Percolation of networks with directed dependency links

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The self-consistent probabilistic approach has proven itself powerful in studying the percolation behavior of interdependent or multiplex networks without tracking the percolation process through each cascading step. In order to understand how directed dependency links impact criticality, we employ this approach to study the percolation properties of networks with both undirected connectivity links and directed dependency links. We find that when a random network with a given degree distribution undergoes a second-order phase transition, the critical point and the unstable regime surrounding the second-order phase transition regime are determined by the proportion of nodes that do not depend on any other nodes. Moreover, we also find that the triple point and the boundary between first- and second-order transitions are determined by the proportion of nodes that depend on no more than one node. This implies that it is maybe general for multiplex network systems, some important properties of phase transitions can be determined only by a few parameters. We illustrate our findings using Erdős-Rényi networks.

I. INTRODUCTION

Complex networks science has become an effective tool for modeling complex systems. It treats system entities as nodes and the mutually supporting or cooperating relations between the entities as connectivity links [1–12]. In many systems, nodes that survive and fail together form dependency groups through dependency links. Dependency links denote the damaging or destructive relations among entities [13–21]. Compared to ordinary networks [5,6,10], networks with dependency groups or links are more vulnerable and subject to catastrophic collapse [22,23]. The previous works have studied the network system in which the dependency groups, with sizes either fixed at two [22] or characterized by different classic distributions [23], are formed through undirected dependency links. The outcome when the dependency links are *directed*, however, is more general. For example, in a financial network where each company has trading and sales connections (connectivity links) with other companies, the connections enable the companies to interact with others and function together as a global financial market, and companies that belong to the same corporate group strongly depend on the parent company (i.e., there are directed dependency links), but the reverse is not true [24]. Another example is in a social network in which people (followers) follow trends set by celebrities (pioneers), e.g., popular singers and actors, but the reverse is not true [25].

We use a self-consistent probabilistic framework [26–29] to study the percolation phase transitions in a random network A with both connectivity and directed dependency links. Randomly removing a fraction $1 - p$ of nodes in network A causes (i) connectivity links to be disconnected, causing some nodes and clusters to fail due to the disconnection to the network giant component (percolation process), and (ii) failing

nodes to make their dependent nodes to also fail even though they are still connected to the network giant component via connectivity links (dependency process). Thus, the removal of nodes in the percolation process leads to the failure of dependent nodes in the dependency process, which in turn initiates a new percolation process, which further sets off a dependency process, and so on. We show that this synergy between the percolation process and the dependency process leads to a cascade of failures that continues until no further nodes fail (see Fig. 1).

To fully capture the structure of network A, we introduce the degree distribution $P(k)$ and, in addition, the directed dependency degree distribution $Q(k_o)$, which is the probability that a randomly chosen node has k_o directed dependency links connecting to k_o nodes, which are supporting this chosen node. In our model, when i depends on k_o nodes, we assume that if any one of these k_o nodes fails, node i will fail too (see Fig. 1). Usually, this kind of multiplex has both first- and second-order phase transitions [22,23]. Here we find that $Q(k_o)$ strongly affects the robustness of network A. Specifically, the percolation threshold p_c^{II}, at which network A disintegrates in a form of second-order phase transition, is determined solely by $Q(0)$ for a given $Q(k_o)$, and $Q(0) + Q(1)$ characterizes the boundary between the first-order phase transition and the second-order phase transition regime.

This paper is organized as the follows. In Sec. II we introduce the general framework and develop the analytic formulas to solve the influence of $Q(k_o)$ on the percolation properties of a random network. In Sec. III, we demonstrate these influences using an ER network.

II. GENERAL FRAMEWORK

For a random network A of size N with both connectivity links and directed dependency links [see Fig. 1(a)], as in Ref. [5], we introduce the generating function $G_0(z)$ of the
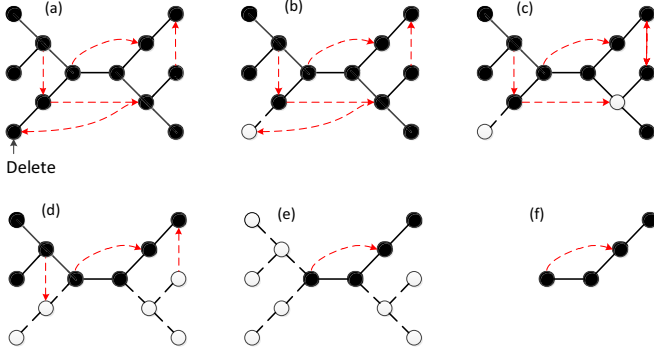
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FIG. 1. Demonstration of the synergy between the percolation process and the dependency process that leads to a cascade of failures. The network contains two types of links: connectivity links (solid black lines) and directed dependency links (dashed red arrows). (a)→(b) Initial failure: a random node is removed. (c)→(e) Synergy between percolation process and dependency process: nodes cutoff from the giant component or depending on failed nodes are removed. (f) Steady state: the surviving giant component contains four nodes.

