# A DIFFERENTIAL GENERATOR FOR THE FREE ENERGY AND THE MAGNETIZATION EQUATION OF STATE☆

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We give an approximate differential generator for the Helmholtz free energy and for the magnetization equation of state. The equation of state is a function only of the limits of the renormalized Hamiltonian parameters as the renormalization parameter $l \to \infty$.

The first renormalization group methods for the calculation of the equation of state utilized entirely field theoretic techniques [1]. Recently, equations of state have been obtained by using a combination of field theoretic and differential renormalization group methods [2, 3]. In this work, we introduce a calculational method which is entirely differential in character. This differential generator for the free energy thus avoids the use of any field theoretic diagram expansions and only convergent integrals are needed.

The differential generator is derived in a manner analogous to that of ref. [4], except that a saddle point expansion is employed in order to obtain a one-particle irreducible generator [1]. Making the approximation that the expansion coefficients are momentum independent [5], we obtain

$$\partial A(\boldsymbol{M}, l)/\partial l = \exp(-dl) \operatorname{tr} \ln [\delta_{ij} + \exp(2l)\, \partial^2 A/\partial M^i \partial M^j]. \tag{1}$$

where $A(\boldsymbol{M}, l)$ is an $l$-dependent Helmholtz potential, $l$ is the renormalization parameter, $d$ is the lattice dimension and $\boldsymbol{M}$ is the physical $n$-component magnetization vector. We have set the critical point exponent $\eta = 0$ in this approximation. The physical Helmholtz potential is given by $A(\boldsymbol{M}) = \lim_{l\to\infty} A(\boldsymbol{M}, l)$. This generator (1) can be related to our previously introduced generator by the scale changes [5], $\boldsymbol{s} = \boldsymbol{M} \exp(\frac{1}{2}(d-2)l)$, $H(\boldsymbol{s}, l) = A(\boldsymbol{M}, l) \exp(dl)$. Even though the generator (1) is approximate, it contains some information from every order in perturbation theory. For example, one can easily show that (1) predicts that for non-Ising systems ($n \neq 1$) the longitudinal susceptibility $\chi_{\parallel}$ diverges on the coexistence surface, $\chi_{\parallel} \propto |\boldsymbol{h}|^{-\nu\epsilon/2}$, where $\boldsymbol{h}$ is the magnetic field and $\epsilon \equiv 4 - d$.

Although it is conceivable that (1) may be solved directly or that numerical methods might be employed, it is more probable that the following iterative technique, analogous to the field-theoretic loop expansion [1] will be useful in applications. To illustrate this, we will derive the equation of state for a ferromagnet correct to one-loop from (1). We start with an initial Hamiltonian $H(\boldsymbol{s}, l) \equiv r(l) s^2/2 + u(l) s^4/4$. This gives the lowest order or "zero-loop" (0-loop) approximation to $A(\boldsymbol{M}, l)$:

$$A_0(\boldsymbol{M}, l) = r(l) \exp(-2l) M^2/2 + u(l) \exp(-\epsilon l) M^4/4. \tag{2}$$

Thus, the 0-loop approximation to $A(\boldsymbol{M})$ is a Landau expansion of the same sort as the initial Hamiltonian but with "renormalized" expansion coefficients, $r_\infty \equiv \lim_{l\to\infty} r(l) \exp(-2l)$ and $u_\infty \equiv \lim_{l\to\infty} u(l) \exp(-\epsilon l)$, with $r_\infty \propto t^\gamma$ and $u_\infty \propto t^{\gamma - 2\beta}$, where $t \propto T - T_c$, and $\gamma$ and $\beta$ are the usual critical point exponents characterizing the susceptibility and coexistence curve, respectively [3]. Therefore, even in this 0-loop approximation, we can obtain the correct critical-point exponents to $O(\epsilon)$. The quantities $r_\infty$ and $u_\infty$ are nonlinear scaling fields [6], which are calculated from the complete nonlinear solutions for $r(l)$ and $u(l)$ given in [7].

