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LETTER TO THE EDITOR

Ghost fields, pair connectedness, and scaling: exact results in one-dimensional percolation†

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Abstract. The percolation problem is solved exactly in one dimension. The functions obtained bear a strong resemblance to those of the n-vector model on the same lattice. Further, we include a ghost field exactly in all dimensions d, thereby treating the 'thermodynamics' of percolation without appealing to the Potts model. In particular, we show for $d = 1$ that the nature of the singularities near the critical percolation probability, $p_c = 1$, is described by $\alpha_p = \gamma_p = 1$, $\beta_p = 0$, and $\delta_p = \infty$. We also calculate the pair connectedness and correlation length explicitly, and find $\eta_p = \nu_p = 1$, in agreement with the hyperscaling relation $d\nu_p = 2 - \alpha_p$. Finally, scaling is demonstrated for both the cluster size distribution and the percolation function analogous to the Gibbs free energy, and the scaling powers are explicitly evaluated; in particular, we find the exponents $\sigma = 1$ and $\tau = 2$.

1. Introduction

Although the percolation problem has been studied extensively in recent years—particularly in $d = 2$ and 3 dimensions—little attention has been paid to the one-dimensional problem (see, e.g., Broadbent and Hammersley 1957, Shante and Kirkpatrick 1971, Essam 1972). The reason that many workers have felt one dimension 'uninteresting' is that percolation occurs only when the lattice is completely full. That is, the critical value for the probability, p, that a site (or bond) is occupied, is unity. Thus the location of the critical point is trivial, just like for thermal phase transitions in $d = 1$. However, the precise form of the thermodynamic functions, and the nature of their singularities, is interesting. In this Letter we show the same is true for percolation.

First we will derive the functions of interest to percolation directly‖, and then we will re-derive these as the *analogues* of thermodynamic functions from a Gibbs potential $G(p, h)$, with h a 'ghost field'. In this way we find all the exponents to have their usual meanings. Moreover, we show explicitly that $G(p, h)$ is a generalised homogeneous function in $(p - p_c)$ and h, and that all critical point exponents are related by the usual scaling laws. We also calculate the pair connectedness and

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‖ Wortis (1974) has solved the dilute Ising model exactly in $d = 1$ dimension. His results, in the limit of zero magnetic field and zero temperature, implicitly solve the one-dimensional percolation problem for the case of zero 'ghost field'. Stauffer and Jayaprakash (unpublished) also use this limit of the dilute Ising model to study one-dimensional percolation.

correlation length explicitly, and demonstrate the analogue of the fluctuation–dissipation theorem for percolation (Essam 1971). Finally, we discuss cluster size scaling.

In percolation, the fundamental quantities of interest are the mean number of clusters (normalised per site) of a given size s, which we shall denote $\langle n_s \rangle$†. All the 'thermodynamic' percolation functions can be determined from a knowledge of these $\langle n_s \rangle$ for all s. The mean number of finite clusters of any size is given by

$$G(p) = \sum_s{}' \langle n_s \rangle, \tag{1}$$

where the prime on the sum denotes the exclusion of the infinite cluster (if one exists). The unrestricted first moment of $\langle n_s \rangle$ provides an important sum rule:

$$\sum_s s\langle n_s \rangle = \frac{\text{total number of occupied sites contained in all clusters on the lattice}}{\text{total number of sites on the lattice}}$$

$$= \begin{cases} p & \text{(site percolation)} \\ 1 & \text{(bond percolation).} \end{cases} \tag{2}$$

We may write $\Sigma_s s\langle n_s \rangle = \Sigma'_s s\langle n_s \rangle +$(fraction of sites which belong to the infinite cluster). Hence we see immediately that $P(p)$, the fraction of *occupied* sites that belong to the infinite cluster, is given by

$$P(p) = 1 - \frac{\Sigma'_s s\langle n_s \rangle}{\Sigma_s s\langle n_s \rangle}, \tag{3}$$

where the denominator is given by equation (2). The quantity $s\langle n_s \rangle$ is clearly the probability P_s that a given site in the lattice belongs to a cluster of size s. Thus, following Fisher and Essam (1961), we define the mean size of the (finite) clusters as

$$S(p) \equiv \frac{\Sigma'_s sP_s}{\Sigma'_s P_s} = \frac{\Sigma'_s s^2 \langle n_s \rangle}{\Sigma'_s s\langle n_s \rangle}. \tag{4}$$

