RENORMALIZATION GROUP CALCULATION FOR CRITICAL POINTS OF HIGHER ORDER WITH GENERAL PROPAGATOR[☆]

J.F. NICOLL, G.F. TUTHILL, T.S. CHANG and H.E. STANLEY

Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

Received 9 June 1976

We give first order perturbation results for the critical point exponents at order O critical points with *anisotropic* propagators. The exponent η is calculated to second order for *isotropic* propagators, and all O; $1/n$ expansion results are given for $O = 2$.

Recently, Hornreich, Luban and Shtrikman have used renormalization group techniques to discuss the onset of helical order in magnetic systems [1, 2]. In particular, the existence of new types of critical behavior has been postulated for the "Lifshitz" point where the transition from a uniformly ordered to a helically ordered state occurs. At such a point, the propagator differs from the usual Wilson form, $G^{-1} = k^2 + r$. Refs. [1–2] consider propagators of the form $G^{-1} = k_1^2 + k_2^4 + r$ where k_i is a d_i-dimensional wave vector, and $d_1 + d_2 = d$, the dimension of the lattice.

Here we consider critical propagators ($r = 0$) of the form

$$G^{-1} = \sum_{i=1}^{J} |k_i|^{\sigma_i}, \tag{1}$$

where each k_i is a d_i-dimensional vector, so that $\Sigma_{i=1}^{J} d_i = d$. The σ_i are termed "propagator exponents". Renormalization group techniques are applied to systems described by (1) in a manner parallel to earlier work [3–4].

For an isotropic n-vector Wilson model at an Oth order [5] critical point the borderline dimension d_b (above which mean field behavior holds) is determined by

$$\sum_{i=1}^{J} d_i/\sigma_i = O/(O-1). \tag{2}$$

Catastrophic infrared divergences set in at dimensions below d_{min} at which

$$\sum_{i=1}^{J} d_i/\sigma_i = 1. \tag{3}$$

For $\sigma_i = 2$ these conditions reduce to those given previously [5, 6]. In some cases (such as isotropic propagators [1–3]) d_{min} may be larger than three. A more interesting physical case is obtained if only one component of k enters G^{-1} as k^{2L} and the remaining components have k^2 dependence. Eqs. (2)–(3) then give $d_b = (3O-1)/(O-1) - 1/L$ and $d_{min} = 3 - 1/L$. Thus, we have $d_b \geqslant 3 > d_{min}$ for all $O \leqslant 2L + 1$.

For anisotropic systems, the critical point exponents $\{\eta_i\}$ are defined by examining the behavior of the critical two-point function for a wave-vector lying entirely in one of the d_i-dimensional subspaces:

$$\Gamma_2(k_i) \propto |k_i|^{\sigma_i - \eta_i}. \tag{4}$$

There will also be different values of the correlation length exponent ν_i in each of the subspaces. The following relationships between the exponents hold generally

$$2 - \alpha = \sum_{i=1}^{J} d_i \nu_i; \quad \gamma = (\sigma_i - \eta_i)\nu_i;$$
$$\delta = \frac{\Sigma_{i=1}^{J} d_i/(\sigma_i - \eta_i) + 1}{\Sigma_{j=1}^{J} d_j/(\sigma_j - \eta_j) - 1}. \tag{5}$$

Denoting the largest of the propagator exponents as $\sigma_>$, we define the unperturbed or Gaussian eigenvalues λ_p (corresponding to s^{2p}, cf [3])

$$\lambda_p \equiv \left[\sum_{i=1}^{J} d_i \sigma_> / \sigma_i\right](1-p) + p\sigma_>. \tag{6}$$

☆ Work supported by the Air Force Office of Scientific Research and the National Science Foundation.

The expansion parameter for $d < d_b$ is $\epsilon_O \equiv \lambda_O$. The corrected eigenvalues λ'_p for this general anisotropic case are found to be

$$\lambda'_p = \lambda_p - 2\epsilon_O \langle O, p; p\rangle_n / \langle O, O; O\rangle_n , \tag{7a}$$

where [3, 7]

$$\langle O, p; p\rangle_n \equiv \sum_{j=0}^{[O/2]} \binom{p}{j} \binom{p+\frac{1}{2}n-1}{j} \binom{2p-2j}{O-2j} . \tag{7b}$$

The calculation of the $\{\eta_i\}$ is more difficult except for the somewhat unphysical isotropic case (if $G^{-1} = k^\sigma$, then $d_{\min} = \sigma$). We find that for $\sigma \neq 2L$, there is no shift in the propagator exponent, i.e. $\eta = 0$ to $O(\epsilon_O^2)$. For $\sigma = 2L$ (the generalized Lifshitz point [3]), we find at an Oth order critical point [8]

$$\eta_O = \frac{4(-1)^{L+1}\,\epsilon_O^2 (O\Gamma^2(\frac{1}{2}d_b))\,C_n}{L\binom{2O}{O}^3 \Gamma(\frac{1}{2}d_b - L)\Gamma(\frac{1}{2}d_b + L)} , \tag{8a}$$

with

$$C_n \equiv \left[\frac{\langle O, O; O\rangle_n}{\langle O, O; O\rangle_{n=1}}\right]^2 \prod_{j=1}^{O-1} \frac{2j+n}{2j+1} . \tag{8b}$$

Here, $d_b = 2LO/(O-1)$ and $\epsilon_O = (d_b - d)(O-1)$. For the ordinary critical point ($O = 2$), we write simply $\epsilon_2 = \epsilon = 4L - d$. Eq. (8) reduces to

$$\eta = \frac{(-1)^{L+1}\,\Gamma^2(2L)(n+2)\,\epsilon^2}{\cancel{L\,\Gamma(L)\,\Gamma(3L)\,(n+8)^2}} + O(\epsilon^3). \tag{9}$$

For this case we have also calculated the leading term in the $1/n$ expansion for all L (ref. [2] considered the $L = 2$ case). The result is

$$\eta_2(2L) = \frac{(-1)^{L+1}}{L} \frac{\epsilon \sin \pi\epsilon/2}{\pi/2} \frac{\Gamma(d-2L)\Gamma(2L)}{\Gamma(\frac{1}{2}d+L)\Gamma(\frac{1}{2}d-L)} \frac{1}{n} + O(1/n^2). \tag{10}$$

In eq. (10), ϵ is not restricted to be small. Agreement between (9) and (10) is obtained for $\epsilon \ll 1$ and $n \gg 1$. Work is in progress on the general anisotropic case.

References

[1] R.M. Hornreich, M. Lubon and S. Shtrikman, Phys. Rev. Lett. 35 (1975) 1678.

[2] R.M. Hornreich, M. Luban and S. Shtrikman, Phys. Lett. 55A (1975) 269.

[3] J.F. Nicoll, T.S. Chang and H.E. Stanley, Phys. Rev. A 13 (1976) 1251.

[4] K.G. Wilson and J. Kogut, Phys. Rep. 12 (1974) 73 and references contained therein.

[5] T.S. Chang, G.F. Tuthill and H.E. Stanley, Phys. Rev. B9 (1974) 4882.

[6] J.F. Nicoll, T.S. Chang and H.E. Stanley, Phys. Rev. Lett. 33 (1974) 427.

[7] F.J. Wegner, Phys. Lett. 54A (1975) 1.

[8] The calculations are a simple extension of the method described in G.F. Tuthill, J.F. Nicoll and H.E. Stanley, Phys. Rev. B11 (1975) 4579.