degree distribution $P(k)$,

$$G_0(z) = \sum_k P(k)z^k. \tag{1}$$

Analogously, we have the generating function of the related branching processes [5],

$$G_1(z) = \frac{G_0'(z)}{G_0'(1)} = \sum_k \frac{kP(k)}{\langle k \rangle} z^{k-1}. \tag{2}$$

Similarly, we introduce the generating function for the directed dependency degree distribution $Q(k_o)$ as

$$D(z) = \sum_{k_o} Q(k_o)z^{k_o}. \tag{3}$$

We designate $h(s)$ the probability distribution of the number of nodes approachable along the directed dependency links starting from a randomly chosen node in network A. This allows us to write the generating function $H(z)$ for $h(s)$, i.e.,

$$H(z) = \sum_s h(s)z^s. \tag{4}$$

According to Ref. [30], $H(z)$ also satisfies a self-consistent condition of the form

$$H(z) = zD[H(z)]. \tag{5}$$

A random removal of a fraction $1 - p$ of nodes triggers a cascade of failures. When no more nodes fail, network A reaches its final steady state. At this steady state, we use the probabilistic approach [29] and *define x to be the probability that a randomly chosen connectivity link leads to the giant component at one of its ends*. If we randomly choose a connectivity link l and find an arbitrary node n by following l in an arbitrary direction, the probability that node n has degree k is

$$\frac{kP(k)}{\sum_k kP(k)} = \frac{kP(k)}{\langle k \rangle}. \tag{6}$$

For node n, the root of a directed cluster of size s, to be part of the giant component, at least one of its other $k - 1$ outgoing connectivity links (other than the link first chosen) leads to the giant component, provided that every other $s - 1$ node is also in the giant component because the disconnection of any one of these $s - 1$ nodes to the giant component will cause node n to lose support and fail. Computing this probability, we can write out the self-consistent equation for x as

$$x = p\left\{ \sum_k \frac{kP(k)}{\langle k \rangle}[1 - (1-x)^{k-1}] \right\}$$
$$\times \sum_s \left\{ h(s)\left\{ p\sum_k P(k)[1 - (1-x)^k] \right\}^{s-1} \right\}, \tag{7}$$

where p is the probability that a node survives the initial removal process, $1 - (1-x)^{k-1}$ is the probability that at least one of the other $k - 1$ connectivity links of node n leads to the giant component, $h(s)$ is the probability that node n is the root of a directed cluster of size s, and $\{p\sum_k P(k)[1 - (1-x)^k]\}^{s-1}$ is the probability that every other $s - 1$ node in the directed cluster supporting node n is also in the giant component. Using the generating functions defined in Eqs. (1), (2), and (4), we transform Eq. (7) into the compact form

$$x = \frac{1 - G_1(1-x)}{1 - G_0(1-x)}H\{p[1 - G_0(1-x)]\}, \tag{8}$$

which, by viewing $p[1 - G_0(1-x)]$ as a whole and using the property of $H(z)$ outlined in Eq. (5), can also be written as

$$x = p[1 - G_1(1-x)]D(H\{p[1 - G_0(1-x)]\}) \equiv F(x,p). \tag{9}$$

For a given p, x can be numerically calculated through iteration with a proper initial value.

Correspondingly, using similar arguments, the probability $P_\infty(p)$ that a randomly chosen node n in the steady state of network A is in the giant component is

$$P_\infty(p) = p\left\{ \sum_k P(k)[1 - (1-x)^k] \right\}$$
$$\times \sum_s h(s)\left\{ p\sum_k P(k)[1 - (1-x)^k] \right\}^{s-1}$$
$$= H\{p[1 - G_0(1-x)]\}, \tag{10}$$

where $1 - (1-x)^k$ is the probability that at least one of the k connectivity links of node n leads to the giant component. Note that $P_\infty(p)$ is also the normalized size of the giant component of network A at the steady state.

