

PHYSICAL REVIEW LETTERS

VOLUME 51

3 OCTOBER 1983

NUMBER 14

Novel Superuniversal Behavior of a Random-Walk Model

H. Eugene Stanley, Kiho Kang, Sidney Redner, and Robin L. Blumberg

Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215

(Received 21 June 1983)

A model of interacting random walks is proposed in which each new site visited has a weight factor p . For $1 \leq p \leq \infty$, the model interpolates between purely random walks and self-avoiding walks. When $0 < p < 1$, the model describes attracting random walks (and also noninteracting random walks on a lattice with static traps), and shares some of the intriguing features of random walks on percolation fractals—e.g., dimension-independent exponents.

PACS numbers: 05.40.+j, 05.60.+w, 64.60.Kw, 66.30.-h

Random walks have been employed to model linear polymers and diffusion in randomly porous media.¹⁻⁶ It is becoming clear that the basic scaling laws of these two systems are fundamentally different from those for purely noninteracting random walks, and much effort has been directed to discovering how these laws are modified. For polymers, repulsive correlations have been introduced to model the “excluded-volume” effect.¹ For the porous-media problem, much recent work has focused on examples where the medium in question is itself a random fractal, such as the incipient infinite cluster at the percolation threshold.²⁻⁶ It seems that certain physical laws are hyperuniversal—i.e., dimension independent—but the elucidation of these remarkable phenomena has been hampered because numerical studies thus far are limited to two and three dimensions.³

In this Letter, we propose a model of correlated random walks that displays some of the intriguing features of diffusion on random fractals such as percolation clusters, and also describes polymer chains with *either* repulsion or attraction. The model is parametrized by a weight factor p for each new site that the random walker visits, so that a random walk that has

visited s distinct sites has a statistical weight of $p^s \equiv \exp(-Ks)$. One may regard K as a (dimensionless) “energy” or “coupling constant” for the random walker to find a new site. Accordingly, one may introduce a “partition function” for all random walks of N steps,

$$Z(N, K) = \sum_{\text{all } z^N \text{ walks}} e^{-Ks}, \quad (1a)$$

where z is the lattice coordination number. Alternatively, we may write

$$\begin{aligned} Z(N, p) &= \sum_{s=2}^{N+1} c(N, s) p^s \\ &= p^{-(N+1)} \sum_{s=2}^{N+1} c(N, s) p^{N+1+s}, \end{aligned} \quad (1b)$$

where $c(N, s)$ is the number of N -step walks that visit s sites. The second equality of (1b) indicates that an equivalent formulation of our model is that the weight of each self-intersection is p^{-1} . This is in contrast to the Domb-Joyce model of self-interacting polymer chains.⁷

The special case $p = 1$ [$K = 0$] corresponds to the purely random walk where $Z(N, K = 0) = z^N$. If $p > 1$ [$K < 0$], walks that visit a new site at each step are weighted most heavily and the walk

is self-repelling (Fig. 1). As $p \rightarrow \infty$, the term $c(N, N+1)$ dominates in Eq. (1b) and we recover the number of self-avoiding walks of N steps.

The regime $p < 1$ [$K > 0$] corresponds to a walker which prefers to return to previously visited sites. This defines a "self-attracting" random walk, a situation which has received little study thus far. As we shall show below, aspects of our model are identical to properties of diffusion on a lattice with randomly distributed, perfectly absorbing static "traps" at concentration $1-p$.⁶ Here we present calculations which indicate that certain critical properties of attracting walks at $p = p_c$ are hyperuniversal—that is, dimension independent. This is reminiscent of recent intriguing observations of Alexander and Orbach² for diffusion at the percolation threshold, although diffusion on percolation clusters corresponds to a situation where the impurities are "barriers" rather than traps. Finally, there is a connection between our model and that of diffusion in a random potential.

