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Fractal-to-Euclidean crossover of the thermodynamic properties of the Ising model

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In this paper we develop a diagrammatic technique for determining the exact recursive relations for the partition functions of the Ising model in a field, situated on finitely ramified deterministic fractal lattices. Applying this method on the members of the two-dimensional Sierpiński gasket type of fractal lattices, which are characterized by generators of side length b, we first recover the known exact-space renormalization-group results for $b = 2$ and 3, in the case of zero field, and for $b = 2$ when $H \neq 0$. Then we obtain new results for all b up to $b = 15$, for $H = 0$, and up to $b = 8$ for $H \neq 0$. These results enable us to initiate the study of the crossover of thermodynamic properties of the Ising model caused by changing of the underlying fractal lattices towards the Euclidean (triangular) lattice. Accordingly, we calculate the temperature dependence of the specific heat, for $b \leq 15$, and susceptibility for $b \leq 8$, and compare these functions with the known results for the triangular lattice. This comparison demonstrates the difference between the standard thermodynamic limit and the fractal-to-Euclidean crossover behavior.

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I. INTRODUCTION

It has been generally known that a magnetic model system on a finite Euclidean lattice cannot display critical behavior at nonzero temperatures. Critical behavior can appear only in the thermodynamic limit, that is, when the underlying lattice becomes infinitely large. On the other hand, it has been demonstrated, via specific calculations, that the standard Ising model on finitely ramified fractals cannot have a nonzero critical temperature [1–3]. Similarly, in the case of the classical O(n) model, with $n \geq 2$, on fractals with the spectral dimension $d_s \leq 2$, it has been rigorously proved that there is no spontaneous magnetization at any finite temperature [4]. Thus one may rightfully study what happens at the fractal-to-Euclidean crossover. In other words, one may pose the question whether the Ising critical behavior on a Euclidean lattice can be retrieved via a limit of infinite number of finitely ramified fractals whose properties gradually acquire the corresponding Euclidean values. Moreover, one can compare this fractal limit with the standard thermodynamic limit.

Few specific results, related to the preceding question, have been established so far. For instance, it was shown [5] that the decay of the spin-spin correlations with increasing temperature is extremely slow on the Sierpiński-gasket (SG) fractal lattice, causing nontrivial thermodynamic behavior of the Ising model on all finite fractal lattices, even if their size is comparable to the size of the observable universe. Next, an interest was recently shown [6] for the behavior of antiferromagnetic Ising model on fractal lattices in the context of spin glasses and the high ground state frustration which leads to nonzero macroscopic residual entropy. In fact, it was found that contrary to the conjecture that existence of fractal holes should relieve frustration, the residual entropy per spin of the Ising antiferromagnet on the Sierpiński gasket is actually higher than that of the antiferromagnet on the triangular lattice. The effect of the presence of holes may be addressed by studying families of fractal lattices which provide a crossover towards the corresponding Euclidean triangular lattice. The fractal-to-Euclidean crossover behavior of the residual entropy was studied on the family of the SG type of fractal lattices in both the zero field, and in the maximum critical field [8]. In both cases a smooth crossover behavior of the residual entropy was established, and a simple crossover formula was found. Finally, the question of the fractal-to-Euclidean crossover appears to be rather nontrivial in the case of the O(n) model for $n = 0$, that is, in the case of self-avoiding walks (SAW's) on the SG fractals. Namely, the exact renormalization-group (RG) results indicated that the SAW critical exponents might tend toward the corresponding Euclidean values [9], whereas the consequent finite-size scaling arguments [10] showed that it cannot be generally true. The latter dichotomy is still a subject of an active research.

It is thus desirable to address the question of the fractal-to-Euclidean crossover behavior of Ising model thermodynamic response functions, which has up to date remained open. In fact, in the case of the SG fractal family which approaches the Euclidean triangular lattice in the limit when generator side length b tends to infinity, the real-space renormalization-group relations have thus far been established only for $b = 2$ and $b = 3$ for $H = 0$ [2,3], and only for the first member of the family ($b = 2$) for $H \neq 0$ [2]. It may be appropriate to mention here that in the reference [2] RG relations contained a small incorrectness when applied for finite size fractal lattices and become exact only in the limit of infinitely large fractals. To study the crossover of thermodynamic behavior, in this paper we develop a diagrammatic technique for determining the exact recursive relations for the partition functions. We first establish the equivalence of this procedure to the standard renormalization-group decimation technique [2,3] by recovering the exact RG formulas for $b = 2, 3$, in the case of $H = 0$, and for $b = 2$, in the case $H \neq 0$. Then, as the main contribution of this paper, we obtain new results for $4 \leq b \leq 15$, in the case $H = 0$, and for $3 \leq b \leq 8$, when $H \neq 0$. Using these results we calculate the temperature dependence of specific heat for $b \leq 15$ and susceptibility for $b \leq 8$, and compare them to the known results for the triangular lattice.

