the results of Wilson's finite-difference generator. For nonlinear solutions good to first order in ϵ, the momentum-independent equations of Ref. 1 reduce to

$$\dot{r} = 2r + \frac{u}{1+r}\frac{d}{2}\frac{n+2}{n}, \tag{6a}$$

$$\dot{u} = (4-d)u - \frac{u^2}{(1+r)^2}\frac{d}{2}\frac{n+8}{n}, \tag{6b}$$

where r and u are the momentum-independent two- and four-spin coefficients in Wilson's reduced Hamiltonian.[2]

The character of Eqs. (6) is more easily seen after a transformation which maps the solution trajectories of interest into a finite region. We define new variables $\bar{r}$ and $\bar{u}$ by

$$\bar{r} \equiv r/(1+r), \tag{7a}$$

$$\bar{u} \equiv u/(1+r)^2. \tag{7b}$$

The fundamental equations now take the form

$$\dot{\bar{r}} = 2(1-\bar{r})[\bar{r} + \bar{u}d(n+2)/4n], \tag{8a}$$

$$\dot{\bar{u}} = \bar{u}[\epsilon - \bar{u}(3d/2n)(n+4) - 4\bar{r}]. \tag{8b}$$

There are three fixed points of physical interest ($u \geq 0$): the "finite" Gaussian point at $\bar{r} = \bar{u} = 0$; the "infinite" Gaussian point at $\bar{r} = 1$, $\bar{u} = 0$; and the Wilson-Fisher[7] point at $\bar{r} = -\epsilon(n+2)/2(n+8)$, $\bar{u} = \epsilon 2n/d(n+8)$.

Equations (8) are already in diagonal form around the infinite Gaussian fixed point ($\bar{r} = 1$, $\bar{u} = 0$). It is also useful to diagonalize (8) around the finite Gaussian fixed point ($\bar{r} = \bar{u} = 0$). Defining new variables x and y by

$$x \equiv \bar{r} + [\bar{u}/(2-\epsilon)][d(n+2)/2n], \tag{9a}$$

$$\epsilon y \equiv \bar{u}d(n+8)/2n, \tag{9b}$$

we rewrite Eqs. (8) as

$$\dot{x} = 2x\{1 - x - [(n+2)/2(n+8)]\epsilon y\}, \tag{10a}$$

$$\dot{y} = y[\epsilon(1-y) - 4x]. \tag{10b}$$

We have neglected terms of order $\epsilon^2 y^2$ in (10) consistent with (6). This approximation puts (8) and (10) into the same form. We also note (cf. Fig. 1) that the various fixed points are located at $x = y = 0$ (finite Gaussian); $x = 1$, $y = 0$ (infinite Gaussian); and $x = 0$, $y = 1$ (Wilson-Fisher).

We may write the solutions to Eqs. (8) in terms of two functions R and U, which satisfy the equations

$$\dot{R} = 2(1-\bar{r})R, \quad \dot{U} = d\bar{u}U. \tag{11}$$

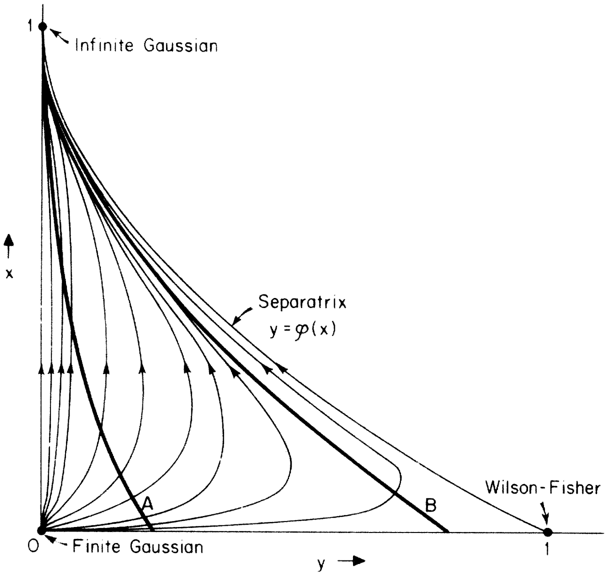

FIG. 1. Qualitative behavior of renormalization-group and temperature trajectories. The light lines depict the renormalization-group trajectories for the parameters x and y [cf. Eqs. (9) and (10)]. The heavy lines labeled A and B depict temperature trajectories for different system Hamiltonians [cf. Eqs. (24)].