2. Direct solution in one dimension

To proceed we need to know $\langle n_s \rangle$. On an infinite linear chain there is exactly one embedding per site of a cluster of size s. Because an s-site cluster must contain exactly s occupied sites and two unoccupied ends, we find that

$$\langle n_s \rangle = \begin{cases} q^2 p^s & \text{(site)} \\ q^2 p^{s-1} & \text{(bond)}‡, \end{cases} \tag{5}$$

where $q \equiv 1-p$.

To evaluate $G(p)$ in (1), we need to specify the range of the sum. In bond percolation a one-site cluster occurs when two adjacent bonds are both vacant, as illustrated here - - - ● - - -. Hence $\langle n_s \rangle = q^2$ for $s=1$. There can be no clusters of

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† The size s here is defined as the 'site size', which is the number of sites in a cluster, even for bond percolation. We can equally well define a quantity $\langle n_b \rangle$, for either site or bond percolation, which is the mean number of clusters of 'bond size' b.

‡ In bond percolation, p is the bond occupation probability; however we shall still count 'site size'. If we wished to count 'bond size', then $\langle n_b \rangle = q^2 p^{b-1}$ for the site problem, while for the bond problem $\langle n_b \rangle = q^2 p^b$.

zero sites, and therefore

$$G(p)=\sum_{s=1}^{\infty} q^2p^{s-1}=1-p \qquad \text{(bond)}. \tag{6a}$$

However, in one dimension the site and bond problems are isomorphic, and therefore we expect the mean number of clusters, $G(p)$, to be the same for both. But from equation (5) we see that if the lower limit on the sum in (1) is $s=1$ for the bond problem, then for the site problem the lower limit must be $s=0$, and thus we must consider *zero-site clusters* in site percolation. That such clusters exist is also easily seen: when two adjacent sites are vacant we have a cluster made simply of one bond (no sites) joining the two vacant sites (thus ○—○). Hence for site percolation $\langle n_s\rangle = q^2$ for $s=0$, and

$$G(p)=\sum_{s=0}^{\infty} q^2p^{s}=1-p \qquad \text{(site)}. \tag{6b}$$

The fraction of occupied sites that belongs to the infinite cluster is (cf equations (2), (3), and (5))

$$P(p)=\begin{cases} 1-\dfrac{q^2\Sigma'_s sp^s}{p} & \text{(site)} \\[2ex] 1-q^2\displaystyle\sum_s{}' sp^{s-1} & \text{(bond),} \end{cases} \tag{7a}$$

and thus for both site and bond percolation,

$$P(p)=1-q^2\sum_s{}' sp^{s-1}=\begin{cases} 0 & p<1 \\ 1 & p=1. \end{cases} \tag{7b}$$

The mean cluster size is (cf equations (4) and (5))

$$S(p)=\frac{q^2\Sigma'_s s^2p^s}{q^2\Sigma'_s sp^s}=\frac{1+p}{1-p} \qquad \text{(site or bond)}. \tag{8}$$

It is interesting to note that the thermal susceptibility of the $d=1$, n-vector model (Stanley 1969), is identical to $S(p)$ with p replaced by y_n, the nearest-neighbour two-spin correlation function at $H=0$.

From (6)–(8) it follows that

$$\alpha_{\mathrm{p}}=1, \qquad \beta_{\mathrm{p}}=0, \qquad \gamma_{\mathrm{p}}=1, \qquad \text{and} \qquad p_{\mathrm{c}}=1, \tag{9}$$

where the percolation exponents are defined by $G(p)\sim\epsilon^{2-\alpha_{\mathrm{p}}}$, $P(p)\sim(-\epsilon)^{\beta_{\mathrm{p}}}$ and $S(p)\sim\epsilon^{-\gamma_{\mathrm{p}}}$, with $\epsilon\equiv(p_{\mathrm{c}}-p)/p_{\mathrm{c}}$. Equation (9) is useful for comparison with renormalisation group calculations (Reynolds *et al* 1977, Marland and Stinchcombe 1977).