We find that there is no giant component at the steady state of network A, i.e., $P_\infty(p) = 0$ when p is smaller than a critical probability p_c^{II} and above the threshold, the giant component appears and its size increases continuously from 0 as p increases. This is typical second-order phase transition behavior and as $p \to p_c^{II}$, $P_\infty(p_c^{II}) = H\{p_c^{II}[1 - G_0(1-x)]\} \to 0$, which suggests $x \to 0$. Thus we can take the Taylor expansion of Eq. (9) with $x \to 0$ to obtain p_c^{II} as (see

Appendix A)

$$p_c^{II} = \frac{1}{Q(0)G_1'(1)} = \frac{\langle k \rangle}{Q(0)\langle k(k-1) \rangle}, \quad (11)$$

which is consistent with our previous result reported in Ref. [25] and depends on $Q(0)$ only but not any other terms from $Q(k_o)$.

In some cases, however, there is no giant component at the steady state of network A, i.e., $P_\infty(p) = 0$ when p is smaller than a critical probability p_c^I but above the threshold, the giant component suddenly appears and its size increases abruptly from 0 as p increases. This is typical first-order phase transition behavior. When $p = p_c^I$, the straight line $y = x$ and the curve $y = F(x, p)$ from Eq. (9) will tangentially touch each other at (x_c, x_c) [25]. Thus, the condition corresponding to the first-order transition is that the derivatives of both sides of Eq. (9) with respect to x are equal,

$$1 = \frac{dF(x, p)}{dx}|x = x_c, p = p_c^I. \quad (12)$$

Due to the complexity of Eqs. (9) and (12), numeric methods are generally used to get p_c^I.

Note $p_c^I = p_c^{II}$ corresponds to the case where the phase transition changes from first order to second order when the conditions for both the first- and second-order transitions are satisfied simultaneously. By substituting p_c^{II} from Eq. (11) into Eq. (12) and further evaluating x, we obtain the boundary between the first-order and second-order phase transitions, which is characterized by (see Appendix B)

$$Q(1) = \frac{Q(0)G_1''(1)}{2G_0'(1)} = \frac{Q(0)\langle k(k-1)(k-2) \rangle}{2\langle k \rangle^2}. \quad (13)$$

Thus, the boundary between first- and second-order transitions is determined only by the proportion of nodes that do not depend on more than one node, i.e., the boundary is solely determined by $Q(0)$ and $Q(1)$ but not any other terms from $Q(k_o)$. This implies that the triple point—the intersection of first-order phase transition, second-order phase transition, and the unstable regime is also determined by $Q(0)$ and $Q(1)$.

When removing any fraction of nodes results in the total collapse of network A, i.e., when $p_c^{II} \geqslant 1$, the network is unstable. By requiring $p_c^{II} = 1$ and using Eq. (11), we can obtain the boundary between the second-order phase transition and the unstable state,

$$Q(0) = \frac{1}{G_1'(1)} = \frac{\langle k \rangle}{\langle k(k-1) \rangle}, \quad (14)$$

which depends solely on the proportion of nodes that do not depend on other nodes at all, i.e., $Q(0)$.

Similarly, by requiring $p_c^I = 1$ in Eq. (12), we use numerical calculations to find the boundary between the first-order phase transition and the unstable state. Therefore, the complete boundary between the unstable state and the phase transition state is achieved by joining these two boundaries together. Moreover, substituting Eq. (13) into Eq. (14), we could obtain the explicit formula of the triple point, which is the intersection of these two boundaries:

$$Q(1) = \frac{\langle k(k-1)(k-2) \rangle}{2\langle k \rangle \langle k(k-1) \rangle}. \quad (15)$$

Note that for scale-free networks with power-law degree distribution $P(k) \propto k^{-\gamma}$ and $\gamma \in (2,3]$, both $\langle k(k-1) \rangle$ and $\langle k(k-1)(k-2) \rangle$ are divergent. This implies that $p_c^{II} = 0$ for any $Q(0)$ according to Eq. (11) and the regime of the second-order phase transition shrinks toward the origin. Thus, for scale-free networks, the situation becomes a little bit simple. Therefore, if $Q(0) > 0$ one could always see the second-order phase transition with $p_c^{II} = 0$ and if $Q(0) = 0$ the system undergoes unstable or first-order phase transition.

