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Multifractal scaling of 3D diffusion-limited aggregation

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We study the multifractal (MF) properties of the set of growth probabilities $\{p_i\}$ for 3D off-lattice diffusion-limited aggregation (DLA). We find that: (i) the $\{p_i\}$ display MF scaling for all moments – in contrast to 2D DLA, where one observes a "phase transition" in the MF spectrum for negative moments; (ii) multifractality is also displayed by the p_i located in a shell of reduced radius $x \equiv r/R_g$, where R_g is the radius of gyration of the cluster and r the radius of the shell; (iii) the average value α_{av} of $\alpha \equiv -\ln p/\ln M$ in a shell of reduced radius x in a cluster of mass M is a function that does not depend on the cluster mass but only on x.

The multifractal (MF) formalism has proven to be a valuable tool in the study of numerous systems of statistical mechanics (for reviews, see, e.g., ref. [1]). In diffusion-limited aggregation (DLA) the set of growth probabilities $\{p_i\}$ forms an MF. For 2D DLA, the $\{p_i\}$ have been studied extensively [2]; it was found numerically that a "phase transition" in the MF spectrum occurs [3], manifesting itself in a breakdown of power-law scaling with cluster mass M for negative moments of the distribution of the $\{p_i\}$.

In the present study, we address the MF spectrum of 3D DLA [4]. Specifically, we study the distribution $N_M(\alpha, x)$. Here, $\alpha_i \equiv -\ln p_i/\ln M$ [#1], $x \equiv r/R_g$ is the ratio of the distance of a growth site from the seed of the cluster, and R_g the radius of gyration of the cluster [5]; and $N_M(\alpha, x)\,\mathrm{d}x\,\mathrm{d}\alpha$ is the number of growth sites with values of α_i in the interval $\alpha < \alpha_i < \alpha + \mathrm{d}\alpha$ and $x < x_i < x + \mathrm{d}x$ averaged over an ensemble of clusters of mass M.

In a recent work [6] the integrated distribution,

$$N_M(\alpha) \equiv \int_0^\infty \mathrm{d}x\, N_M(\alpha, x)\,, \tag{1}$$

[#1] Note that in the literature α is sometimes defined with respect to linear size L, i.e., $\alpha \equiv -\ln p/\ln L$. Similarly, $\ln N_M(\alpha)$ is sometimes rescaled with respect to $\ln L$ (and not $\ln M$ as in this work). For comparisons to such work, our values for α and $f(\alpha)$ must be multiplied by d_f.

was studied. The MF spectra $f_M(\alpha) \equiv \ln N_M(\alpha)/\ln N_a$ for an ensemble of 50 3D off-lattice DLA clusters in the mass range $165 < M < 15\,015$ are displayed in fig. 1. Here, N_a is the average number of growth sites with $p_i > 0$ ("active" growth sites). We observe that $f_M(\alpha)$ tends towards a limiting distribution $f(\alpha)$ for large M. Furthermore, the maximum value α_{max} for which $f(\alpha)$ is defined, is finite. Thus, from the definition of α we conclude that the smallest growth probability p_{min} scales as a power-law of M, $p_{min} \sim M^{-\alpha_{max}}$ [#2].

In contrast, in 2D DLA a breakdown of power-law scaling of p_{min} was observed [3,7]. Ref. [8] suggests a picture for the structure of 2D DLA that regards the cluster as a succession of self-similar "voids", separated by narrow "necks", whose widths scale slower than the linear size of the associated "void". In both 2D and 3D, necks are created by side branches of the cluster that grow closer until their growth probabilities become so small that no further narrowing occurs. The significant difference between 2D and 3D that causes the peculiar scaling behavior of p_{min} in 2D in the "void"–"neck" picture is that in 3D two branches cannot cut off a volume in the same way they can cut off an area in 2D and thus cannot impede growth as strongly as in 2D.