Our conclusions are based on examining several physical quantities: (i) The mean number of sites visited,

$$\begin{aligned} \langle S(N, p) \rangle &= \sum_{\text{walks}} s p^s / \sum_{\text{walks}} p^s \\ &= (-\partial / \partial \ln p) \ln Z(N, p). \end{aligned} \quad (2)$$

Equation (2) may be extended to the higher moments, $\langle S(N, p)^l \rangle$, by taking l derivatives with respect to the parameter $-K = \ln p$, and thus one may calculate any statistical property charac-

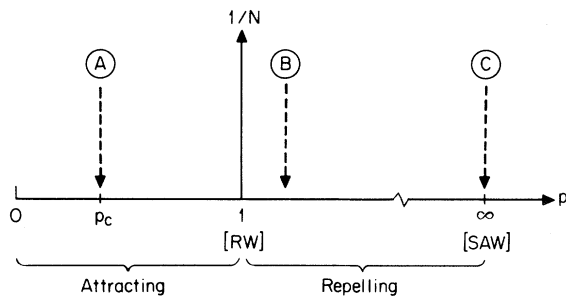


FIG. 1. Phase diagram for the interacting-walk model, showing three typical paths that probe the $N \rightarrow \infty$ critical behavior. The case $p = 1$ corresponds to noninteracting random walks, the range $0 < p < 1$ to self-attracting walks, and the range $1 < p < \infty$ to self-repelling walks (with the special case $p = \infty$ being the self-avoiding walk). For $p < 1$, we interpret the parameter $p = e^{-K}$ to be the probability of a site being occupied, and our model is related to diffusion in a random environment.

terizing the distribution of sites occupied by an interacting random walk. (ii) The probability of return to the origin,

$$P_0(N, p) = \sum'_{\text{walks}} p^s / \sum_{\text{walks}} p^s, \quad (3)$$

where the primed sum is restricted to walks that return to the origin. (iii) The mean square displacement,

$$\langle R(N, p)^2 \rangle = \sum_{\text{walks}} [R(\text{walk})]^2 p^s / \sum_{\text{walks}} p^s, \quad (4)$$

where $[R(\text{walk})]^2$ denotes the end-to-end distance squared of a given walk; higher moments of the displacement may be defined straightforwardly. The behavior of $Z(N, p)$, $\langle S(N, p)^l \rangle$, $\langle R(N, p)^{2l} \rangle$, and $P_0(N, p)$ has not been studied for interacting walks, except for the special case of self-avoiding walks.

To calculate these quantities we use enumeration methods as well as Monte Carlo simulations. By the former method, we have exactly calculated properties up to a given number of steps determined by limitations of computer time, for arbitrary values of *both* p and spatial dimension d . The results behave sufficiently smoothly with N that extrapolation to $N = \infty$ is generally possible. Moreover, we confirmed from the Monte Carlo simulation that the trends established for small N continue to larger N .

To illustrate the enumeration, consider the square lattice. The first step of a random walk can occur in any of four directions so that

$$Z(1, p) = 4p^2. \quad (5a)$$

The second step can continue in the same direction as the first step, or it may make a 90° turn. Both of these cases lead to three sites being visited. However, if the second step makes a 180° turn, then only two sites are visited, leading to

$$Z(2, p) = 12p^3 + 4p^2. \quad (5b)$$

We have continued this procedure to higher order and calculated $Z(N, p)$ through order $N = 10$ valid for *all* spatial dimensions. We have also computed even longer "series" for $d = 1, 2, 3$, and 4 to order $N = 30, 16, 13$, and 11, respectively, and to order 13 for the triangular lattice. We also calculated the mean number of sites visited, the return probability, and the mean square displacement. These series data were also extended up to $N \approx 10^4$ in $d = 1, 2$, and 3 by a Monte Carlo procedure that generalizes to the attract-

ing case the inverse sampling method⁸ introduced for self-avoiding walks. From the analysis of our data, we find several intriguing results, which we shall now describe briefly.

(i) $Z(N, p)$.—We calculated the growth parameter, or connective constant, defined by $\lambda(p) \equiv \lim_{N \rightarrow \infty} N^{-1} \ln Z(N, p)$, for a number of values of p in the range $0 < p < \infty$. For the hypercubic lattices, $\lambda(p) = 2d$ when $p = 1$. However, we find that at the percolation threshold of each lattice, $p_c(d)$, $\lambda(p_c) \cong 3.4$ independent of d [Fig. 2(a)]. This striking result suggests that at p_c , the connectivity of the underlying structure on which the random walk is taking place is dimension inde-