III. RESULTS ON ER NETWORKS

Section II provided the general framework for random networks with an arbitrary degree distribution $P(k)$. We here illustrate it using an Erdős Rényi (ER) network [31–33] with a Poisson degree distribution $P(k) = e^{-\langle k \rangle} \langle k \rangle^k / k!$, where $\langle k \rangle$ is the average degree. We choose this network because it is representative of random networks, and the generating function corresponding to the degree distribution $P(k)$ is $G_0(z) = e^{\langle k \rangle(z-1)}$.

A. Second-order phase transitions

Plugging $G_1'(1) = \langle k \rangle$ into Eq. (11), we get the second-order phase transition point p_c^{II},

$$p_c^{II} = \frac{1}{Q(0)\langle k \rangle}. \quad (16)$$

Therefore, for ER networks, the critical point of second-order phase transition is indeed determined solely by $Q(0)$ and its average degree. We support our analytical results by simulations. We choose $\langle k \rangle = 8$ and $D(z) = Q(0) + Q(1)z + Q(2)z^2$ with $Q(0)$ fixed at 0.4 and $Q(1), Q(2)$ tunable. Figure 2 shows the size of the giant component $P_\infty(p)$ as a function of p with the given $\langle k \rangle$ and $D(z)$. Note that in all cases simulation results (symbols) agree well with numerical results (dotted
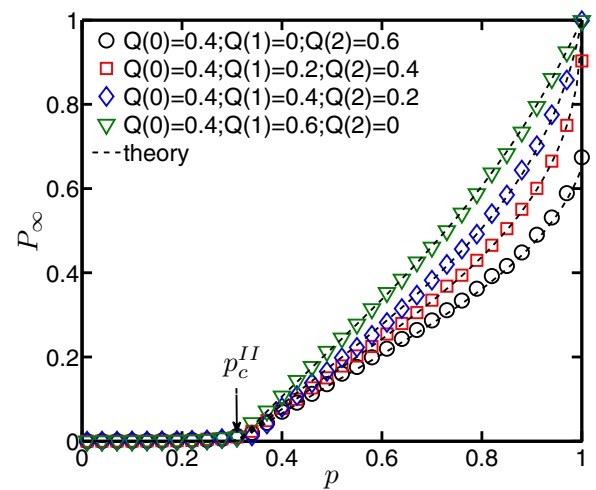

FIG. 2. The size of the giant component $P_\infty(p)$, as a function of the fraction of nodes that remain after random removal, p, for ER networks with $\langle k \rangle = 8$ and $D(z) = Q(0) + Q(1)z + Q(2)z^2$. The symbols represent simulation results of 10^4 nodes and the dashed lines show the theoretical predictions from Eq. (10). The percolation threshold p_c^{II} is uniquely determined by $Q(0)$.

lines) and the curves of $P_\infty(p)$ converge at a fixed value of $p_c^{II} = 0.3125$ as predicted by Eq. (16). This convergence of $P_\infty(p)$ curves is possible because p_c^{II} is determined solely by $Q(0)$, which is fixed to be 0.4 in Fig. 2. Note that if there is no directed dependency links in the network, i.e., $Q(0) = 1$, we will get $p_c^{II} = 1/\langle k \rangle$, which is consistent with the well-known result obtained in Ref. [3].

B. First-order phase transitions

When networks have a greater proportion of directed dependency links, an abrupt transition can occur instead of a continuous transition demonstrated in Fig. 2. To get the p_c^I for the onset of this abrupt transition, we equate the derivatives of both sides of Eq. (9) with respect to x, i.e.,

$$1 = \frac{d\{p(1 - e^{-\langle k \rangle x})D[H(p(1 - e^{-\langle k \rangle x})\}}{dx}\Big|_{x=x_c, p=p_c^I}, \quad (17)$$

where we used the equtions $G_0(z) = G_1(z) = e^{\langle k \rangle(z-1)}$. Using Eqs. (9) and (17), we apply numerical methods to get p_c^I.