The solutions are given by the scaling fields

$$(\bar{u}/R^2)U^{3(n+4)/2n} = \text{const}e^{-dl}, \tag{12a}$$

$$[(1-\bar{r})/R]U^{(n+2)/2n} = \text{const}e^{-2l}. \tag{12b}$$

The advantage of this formulation becomes apparent when we perform a similar calculation for Eqs. (10). Defining F and G through the equations

$$\dot{F} = -2xF, \tag{13a}$$

$$\dot{G} = -\epsilon y G, \tag{13b}$$

we discover that the scaling fields can be written as

$$y/GF^2 = \text{const}e^{(4-d)l}, \tag{14a}$$

$$x/FG^{(n+2)/(n+8)} = \text{const}e^{2l}. \tag{14b}$$

Since both sets of scaling fields describe the same solutions, we may match them to reduce the number of unknown functions. Noting that $U = G^{-2n/(n+8)}$, we find that

$$F = 1 - \bar{r}, \tag{15a}$$

$$R = xG^{-2(n+2)/(n+8)}. \tag{15b}$$

All that remains is the calculation of G. The partial differential equation for G can be solved in terms of the separatrix connecting the Wilson-Fisher point with the infinite Gaussian point; this separatrix is indicated as $y=\varphi(x)$ in Fig. 1. The function φ satisfies

$$2x\{(1-x)-[(n+2)/2(n+8)]\epsilon\varphi\}d\varphi/dx = \varphi[(1-\varphi)-4x]. \qquad (16)$$

On this separatrix G is identically zero. Using Eqs. (16) we may write G as

$$G=(1-y/\varphi)e^{g}, \qquad (17a)$$

where g satisfies

$$\dot{g}=-\frac{n+2}{n+8}\epsilon x\frac{y}{\varphi}\frac{d\varphi}{dx}. \qquad (17b)$$

Solving Eqs. (16) and (17b) together we find (to order ϵ)

$$\varphi=(1-x)^{d/2}\exp[\tfrac{1}{2}\epsilon x(4-n)/(n+8)], \qquad (18a)$$

$$G=\left(1-\frac{y}{\varphi}\right)\exp\left[\frac{n+2}{n+8}\epsilon x\frac{y}{\varphi}\right]. \qquad (18b)$$

Equations (6) are now completely solved (to order ϵ). We define the Gaussian and Wilson-Fisher scaling fields by

$$S_G=xG^{-(n+2)/(n+8)}/(1-\bar{r}), \qquad (19a)$$

$$S_{\rm WF}=xy^{-(n+2)/(n+8)}/(1-\bar{r})^{(4-n)/(n+8)}. \qquad (19b)$$

The behavior of any function whose renormalization behavior is known can be expressed in terms of a generalized homogeneous function. If Q is a function that satisfies the renormalization transformation

$$\dot{Q}=a_Q Q, \qquad (20)$$

then Q satisfies

$$Q(\lambda^{a_H}H,\lambda^{a_G}S_G,\lambda^{a_{\rm WF}}S_{\rm WF}) = \lambda^{a_Q}Q(H,S_G,S_{\rm WF}), \qquad (21)$$

where H is the ordering field, and[8]

$$a_H=1+d/2,\quad a_G=2,$$
$$a_{\rm WF}=2-\epsilon(n+2)/(n+8). \qquad (22)$$

In particular, the correlation length satisfies (20) with $a_Q=-1$; the Gibbs potential satisfies (20) with $a_Q=d$.[9] An example of a correlation length which satisfies (21) is

$$\xi=\left[\frac{y^{(n+2)/(n+8)}(1-\bar{r})^{(4-n)/(n+8)}}{x}\right]^{1/a_{\rm WF}} + A\left[\frac{(1-\bar{r})G^{(n+2)/(n+8)}}{x}\right]^{1/a_G}. \qquad (23)$$

For any nonzero y (at the critical temperature), the Wilson-Fisher term will dominate asymptotically near the $x=0$ ($T=T_c$) singularity [provided that $a_{\rm WF}<a_G$, i.e., $\epsilon(n+2)/(n+8)>0$], giving $\nu=\frac{1}{2}+\epsilon(n+2)/4(n+8)$. However, for finite x ($T\neq T_c$) the Gaussian term may become important. This would give mean-field behavior, characterized by the exponent $\nu=\frac{1}{2}$. The "rate" of the crossover (between critical and mean-field behavior) depends on the magnitude of the constant A and on the explicit temperature dependences of x and y.