pendent.⁹

(ii) $\langle S(N, p)^l \rangle$.—We calculated the moments $\langle S(N, p)^l \rangle$ for the hypercubic lattices at p_c . At this point, we find $\langle S(N, p)^l \rangle \sim N^{\alpha l}$ with $\alpha \cong 0.6$ for $1 \leq d \leq 10$. For the square lattice, Monte Carlo data for N up to 10^3 confirm the trend indicated by the enumeration results. Our estimate for α agrees with the prediction $\alpha = \frac{2}{3}$ for diffusion on percolation clusters at the percolation threshold.⁵ Interestingly, the data for $\langle S(N, p) \rangle$ for all the hypercubic lattices lie on essentially the same locus of points when $p = p_c$ [Fig. 2(b)]. This again suggests that certain features of diffusion at p_c are dimension independent.

(iii) $\langle R(N, p)^{2l} \rangle$.—For the repelling case ($p > 1$), the mean square displacement crosses over from purely random-walk behavior for small N to self-avoiding walk behavior for larger N , as expected by scaling. This crossover should occur for $1 \leq d < 4$, and we have explicitly observed the effect in one dimension. In the attracting case, the situation seems to be more subtle and more interesting. For both two and three dimensions, the mean square displacement appears to saturate at a finite value when $p = p_c$ and also for values of $p > p_c$. Further work on this subject is in progress. In one dimension, however, the mean square displacement is an increasing function for all values of p studied ($0.05 < p \leq 1$).

In light of the resemblance of our result (ii) for attracting walks at p_c with those predicted for random walks on percolation clusters, it is worthwhile to discuss the nature of the relationship between the two models. Diffusion on percolation clusters is defined by a *quenched* average: First, a random cluster is generated, and then an average over many random walks on this cluster is taken. Finally, the average over all clusters is performed. Our interacting-walk model for $p < 1$ also resembles diffusion in a random environment, but in an *annealed* average sense. Each new site visited causes an additional factor of p to be associated with the walk. This factor may be regarded as the probability that the site is occupied when an average over the walk and the random environment is performed simultaneously.

However, it is possible to show that this annealed average corresponds to an actual quenched average in a model of diffusion with impurity sites being randomly distributed traps.⁶ In this case, the quantity $Z(N, p)$ gives the average number of random walks which “survive” to N steps when the trap concentration is $1 - p$. To see this,

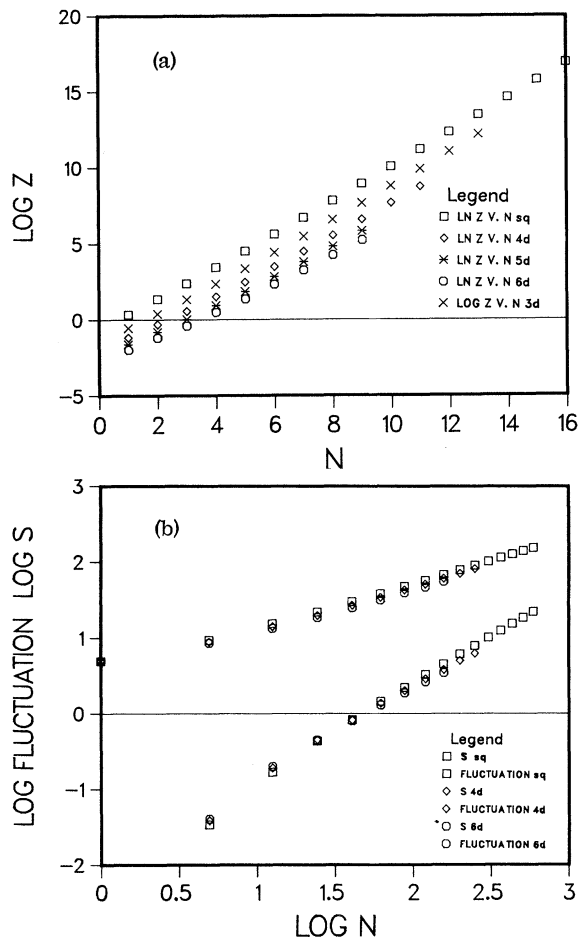


FIG. 2. (a) Exact results for $\ln Z(N, p)$ at the respective percolation thresholds, $p_c(d)$, for hypercubic lattices in $d = 2$ (squares), 3 (crosses), 4 (diamonds), 5 (asterisks), and 6 (hexagons). The data for different d become asymptotically parallel, suggesting that the growth parameters $\lambda(p_c)$ are identical for all d . (b) Exact results for $\ln \langle S(N, p) \rangle$ (upper) and $\ln \langle S^2 \rangle - \langle S \rangle^2$ (lower). The slope of straight lines that fit the data are roughly $\frac{2}{3}$ and $\frac{1}{3}$, respectively, for all d .