We now consider the normalized distributions $n_M(\alpha, x)$,

$$n_M(\alpha, x) \equiv N_M(\alpha, x) \Big/ \int d\alpha \; N_M(\alpha, x) \, . \tag{2}$$

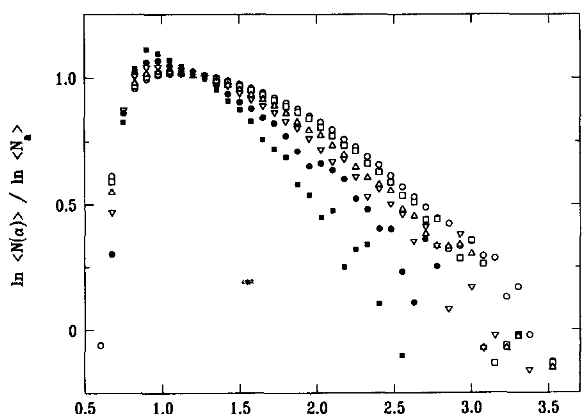

Fig. 1. $f_M(\alpha) \equiv \ln N_M(\alpha)/\ln N_a$ for 3D DLA (from ref. [6]). Data are obtained by averaging over a set of 50 off-lattice 3D DLA clusters with $M = 165$ (■), 435 (●), 1117 (▽), 2892 (△), 7502 (□), 15015 (○). The data shows a tendency to converge for large M. More evidence for the absence of a phase transition in the MF spectrum of 3D DLA is given in ref. [6].

[#2] In ref. [6], α_{max} was determined more precisely by studying explicitly the mass dependence of p_{min}. There it was found that $\alpha_{max} = 4.3 \pm 0.2$.

We obtain the histogram $N_M(\alpha, x)$ by counting the growth sites in a shell around the cluster seed with radius $r = xR_g$ and small finite width $\Delta x = 0.1$. Two questions are addressed: (i) what is the behavior of $n_M(\alpha, x)$ when x is fixed and M varies, and (ii) what is the nature of the x-dependence of $n_M(\alpha, x)$ at fixed M?

(i) fixed x, varying M. In fig. 2 we show $\ln n_M(\alpha, x)/\ln M$ for the shell $x = 0.4$ and $435 < M < 15\,015$. We observe a convergence towards a limit function, which is related to the MF spectrum of the $\{p_i\}$ in the considered shell [#3]. We observe similar behavior for other values of x.

(ii) fixed M, varying x. In fig. 3 we display $n_M(\alpha, x)$ for $M = 15\,015$ and several x, $0.2 < x < 2.0$. It is clear that the distribution depends on the location of the investigated shell. For shells in the interior of the cluster (small x), α is shifted to larger values (smaller p_i) due to the screening effects of the exterior parts of the cluster.

Since the structure in the interior of the DLA cluster will undergo only very few changes as growth proceeds, we argue that the shape of $n_M(\alpha, x)$ deep in the cluster is determined by the "frozen" structure and is only weakly

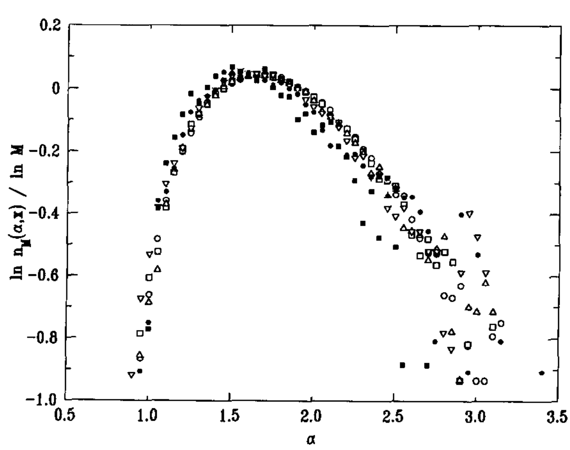

Fig. 2. $\ln n_M(\alpha, x)/\ln M$ for $x = 0.4$. Data are averaged over 6 ensembles of 50 clusters with $M = 435$ (■), 1117 (●), 2892 (▽), 7502 (△), 10330 (□), 15015 (○). Observe that for large M the data collapses to a common form. Other values of x yield similar plots, but more asymmetric distributions due to the existence of a minimum value of α [9].

[#3] Note that the p_i in one shell are not normalized and thus do not constitute a measure. Proper normalization would effect a shift of the $n_M(\alpha, x)$ curves in fig. 2 to the left. Furthermore, the general relation of the logarithm of a normalized distribution $n(\alpha)$ as in figs. 2, 3 and 5 to $f(\alpha)$ is $f(\alpha) = \ln m(\alpha)/\ln M + 1$.

