Dynamics of Spreading Phenomena in Two-Dimensional Ising Models

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We consider the time evolution of two Ising systems that differ at time $t=0$ in the orientation of only one spin. The detailed time development is calculated from two algorithms: (i) Glauber dynamics and (ii) Q2R dynamics (a deterministic cellular automaton). We find that for both algorithms spreading of "damaged regions" is greatly hindered below a threshold temperature T_s (or energy), which agrees numerically with the Curie point. For Glauber dynamics T_s is found to be a sharp phase transition point; for Q2R dynamics we find a kinetic slowing down which is reminiscent of a (spin-) glass transition.

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How a perturbation spreads throughout a cooperative system composed of interacting subunits is a question that arises in many fields of research (see, e.g., Kauffman,[1] Derrida and co-workers,[1] Packard and Wolfram,[2] and de Arcangelis,[3] and references therein). Here we study the spread of a small perturbation, termed the damage, in a cooperative system—the two-dimensional Ising model. To do this, we first simulate a system until it is in equilibrium. Then, at time $t=0$, we make a replica of this equilibrium configuration, and create a single spin flip ("initial damage") in the center of the replica. Thenceforth, both the original system ("control") and its damaged replica evolve by use of identical dynamics—e.g., the same random numbers are used for both systems.

As t evolves, our initial single-site damage generally results in a *region* of the cooperative system in which the spins $s_i(t)$ differ in orientation from the corresponding spins in the control system. We call this region the "damage," and measure this damage quantitatively by counting the number of spins in the perturbed system that differ from their counterparts in the control system. We find that sometimes the damage remains localized to a relatively small region of the lattice, and sometimes it spreads through the entire cooperative system (since each spin is interacting with its neighbors). In our comparison of the control and its replica, overall averages —like the *net* number of up spins or magnetization —will generally remain unaffected by small initial damage; thus our study concerns microscopic details of a spin configuration rather than macroscopic averages.

Specifically, our two systems consist of L^2 spins situated on the vertices of a $L\times L$ square lattice. Each pair of neighboring spins has an exchange energy Js_is_j, which is J or $-J$ depending on whether the two spins are oriented parallel or antiparallel. We use two different algorithms: (a) Glauber dynamics, in which we go through the system like a typewriter, and *flip* a spin with probability $\exp(-\Delta/T)/[\exp(\Delta/T)+\exp(-\Delta/T)]$, where 2Δ is the energy difference between the original and the flipped configurations, and (b) Q2R dynamics, where the system is scanned in an alternating "chessboard" manner, and a spin is flipped if this flip does not change the energy.[4]

We calculate the detailed time development until that time $t=\tau$ when the damaged region touches the upper or lower boundary of the lattice (time is measured in units of updates per site). At this "touching time" τ we calculate the following quantities, all of which depend on L: (i) M_τ, the actual number of damaged sites at time $t=\tau$; (ii) $M_{\rm tot}$, the total number of sites that have been damaged at least once at some time $t\le\tau$; and (iii) M_1, the number of sites that were damaged for the first time at $t=\tau$.

We define the critical or "spreading" point T_s to be that temperature *above* which a single-spin flip causes —with nonzero probability—a damage that spreads indefinitely. At $T=T_s$ we propose the asymptotic power laws $M_\tau\propto L^{d_\tau}$, $M_{\rm tot}\propto L^{d_{\rm tot}}$, $M_1\propto L^{d_1}$. Also, we expect $\tau\propto L^{1/z}$. An order parameter ψ for the spreading phase transition can be defined as the fraction of damaged sites if the initially damaged fraction at $t=0$ is infinitesimally small,[1] after taking the average over all starting configurations and extrapolating to the limits $t\to\infty$, $L\to\infty$.

Figure 1(a) shows ψ for different values of T (Glauber method); ψ approaches zero quite sharply at a temperature T_s that cannot be reliably distinguished from T_c. For Q2R dynamics, ψ increases smoothly with increasing energy [Fig. 1(b)], and even at low energies we find $\psi>0$. (The energy at $T=T_c$ is $E_c/J=-\sqrt{2}$.) For the Glauber method, the simulation starts with all spins up, and so the magnetization at finite times is still positive; hence, when studying behavior at criticality, we do our calculations at $T=1.025T_c$ in order to reduce these errors.