3. The generating function and 'thermodynamics'

Just as in thermodynamics the Gibbs free energy is the 'generating function' from which one derives all other thermodynamic functions by taking appropriate derivatives with respect to either the temperature T or the magnetic field H, G is the generating function for the percolation problem†. The role of T as a randomising or disordering field is played by $q = 1-p$. It is natural to ask what plays the role of H. For the thermal problem, the 'ghost spin' (Griffiths 1967) may be used as a means of including H. This idea has been carried over to the percolation problem by Kasteleyn and Fortuin (1969). What we shall do reduces to the Kasteleyn and Fortuin definition, though it is somewhat more general, applying also to problems which are not directly related to the Potts model (e.g. bond percolation with bond size counting or site percolation with either site size or bond size counting).

First, change the original percolation problem slightly by adding one additional 'ghost site' somewhere, not on the lattice (see figure 1). Then connect bonds from the ghost site to each lattice site‡ with a probability h. This fundamentally changes the connectivity of the lattice for $h \neq 0$. In particular we have an infinite cluster even for $p < p_c$ (i.e. all the sites that have an h-bond to the ghost site are connected through the ghost site). This is precisely the effect of turning on a magnetic field in a thermal problem, thereby creating a magnetisation even for $T > T_c$. (Unlike the usual thermal problems, however, in percolation we cannot turn on the field in a direction to destroy the order. This can be interpreted as there being only one direction for the field, which is as it must be, since percolation is the $Q \to 1$ limit of the Q-state Potts model.)

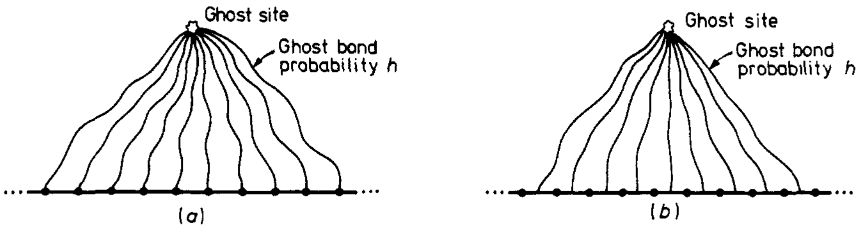

Figure 1. The ghost-site joins to the lattice with ghost bonds. These are randomly 'connective' with probability h, and 'non-connective' with probability $1-h$. In (a) the ghost bonds join to the sites of the lattice, thereby keeping track of the cluster 'site size', while in (b) they join to the bonds of the lattice, and so keep track of cluster 'bond size'. In both (a) and (b), the lattice itself may be either site or bond connective—corresponding at $h = 0$ to the usual site or bond percolation problems.

We now examine the effect of h on the $\langle n_s \rangle$. For a cluster to contain s sites it must have the same structure as before, and in addition must have *no* h-bonds connected to the ghost site (or else it would be part of the infinite cluster). Thus the total number of finite clusters is given by

$$G(p, h) = \sum_s{}' \langle n_s \rangle = \sum_s{}' \langle n_s \rangle_{h=0} (1-h)^s, \tag{10}$$

since $(1-h)^s$ is the probability that *none* of the sites in an s-site cluster is connected to

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† Coniglio (1976) defines a generating function $\tilde{G}(x) = \Sigma_s \langle n_s \rangle x^s$. Earlier, Fisher and Essam (1961) had used a similar form without the x, recognising that x was contained implicitly in $\langle n_s \rangle = D_s(q) p^s$, by treating p and q as independent. Here $D_s(q)$ is the perimeter polynomial. In our formulation, the generating function also needs no artificial parameter x, nor do we need to treat p and q as independent.