The next successive approximation to $A(\boldsymbol{M}, l)$ is obtained by replacing $A(\boldsymbol{M}, l)$ by $A_0(\boldsymbol{M}, l)$ on the right hand

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side of (1),

$$\frac{\partial A(M,l)}{\partial l} = \mathrm{e}^{-dl}\left\{\frac{3\mathrm{e}^{(2-\epsilon)l}u(l)M^2}{1+r(l)} - \frac{9}{2}\frac{\mathrm{e}^{(4-2\epsilon)l}u^2(l)M^4}{(1+r(l))^2}\right\}$$
$$+\,\mathrm{e}^{-dl}\left\{\ln(1+r(l)+3\mathrm{e}^{(2-\epsilon)l}u(l)M^2) - \frac{3\mathrm{e}^{(2-\epsilon)l}u(l)M^2}{1+r(l)} + \tfrac{9}{2}\,\mathrm{e}^{(4-2\epsilon)l}\frac{u^2(l)M^4}{(1+r(l))^2}\right., \tag{3}$$

where for simplicity we have set $n = 1$. The first two terms are precisely those which contribute to the equations for $r(l)$ and $u(l)$ and are, therefore, equal to $\partial A_0/\partial l$. If we write $A = A_0 + A_1$, we obtain a closed form expression for $A_1(M,l)$:

$$A_1(M,l) = \int_0^l \mathrm{d}l\,\mathrm{e}^{-dl}\left\{\ln(1+r(l)+3\mathrm{e}^{(2-\epsilon)l}u(l)M^2) - \left[\frac{3\mathrm{e}^{(2-\epsilon)l}u(l)M^2}{1+r(l)} - \tfrac{9}{2}\,\mathrm{e}^{(4-2\epsilon)l}\frac{u^2(l)M^4}{(1+r(l))^2}\right]\right\}. \tag{4}$$

The 1-loop contribution to the free energy is obtained by taking the $l \to \infty$ limit of (4).

Once the free energy is known, the magnetization equation of state is simply obtained by differentiation with respect to $M$. In fact, it is simpler to derive the equation of state by differentiating (1) first and then evaluating the limit $l \to \infty$. We find that the equation of state depends on $r_\infty$ and $u_\infty$ only.

$$h = r_\infty M + u_\infty M^3 + M r_\infty |r_\infty|^{-\epsilon/2}\left[(n-1)u_\infty\left\{\left(1+\frac{M^2u_\infty}{r_\infty}\right)\ln\left|1+\frac{M^2u_\infty}{r_\infty}\right| - \frac{M^2u_\infty}{r_\infty}\right\}\right.$$
$$\left. +\,3u_\infty\left\{\left(1+\frac{3M^2u_\infty}{r_\infty}\right)\ln\left|1+\frac{3M^2u_\infty}{r_\infty}\right| - \frac{3M^2u_\infty}{r_\infty}\right\}\right], \tag{5}$$

for general $n$. In (5), the nonlinear crossover information is contained for *all* values of $r$, not just in the critical region ($r \ll 1$). Since the behavior for large $r$ of the equation of state should not be expected to be universal, we will consider the form of (5) for small $r$. Inserting the expressions for the nonlinear scaling fields $r_\infty$ and $u_\infty$ from [3], we obtain the equation of state for the critical regime:

$$\frac{h}{M^\delta} = \mathrm{sgn}t\,|x|^\gamma + |x|^{\gamma-2\beta} + \frac{\epsilon}{2(n+8)}|x|^{\gamma-2\beta}\{[(n-1)(\mathrm{sgn}t\,|x|^{2\beta}+1)\ln|1+\mathrm{sgn}t\,|x|^{-2\beta}|]$$
$$+\,3\{[(3+\mathrm{sgn}t\,|x|^{2\beta})\ln|1+3\mathrm{sgn}t\,|x|^{-2\beta}|] - \ln 4 + [\ln 4 - \ln 27]\,[1+\mathrm{sgn}t\,|x|^{2\beta}]\}\}. \tag{6}$$

Here, $2\beta = 1 + 3\epsilon/(n+8)$, $\delta = (d+2)/(d-2)$ and $x = t/M^{1/\beta}$. Eq. (5) agrees with [1] with $n = 1$. It is expressed in Griffiths asymptotic form [8].

As pointed out earlier, our generator contains some information from every order in perturbation theory. It will be of interest to investigate if a scheme is available in the differential generator approach similar to the diagram resummation techniques in field theory [1].

Elsewhere, we have generalized (1) to systems with global constraints and arbitrary spin-groupings [9]. We have applied our differential technique to study the global behavior of the equation of state of a compressible ferromagnet with a generalized "Lifshitz" propagator [9, 10].

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