Figure 2 shows log-log plots for the averages M_τ and

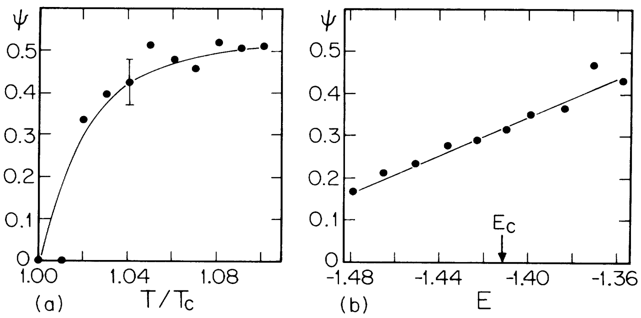

FIG. 1. Dependence of order parameter ψ on (a) temperature (Glauber dynamics, 10^3–10^4 updates/site) and (b) energy (Q2R dynamics, 10000 updates/site). In the Glauber case for the nonzero values of ψ we took an average over those runs where ψ did not vanish.

$M_{\rm tot}$ for the Q2R model at the Curie point; for large L these plots approximate straight lines, thus confirming our scaling assumptions above. Similar results were obtained for Glauber dynamics. The fractal dimensions d_τ and $d_{\rm tot}$ cannot be distinguished from the Euclidean dimension d. In our averages to calculate the damage at the critical point we ignore those runs where damage caused by the single spin flip at $t=0$ does not touch the boundaries within the observation time.

There are definite differences between the spreading behavior for Glauber and Q2R dynamics. These differences show up in both the low- and high-temperature regions and reveal that the character of the spreading transition is not the same for the two models we study. In the Glauber approach we find for $T<T_s$ that the amount of damage remains finite. This is understandable, since at low temperatures one has only a few isolated down spins in the sea of up spins and they have a very short lifetime. If the damaged spin points up, then the corresponding spin will point up very soon in the control system. If the damaged spin points down, the damage will heal within this short lifetime. Multispin events may change this simple picture, but at sufficiently low temperatures ψ should vanish in the thermodynamic limit.

For Glauber dynamics, ψ increases rapidly at T_c: At $T=0.99T_c$ we found 27, 15, and 10 damaged regions out of 1000 to touch the boundaries at $L=40$, 60, and 80, respectively. For $L=100$, we found 14 such events out of 2000. Thus the touching events below T_c appear to be finite-size effects in the Glauber case. Using the plot of $\log M_{\rm tot}$ vs $\log t$, we determined the exponent α in $M_{\rm tot}\propto t^\alpha$. For $T>T_c$, we obtained $\alpha=2$ asymptotically, but for $T=0.99T_c$ the slope does not seem to converge to this value; we get an effective exponent α near 1.2.

Since Q2R dynamics is reversible, such systems can never "heal" entirely. However, the damage can be localized for very long times; such behavior is reminiscent of the latent period of a disease. Furthermore, we observe that the spreading proceeds up to E_c often in a stepwise fashion (Fig. 3) showing that the above mechanism works also in the later stages. Although the microscopic mechanism of the restarting of spreading is not clear, the migration of clusters of damaged spins must play a role. For low energies, the damage is not completely frozen in, but only takes a very long time to spread—similar to relaxation times in glasses or spin-glasses. Above E_c, the spreading occurs much faster than below E_c. For Q2R dynamics, we could not observe systematic finite-size effects in the survival fraction for

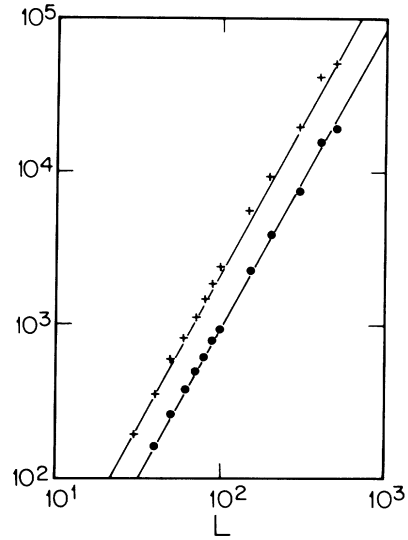

FIG. 2. Variation with system size of M_τ, actual damage at touching time τ (circles), and $M_{\rm tot}$, total damage from $t=0$ up to $t=\tau$ (plusses), for Q2R dynamics at the Curie point.

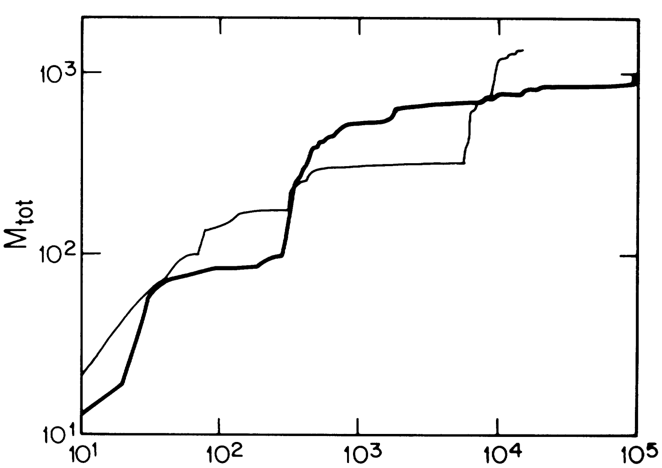

FIG. 3. Detailed time development of damaged sites for a fixed (large) value of L for Q2R dynamics. The two curves show (for two independent runs, both with $T/T_c\simeq0.95$ and $L=500$) the stepwise character of the spreading, with long latent periods. (At high energies we find $M_{\rm tot}\propto t^2$, as expected.)