With $D(z) = Q(0) + Q(1)z + Q(2)z^2$, Fig. 3 shows the size of the giant component $P_\infty(p)$ as a function of p by comparing simulation results and theoretical predictions. Note that they agree with each other very well. Figure 3 shows that with $\langle k \rangle = 5$ and $Q(0) + Q(1) = 1$, when $Q(0) = 0.4, P_\infty(p)$ undergoes a second-order phase transition at $p_c^{II} = 0.5$ ($\square$), but when $Q(0) = 0.2, P_\infty(p)$ exhibits behavior of a first-order phase transition at p_c^I, satisfying Eq. (17) ($\star$). In addition, when $\langle k \rangle = 10, Q(0) = 0.2, Q(1) = 0.7$ and $Q(2) = 0.1, P_\infty(p)$ undergoes a second-order phase transition at $p_c^{II} = $

0.5 ($\bigcirc$), but when $Q(0) = 0.1, Q(1) = 0.8$ and $Q(2) = 0.1$, $P_\infty(p)$ undergoes a first-order phase transition at p_c^I predicted by Eq. (17) ($\triangledown$).

C. Boundaries of phase diagram

We fix the average degree $\langle k \rangle$ and from Eq. (16) we conclude that the smaller the $Q(0)$ in the network, the bigger the p_c^{II} value. If $Q(0)$ is properly small that $p_c^{II} \approx 1$, which corresponds to the case in which the removal of any fraction of nodes causes a second-order phase transition that totally disintegrates network A. Thus, by requiring $p_c^{II} = 1$, and using Eq. (16) we obtain the boundary between the second-order phase transition and the unstable state,

$$\frac{1}{\langle k \rangle Q(0)} = 1. \quad (18)$$

In addition, using Eq. (13), we obtain the boundary between the first-order and second-order phase transitions of network A,

$$Q(1) = \frac{\langle k \rangle Q(0)}{2}. \quad (19)$$

Using $D(z) = Q(0) + Q(1)z + Q(2)z^2 + Q(3)z^3$ where $Q(0) + Q(1) = 0.9$ and $\langle k \rangle = 10$, Fig. 4 plots $P_\infty(p_c)$ as a function of $Q(1)$ by comparing simulation and numerical results. The critical value of $Q(1)_c$ falls onto $Q(1)_c = 0.75$ as predicted by Eq. (19), delimiting two different transition regimes. Specifically, if $Q(1) < 0.75$, $P_\infty(p_c) = 0$, which indicates the presence of a second-order phase transition, but if $Q(1) > 0.75$, $P_\infty(p_c) > 0$, which indicates the presence of a first-order phase transition. We also consider a special case in which $Q(0) + Q(1) = M$ and use Eq. (19) to determine
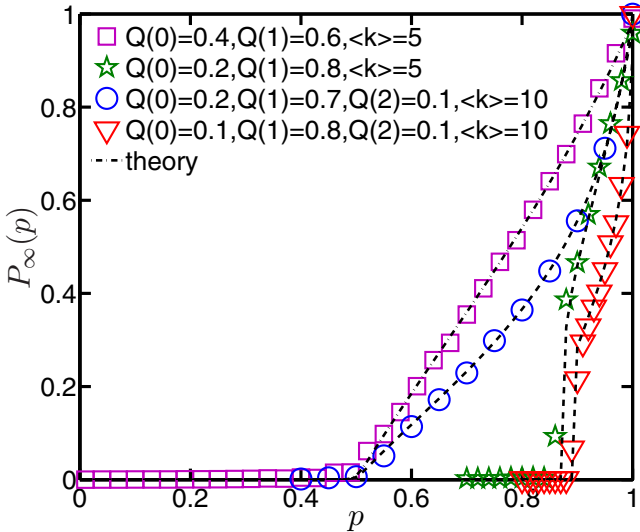