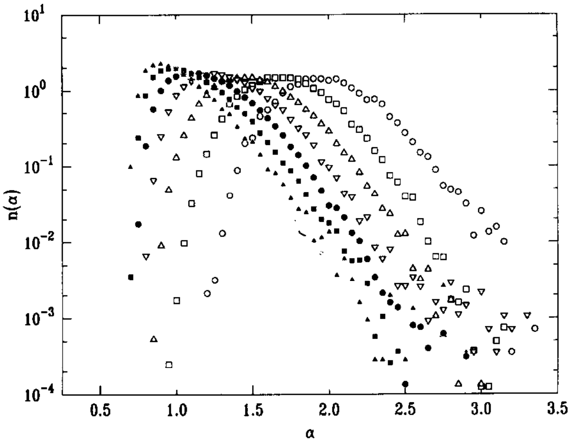

Fig. 3. $n_M(\alpha, x)$ for $M = 15\,015$ and varying x. We display $x = 0.2$ (○), 0.4 (□), 0.6 (△), 0.8 (▽), 1.0 (●), 1.2 (■), 1.4 (▲). Distributions in the interior of the cluster display larger values of α (smaller p_i).

dependent on x. We test this argument by first calculating the average value $\alpha_{av}(x)$ in shells with reduced radius x and clusters of mass $435 < M < 15\,015$ (fig. 4). The average $\alpha_{av}(x)$ apparently converges to a monotonously decreasing, mass-independent function of x. Now we shift the $n_M(\alpha, x)$ by $\alpha_{av}(x)$ to smaller values of α. The result is displayed in fig. 5. We observe a good data collapse for the right-hand side of the distributions. The collapse for the

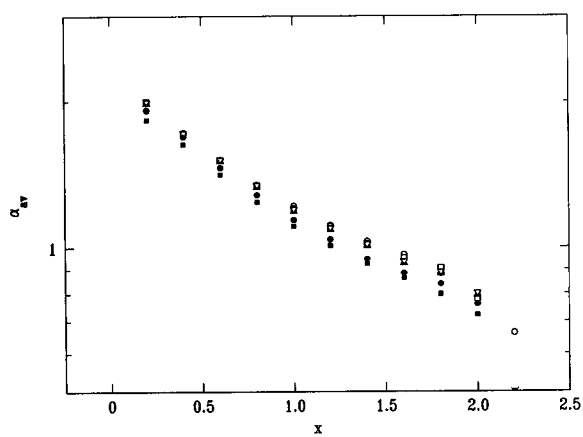

Fig. 4. $\alpha_{av}(x)$ as a function of x. Different symbols denote averages over cluster ensembles of different M. Here, $M = 435$ (■), 1117 (●), 2892 (▽), 7502 (△), 10330 (□), 15015 (○). The last 4 data points for each x lie very close, indicating a scaling behavior $\alpha_{av}(x)$ independent of M. However, the functional form of $\alpha_{av}(x)$ remains unclear.

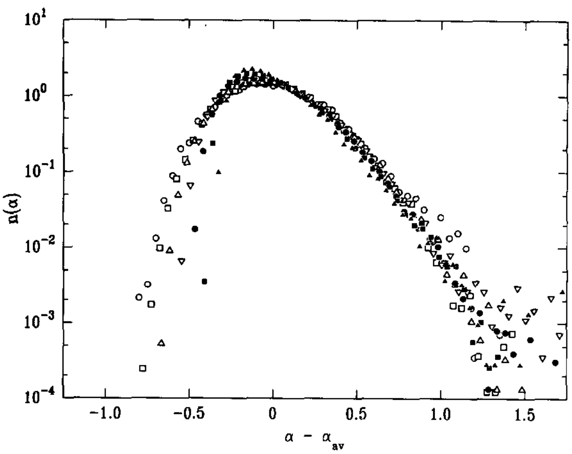

Fig. 5. $n_M(\alpha, x)$ vs. $\alpha - \alpha_{av}(x)$ for $M = 15\,015$. Here, the numerical value of $\alpha_{av}(x)$ is used to shift the $n_M(\alpha, x)$ to obtain a "data collapse". The symbols denote the distributions for $x = 0.2$ (○), 0.4 (□), 0.6 (△), 0.8 (▽), 1.0 (●), 1.2 (■), 1.4 (▲).

left-hand side becomes worse as x increases due to the existence of a minimum value of α [9].

In summary, we find MF scaling in a geometric cut through the aggregate. We also observe (i) a common shape of the distributions $n_M(\alpha, x)$ for small x – the "frozen" part of the cluster – and (ii) the existence of a mass-independent function $\alpha_{av}(x)$ describing the average growth probability at different locations in the cluster.

Thus, the scaling properties of 3D DLA can be adequately described in the framework of the MF formalism, in contrast to 2D DLA, where the MF spectrum is more complicated.

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