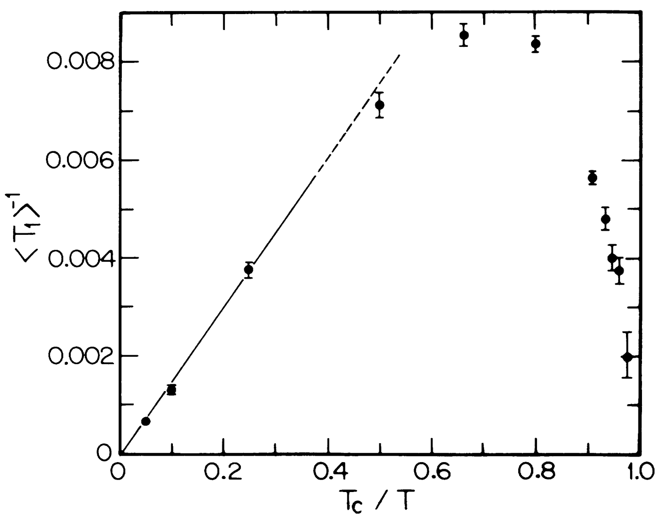

FIG. 4. Average time for *all* 1600 sites in a 40×40 square lattice to be damaged at least once during the time development of the Glauber simulation. Initially only one site is damaged.

low energies, indicating that the damage does not become negligible even for $L\rightarrow\infty$.

In the high-temperature phase, Q2R dynamics leads to a rapid spreading of the damage and this seems to be insensitive to the temperature T. On the other hand, we have for the $T\rightarrow\infty$ limit of Glauber dynamics only one single localized damage because the spins are decoupled. The behavior of the Glauber model looks clearer if one studies the average time $\langle T_1\rangle$ needed to damage every spin in the sample at least once. This time has a minimum at about $1.5T_c$, diverges linearly with T for $T\rightarrow\infty$, and seems to diverge also for $T\rightarrow T_c^+$ (cf. Fig. 4, where we plot $1/\langle T_1\rangle$ against $1/T$).

For Glauber dynamics $\psi=0$ *below* the critical temperature for our spreading problem, whereas $M=0$ *above* the critical temperature for magnetic phase transitions. After submission of this work, we received a preprint from Derrida and Weisbuch (DW)[5] who found, for the $d=3$ Glauber model, the opposite result that damage spreads at *low* temperatures. The reason for this apparent discrepancy is a different way in which the old orientation of the spin is used to calculate its new orientation: DW "orient" the spin and we "flip" the spin. Specifically, if a random number is smaller than a Boltzmann factor, DW set $s_i=1$, whereas we flip the spin. Both algorithms give the same result if only one lattice is simulated, but for the comparison of two lattices the DW method lets the damage heal quickly at high T while our method keeps the initially damaged site always damaged at $T=\infty$. To test these ideas, we used the DW method for $d=2$ and found no damage spread for $T/T_c=0.8$ and 1.2. We did find a nonzero damage below T_c when we started (as did DW) with a random distribution of spins, and with each spin in one lattice oriented opposite to the corresponding spin in the other lattice. Their "final damage" is thus essentially the magnetization M[5]: The DW method provides another way to obtain the Curie point, but does not seem to give new information in the sense of Kauffman's stability analysis. The reason for the discrepancy between our results and those of DW is thus not the difference in dimension, since Costa[6] recently found similar results for $d=3$ Glauber dynamics (except that $T_s/T_c\simeq0.96$).

In summary, we studied the dynamics of spreading phenomena in a two-dimensional[7] cooperative model with the standard lattice-gas or Ising interaction between the constituent subunits. The spread of an initial perturbation is studied with two different forms of dynamics. One difference between the two investigated cases is that for Glauber dynamics we found a sharp phase transition while for Q2R dynamics a kinetic effect seems to be present, analogous to the case of a (spin-) glass transition.

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[7]For $d=1$, the Q2R model can be solved exactly; we find that the damage always spreads as $4t+\text{const}$. For $d=3$, we find that the Q2R model behaves similarly to $d=2$, except that the broad spreading transition occurs below the Curie point. For Glauber models in $d=3$, and also for Kauffman models on square, triangular, and simple cubic lattices, analogous questions have been studied (Ref. 3), and could be studied for numerous other interacting discrete systems.