FIG. 3. The size of the giant component $P_\infty(p)$ as a function of the fraction of nodes that remain after random removal, p, for ER networks. Here we used $D(z) = Q(0) + Q(1)z$ with $\langle k \rangle = 5$ ($\square$ and $\star$) and $D(z) = Q(0) + Q(1)z + Q(2)z^2$ with $\langle k \rangle = 10$ ($\bigcirc$ and $\triangledown$). The symbols represent simulation results of 10^4 nodes and the dashed lines are the theoretical predictions from Eq. (10). With a relatively larger $Q(0)$, the network undergoes a second-order phase transition at p_c^{II}, which only depends on $Q(0)$. However, for relatively smaller $Q(0)$ and larger $Q(1)$ and $Q(2)$, the network undergoes a first-order phase transition.
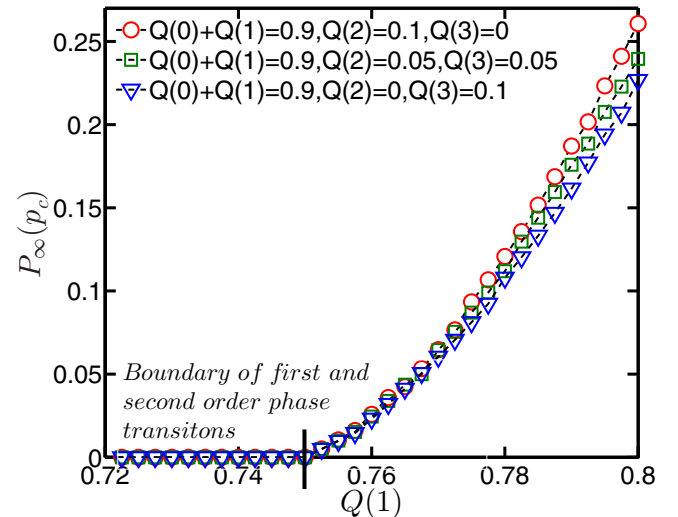

FIG. 4. Comparison between simulation (symbols) and theory (lines) for $P_\infty(p_c)$ as a function of $Q(1)$ for different $D(z)$ ($D(z) = Q(0) + Q(1)z + Q(2)z^2 + Q(3)z^3$) while keeping $Q(0) + Q(1) = 0.9$ and $\langle k \rangle = 10$. At the first-order phase transition point p_c^I, $P_\infty(p_c)$ is nonzero; whereas at the second-order phase transition point p_c^{II} and $P_\infty(p_c)$ is zero. From Eq. (19) the boundary between first-order phase transition and second-order phase transition is only dependent on $Q(0) + Q(1)$, thus at $Q(1)_c = 0.75$ for this case.
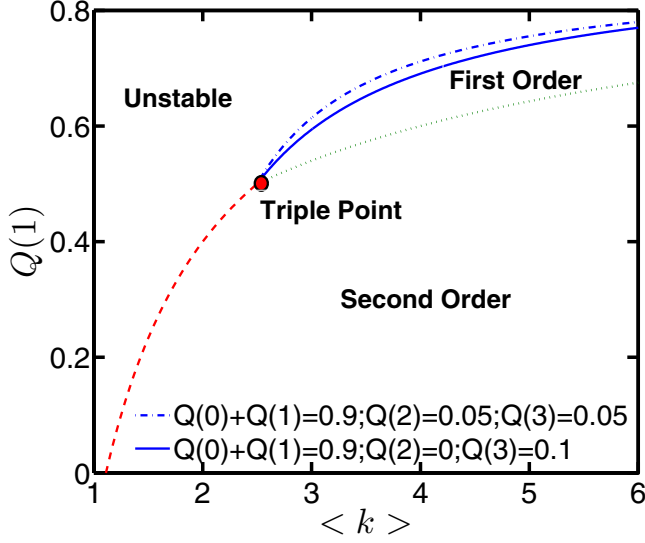
FIG. 5. The fraction of nodes that have one dependent node $Q(1)$ as a function of the average degree $\langle k \rangle$ with $Q(0) + Q(1) = \frac{9}{10}$. The dashed lines are theoretical results obtained from Eqs. (20) (green) and (21) (red) with intersection points at $(\frac{2}{2M-1}, \frac{1}{2})$. The dashed blue line is the boundary between first-order phase transition and unstable system, obtained numerically. Here the dashed red and green lines only depend on m_0, whereas the blue lines (both solid and dashed) depend on the specific details of $Q(k_o)$ other than M.

the boundary between the first-order phase transition and the second-order phase transition,

$$Q(1) = \frac{M \langle k \rangle}{\langle k \rangle + 2}. \qquad (20)$$

In addition, in terms of M, Eq. (18) delivers the boundary between the second-order phase transition and unstable state,

$$Q(1) = \frac{M \langle k \rangle - 1}{\langle k \rangle}. \qquad (21)$$

Thus, in the coordinate system of $\langle k \rangle$-$Q(1)$, using Eqs. (20) and (21) we can plot the phase diagram of network A under random failures, with these two boundaries converging at the triple point $(\frac{2}{2M-1}, \frac{1}{2})$ (the solid red dot in Fig. 5). Because $\langle k \rangle > 0$ always holds, when $M \leqslant \frac{1}{2}$ this intersection point is nonphysical, indicating that the network will not be subject to first-order phase transitions under attack regardless of the form of $P(k)$, but if $M > \frac{1}{2}$, the network will be subject to first-order phase transitions.