‡ We automatically keep track of the cluster site size by connecting bonds from the ghost site to every site in the lattice (figure 1(a)). Conversely, if we are interested in bond size, the ghost site should be connected to every bond in the lattice (figure 1(b)). Thus the coupling of the ghost site depends on the cluster counting procedure, not on whether we are doing bond or site percolation.

the ghost site. Equation (10) is the analogue of the Gibbs free energy (and also serves as a generating function for the percolation functions $P(p, h)$ and $S(p, h)$). One 'field' derivative of $G(p, h)$ gives us the 'order parameter' for the percolation problem,

$$P(p, h)=1+\frac{(1-h)}{p_s}\frac{\mathrm{d}G(p, h)}{\mathrm{d}h}. \tag{11}$$

Here p_s is the probability that a site is occupied, which is simply p in the site problem and unity in the bond problem. The percolation 'susceptibility', which is the mean cluster size, is related to $G(p, h)$ by two 'field' derivatives,

$$S(p, h)=1-(1-h)\frac{\mathrm{d}^2G(p, h)/\mathrm{d}h^2}{\mathrm{d}G(p, h)/\mathrm{d}h}. \tag{12}$$

In particular, for $d = 1$ we find from equation (10)

$$G(p, h)=\begin{cases}\sum_{s=0}^{\infty} q^2p^s(1-h)^s=\dfrac{(1-p)^2}{1-p(1-h)} & \text{(site)}\\[2ex] \sum_{s=1}^{\infty} q^2p^{s-1}(1-h)^s=\dfrac{(1-p)^2(1-h)}{1-p(1-h)} & \text{(bond).}\end{cases} \tag{13}$$

From equation (11), we find for *both* site and bond percolation

$$P(p, h)=1-\frac{(1-h)(1-p)^2}{[1-p(1-h)]^2}. \tag{14}$$

And, similarly, from equation (12)

$$S(p, h)=\frac{1+p(1-h)}{1-p(1-h)}. \tag{15}$$

Equations (13)–(15) reproduce the results of equations (6)–(8) when $h \to 0$. Furthermore, equation (14) enables us to calculate an additional exponent δ_p, which is defined by

$$P(p=p_\mathrm{c}, h)\sim h^{1/\delta_\mathrm{p}}. \tag{16a}$$

Since $P(p = 1, h) = 1$ for all h,

$$\delta_\mathrm{p}=\infty \tag{16b}$$

for $d = 1$.

We may also take derivatives of $G(p, h)$ with respect to p. We expect to find functions analogous to the entropy and specific heat in this way. However, $G(p, h = 0)$ is an analytic function, since

$$G(p, h=0)\sim(p-p_\mathrm{c})^{2-\alpha_\mathrm{p}}, \tag{17}$$

with $2-\alpha_\mathrm{p}=1$. Hence we can never see a 'specific heat' singularity $C(p, h=0)\sim A(p-p_\mathrm{c})^{-\alpha_\mathrm{p}}$ because A is identically zero. We must therefore taken the meaning of α_p from the 'Gibbs potential'.

We now show that $G(p, h)$ is asymptotically a generalised homogeneous function (GHF) of $(p-p_\mathrm{c})$ and h. As $p\to p_\mathrm{c}$ and $h\to 0$, we have from (13)

$$G(\epsilon, h)\sim\frac{\epsilon^2}{h+\epsilon} \qquad \text{(site or bond)}, \tag{18}$$

where here $\epsilon = 1-p = q$. This is indeed a GHF, since asymptotically

$$G(\lambda^{a_\epsilon}\epsilon, \lambda^{a_h}h) = \lambda G(\epsilon, h) \tag{19}$$

with scaling powers $a_\epsilon = a_h = 1$, from which our previously derived exponent results again follow: $2-\alpha_p = a_\epsilon^{-1} = 1$, $\beta_p = (1-a_h)/a_\epsilon = 0$, $-\gamma_p = (1-2a_h)/a_\epsilon = -1$, and $\delta_p^{-1} = (1-a_h)/a_h = 0$ (Hankey and Stanley 1972). We thus find all the usual 'weak' (or 'three-exponent') scaling laws; even their explicit form is unchanged, since the exponent terminology is appropriate to the particular derivative of $G(p, h)$ in the corresponding thermal problem.