The temperature dependence of the two- and four-spin coefficients r and u will vary from model to model. For the case of two-spin interaction models, for which the four-spin term is introduced as a phase-space weight factor, the only temperature dependence is in the two-spin term, $r(T)$. It is straightforward to show that, in this case, the temperature trajectories are

$$1-x=(1+r_c)^{-1}(y/y_c)^{1/2}[1+r_c(y/y_c)^{1/2}], \qquad (24a)$$

where r_c is the value of r at the critical temperature,

$$\frac{r_c}{1+r_c}=-\frac{\epsilon y_c}{2-\epsilon}\frac{n+2}{n+8}, \qquad (24b)$$

and y_c is the value of y at the critical temperature. Two temperature trajectories are shown by the heavy lines labeled A and B in Fig. 1. It is clear that, for a given change of x, temperature trajectory A crosses more renormalization-group trajectories than does temperature trajectory B. To make this more quantitative, the renormalization trajectories can be labeled by the renormalization invariant I:

$$I=x(1-\bar{r})^dG^{a_{\rm WF}}/y^2. \qquad (25)$$

The invariant I is zero on the separatrices passing through the Wilson-Fisher fixed point [$x=0$ and $y=\varphi(x)$]. It is infinite on the limiting integral curve ($y=0$) joining the finite Gaussian fixed point to the infinite Gaussian fixed point. It may therefore be used as a measure of the criticality of a system. A small invariant characterizes a system dominated by the Wilson-Fisher fixed point, while a large invariant indicates that the system is dominated by the Gaussian or mean-field behavior. The crossover of a system from critical to mean-field behavior is governed by the rate of growth of the invariant. For the two-spin systems under consideration [temperature trajectories given by (24)] and $n=-2$ (for simplicity)[10]

$$I(T)=\frac{1}{{y_c}^2}\left(\frac{x(T)}{1-x(T)}\right)^{\epsilon}\left(1-\frac{[1-x(T)]^{\epsilon/2}y_c}{e^{\epsilon x(T)/2}}\right)^2. \quad (26)$$

For small y_c, $I(T)$ is a rapidly varying function of $x(T)$; for y_c near 1, $I(T)$ varies very slowly. For $x(T)$ monotonically increasing, $I(T)$ is also monotonic in T, cutting each renormalization-group trajectory exactly once. Similar behavior holds for general n.

If $x(T)\to 1$ as $T\to\infty$, the temperature trajectories all pass through the infinite Gaussian point at $x=1$, $y=0$. This requires that $r(T)\to\infty$ for $T\to\infty$. For realistic Hamiltonians, $r(T)$ has a finite limit at infinite temperature,[11] and the formal crossover properties of the renormalization-group equations are not completely realized. Moreover, even before the limiting values of x and y are approached (whether these limits are at the infinite Gaussian point or not) the correlation length and other thermodynamic functions will be dominated by their high-temperature behavior, rather than by the limiting behavior of an expression such as Eq. (23).

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4843-keV, 1^+ Level of ^{208}Pb†

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From measurements of resonance-fluorescence cross section, angular distribution, and polarization, the 4843-keV level of ^{208}Pb has been shown to have a 1^+ character and a width of 5.1 ± 0.8 eV with all of the decays to the ground state. As the probable lower member of the giant $M1$ excitation, this state is at a significantly lower energy and has a decay strength which is an order of magnitude larger than the predictions of simple shell-model calculations.

Recently I reported on a number of states in nuclei in the lead region which were observed using the resonance-fluorescence technique.[1] Among these was a spin-1 state in ^{208}Pb at 4843 keV with a width of 5 eV. The level was also observed by Earle *et al.*[2] through the $(d,p\gamma)$ reaction. They also gave a spin-1 assignment but were unable to determine the parity. Using a two-slab Ge(Li) polarimeter,[3] I have now measured the linear polarization of the resonantly scattered radiation from this state, and the results show that the parity must be positive. The corresponding ground-state $M1$ radiative strength is 2.3 Weisskopf units, a surprisingly strong $M1$ transition for this low an energy.

The resonance-fluorescence technique has been adequately described in the literature.[4,5] The 4843-keV level of ^{208}Pb was excited by brems-