Figure 5 shows the boundaries in the phase diagram with $M = \frac{9}{10} > \frac{1}{2}$, where the boundaries between first-order phase transitions and the unstable state are determined numerically. Note that, when M is fixed, the boundary between the second-order phase transition and the unstable state (dashed red line) as well as the boundary between the first-order and second-order phase transitions (dashed green line) are also fixed because they depend only on M, but the boundary between the first-order phase transition and the unstable state (dashed blue line) is subject to the details of $Q(k_o)$. For example, when $Q(0) + Q(1) = \frac{9}{10}$, a shuffle of the remaining terms in $Q(k_o)$ causes a shift in the boundary line, shown as the displacement of the solid blue line to the dashed blue line in Fig. 5.

IV. CONCLUSIONS

In summary, we present an analytical formalism for studying random networks with both connectivity links and directed dependency links under random node failures. Using a probabilistic approach, we find that the directed dependency links greatly reduce the robustness of a network. We show that the system disintegrates in a form of second-order phase transition at a critical threshold and the boundary between second-order phase transition and unstable regimes solely determined by the proportion of nodes that do not depend on other nodes. Our framework also provides the solution for the boundary between the first-order and second-order phase transitions, which is characterized by the proportion of nodes that depend on no more than one node.

APPENDIX A

If $p \to p_c^{II}$, $x \to 0$. From Eq. (9) we have

$$1 - G_1(1 - x) = G_1'(1)x - \frac{G_1''(1)}{2}x^2 + O(x^3), \qquad (A1)$$

$$1 - G_0(1 - x) = G_0'(1)x - \frac{G_0''(1)}{2}x^2 + O(x^3), \qquad (A2)$$

and

$$D(H\{p[1 - G_0(1 - x)]\})$$
$$= Q(0) + pQ(1)h(1)G_0'(1)x + O(x^2). \qquad (A3)$$

Using Eqs. (A1), (A2), and (A3), we can write Eq. (9) as

$$x = pQ(0)G_1'(1)x + p[pQ(1)h(1)G_0'(1)G_1'(1)$$
$$- \frac{Q(0)G_1''(1)}{2}]x^2 + O(x^3). \qquad (A4)$$

Since $x \in (0,1)$, we can divide both sides of Eq. (A4) by x and obtain

$$1 = pQ(0)G_1'(1) + p[pQ(1)h(1)G_0'(1)G_1'(1)$$
$$- \frac{Q(0)G_1''(1)}{2}]x + O(x^2). \qquad (A5)$$

As $x \to 0$, taking the limits of both sides of Eq. (A5) we get

$$p_c^{II} = \frac{1}{Q(0)G_1'(1)}. \qquad (A6)$$

APPENDIX B

Putting Eq. (A6) back into Eq. (A5), we get

$$\frac{Q(1)h(1)G_0'(1)}{Q(0)}x = \frac{Q(0)G_1''(1)}{2}x + O(x^2). \qquad (B1)$$

To simplify Eq. (B1), we first take the derivatives of both sides of Eq. (5) with respect to x and obtain

$$H'(z) = D[H(z)] - z \frac{\partial\{D[H(z)]\}}{\partial[H(z)]} H'(z). \quad \text{(B2)}$$

Plugging $z = 0$ into Eq. (B2), we get $H'(0) = D[H(0)] = D(0) = Q(0)$. Using Eq. (4), we easily obtain $H'(0) = h(1)$, and thus $h(1) = Q(0)$, which would reduce Eq. (B1) as

$$Q(1)x = \frac{Q(0)G_1''(1)}{2G_0'(1)} x + O(x^2). \quad \text{(B3)}$$

Up to this point, if $x \to x_t = 0$, network A undergoes a second-order phase transition and thus Eq. (B3) clearly holds, but if $x \to x_t \neq 0$, network A undergoes a first-order phase transition. On the boundary between the first-order and the second-order phase transitions, we get a nonzero x_t, but it is negligibly small. Here, we can treat $O(x_t) \approx 0$ and obtain the condition characterizing this boundary as

$$Q(1) = \frac{Q(0)G_1''(1)}{2G_0'(1)}. \quad \text{(B4)}$$

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