we present the following argument.¹⁰ Let $k_N(\{A\})$ be the number of walks that survive to N steps out of the ensemble of all $(2d)^N$ random walks, on a fixed configuration of traps $\{A\}$. Then the average number of walks that survive to N steps is

$$\langle k_N \rangle = \sum_{\text{all configs.}} k_N(\{A\})P(\{A\}), \quad (6)$$

where $P(\{A\})$ is the probability of the configuration $\{A\}$, and $\sum P(\{A\}) = 1$. We may rewrite Eq. (6) as

$$\langle k_N \rangle = \sum_{\gamma=1}^{(2d)^N} \sum' P(\{A\}), \quad (7)$$

where the first sum is over all $(2d)^N$ walks γ of N steps, and the prime on the second sum indicates that only those configurations in which the walk survives are included. Clearly, this second sum equals $p^{s(\gamma)}$, where $s(\gamma)$ is the number of distinct sites visited by walk γ . Upon summing over all $(2d)^N$ walks, we obtain the function $Z(N, p)$ as defined in Eq. (1), thus demonstrating the equivalence of the partition function of the attracting walk with the survival probability for random walks on a lattice with static traps. This rigorous connection does not extend, however, to related quantities such as $\langle S(N, p) \rangle$, $P_0(N, p)$, and $\langle R(N, p)^2 \rangle$.

In conclusion, we have introduced a new model that is of interest because it interpolates between self-avoiding walks and random walks for $p > 1$, and because it displays intriguing hyperuniversal behavior of $p = p_c(d)$. Our model exemplifies physical features of the static trap problem,⁶ and also of the XY ferromagnet with a random quenched "pinning field."¹¹

We thank A. Coniglio, Z. Djordjevic, D. C.

Hong, I. Majid, H. Orland, and M. J. Stephen for useful discussions and M. Paci for graphics assistance. This work was supported in part by grants from the U. S. Army Research Office, the U. S. Office of Naval Research, and the National Science Foundation.

¹P. G. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell Univ. Press, Ithaca, N.Y., 1979).

²S. Alexander and R. Orbach, *J. Phys. (Paris)*, Lett. **43**, L625 (1982).

³R. Pandey and D. Stauffer, *Phys. Rev. Lett.* **51**, 527 (1983); S. Havlin, D. Ben-Avraham, and H. Sompolinsky, *Phys. Rev. A* **27**, 1730 (1983).

⁴Y. Gefen, A. Aharony, and S. Alexander, *Phys. Rev. Lett.* **50**, 77 (1983).

⁵R. Rammal and G. Toulouse, *J. Phys. (Paris)*, Lett. **44**, L13 (1983).

⁶G. Zumofen and A. Blumen, *Chem. Phys. Lett.* **83**, 372 (1981); P. Grassberger and I. Procaccia, *J. Chem. Phys.* **77**, 6281 (1982).

⁷C. Domb, *J. Stat. Phys.* **30**, 425 (1983), and references therein. The Domb-Joyce model weights each self-intersection with a Boltzmann factor proportional to $\exp[\sum_{i<j} \delta(i-j)]$. Thus a walk in the shape of a figure 8 contributes a weight p^{-2} in our model, and a weight proportional to p^{-3} in the Domb-Joyce model. In general, if a walk visits the same point n times, then the weight varies as p^{-n} in our model (as the density), but as $p^{-n(n-1)/2}$ in the Domb-Joyce model (as the density squared).

⁸M. N. Rosenbluth and A. W. Rosenbluth, *J. Chem. Phys.* **23**, 356 (1954).

⁹Such dimension-independent behavior is also expected from a "links, nodes, and blobs" picture of the infinite percolating cluster at $p = p_c(d)$; for a recent review, see, e.g., H. E. Stanley and A. Coniglio, to be published.

¹⁰K. Kang, unpublished.

¹¹R. Pelcovits and D. Mukamel, to be published.