4. The pair connectedness

The pair connectedness $C_2(r)$ is the conditional probability that given an occupied site at the origin, a site at a distance r away is in the same cluster†; thus this function is analogous to the two-spin correlation function in the magnetic problem. For $d = 1$ and $h = 0$,

$$C_2(r) = p^r \qquad \text{(site or bond)}, \tag{20}$$

since all the sites (or bonds) between the origin and point r must be occupied for them to be in the same cluster. (Here we note that the correlation function for the $d = 1$, n-vector model is identical to the pair connectedness with p replaced by y_n). The critical exponent η_p, characterising the decay as $r \to \infty$ of the pair connectedness at $p = p_c$, is defined by $C_2(r) \sim r^{-(d-2+\eta_p)}$, and hence from (20) it follows that

$$\eta_p = 1. \tag{21}$$

We may now define the correlation length, which for percolation is the RMS cluster diameter, from the normalised second moment of $C_2(r)$:

$$\xi_p^2 = \frac{\sum_{r=-\infty}^{\infty} r^2 C_2(r)}{\sum_{r=-\infty}^{\infty} C_2(r)} = \frac{2p}{(1-p)^2} \qquad \text{(site or bond)}. \tag{22}$$

Since $\xi_p \sim |p-p_c|^{-\nu_p}$, we find

$$\nu_p = 1, \tag{23}$$

which is in agreement with the renormalisation group results of Kirkpatrick (1977), Reynolds *et al* (1977), and Marland and Stinchcombe (1977). Alternatively, one can define a correlation length from $C_2(r) \sim e^{-r/\xi_p}$. Noting that (cf equation (20))

$$C_2(r) = \exp\{r \ln[1-(1-p)]\} \sim \exp[-r(1-p)],$$

we have $\xi_p \sim (1-p)^{-1}$.

In the usual thermal problem, the fluctuation–dissipation theorem relates the sum over r of $C_2(r)$ to the susceptibility. For percolation, Essam (1971) has given a graph theoretical proof of the analogous theorem. In particular, for $d = 1$ we find

$$\sum_{r=-\infty}^{\infty} C_2(r) = 1 + 2\sum_{r=1}^{\infty} p^r = \frac{1+p}{1-p} = S(p). \tag{24}$$

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† Essam (1971) defines the pair connectedness as the probability that two *occupied* sites belong to the same cluster. For site percolation (only), his definition differs from ours by a factor of p. We prefer our definition because of the analogy with the correlation function of thermal phenomena.

5. Scaling of the cluster size distribution $\langle n_s \rangle$

Stauffer (1975a,b) has proposed that for small values of ϵ, and for large s, the asymptotic form of $\langle n_s \rangle$ should obey the scaling relation

$$\langle n_s(\epsilon)\rangle \sim s^{-\tau} f(\epsilon s^{\sigma}). \tag{25}$$

It is of interest to ask whether or not the $d = 1$ functions of equation (5) obey the Stauffer scaling assumption. For small ϵ, $p^s \simeq \exp[s \ln(1-\epsilon)]$, and hence for both site and bond percolation

$$\langle n_s(\epsilon)\rangle \sim s^{-2} f(\epsilon s) \tag{26a}$$

for *all* s, with

$$f(x) \equiv x^2 \,\mathrm{e}^{-x}. \tag{26b}$$

Thus, in (25), $\sigma = 1$ and $\tau = 2$.

In summary, we have discussed percolation, including a ghost field, without appeal to the Potts model. In particular, for $d = 1$ we have solved for the 'thermodynamic' functions, the 'correlation' function, and its associated correlation length. We have calculated the critical exponents, and explicitly demonstrated the validity, for $d = 1$, of the 'thermodynamic' scaling hypothesis (Essam and Gwilym 1971) and the cluster size scaling hypothesis (Stauffer 1975a,b).

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