This paper is organized as follows. In the next section we develop the technique for determining the exact recursive relations for the partition functions of both ferromagnetic and antiferromagnetic Ising systems on finitely ramified fractal lattices, and recover the previously known RG formulas for the first two members of the SG family. In Sec. III we present the new results for higher members of the family, together with an overall discussion.

II. DIAGRAMMATIC TECHNIQUE FOR DETERMINING THE EXACT RECURSIVE RELATIONS FOR THE PARTITION FUNCTIONS OF THE ISING MODEL ON FINITELY RAMIFIED FRACTALS

A. Sierpiński gasket ($b = 2$)

The real-space RG recursive relations for the Ising model on the Sierpiński gasket have been established for both zero and nonzero magnetic field [1,2]. We will use this particular case as a test ground for developing a convenient diagrammatic technique for obtaining the recursive relations in a new way. This new technique will later prove useful in the case of higher members of the SG fractal family, and it can also be applied in the case of other finitely ramified fractals. In fact, it will turn out that even for the Sierpiński gasket in the field, our technique yields relations which are exact on *all* stages of construction of the fractal, while the previously established relations [2] become *exact* only in the case of infinitely large fractal.

We consider Ising systems with the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle \mathrm{NN} \rangle} S_i S_j - H \sum_i S_i, \tag{1}$$

where J is the nearest neighbor interaction parameter, $S_i = \pm 1$ is the Ising spin variable at the site i, H is the external magnetic field, and $\langle \mathrm{NN} \rangle$ denotes summation over the nearest neighbor pairs.

The first few stages of construction of the first three members of the SG fractal family are shown in Fig. 1. The lattice at stage n is obtained by joining n_c structures of $(n-1)$th stage, *exclusively* at their vertices. For a given member of the SG family, n_c is independent of the stage n. We will further distinguish between *vertex* spins at a given stage n (depicted by full circles in Fig. 1) and the *interior* spins (depicted by open circles). We will term interior spins all those spins that are not at vertices at the nth stage of construction, but are the vertex spins of the n_c constituent $(n-1)$th-stage structures. In the $b = 2$ case, at each stage (n) there are $n_c = 3$ constituent $(n-1)$th-stage structures, and $N_B = 3$ interior spins. For all members of the SG family, at each stage there are $N_A = 3$ vertex spins.

At any stage of construction of the Sierpiński gasket, one can define eight partial partition functions $\{Z_i,\ i = 1, \dots, 8\}$ that correspond to eight possible configurations of the three vertex spins, while the summation is performed over all the other spins. From the symmetry of the lattice, however, it follows that at each stage there are only four independent partial partition functions Z_1,

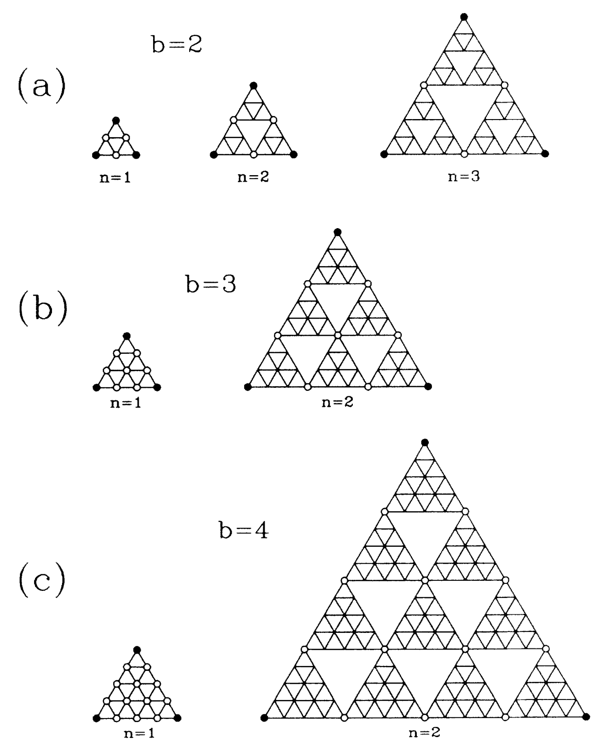

FIG. 1. First three members of the two-dimensional Sierpiński-gasket type of fractal lattices constructed from generators of side length (a) $b = 2$, (b) $b = 3$, and (c) $b = 4$. For $b = 2$ we depict the first three stages of construction, whereas for $b = 3, 4$ the first two stages are shown (for $n = 1$ the fractal structures are actually finite equilateral wedges of the triangular lattice). Full circles indicate the *vertex* spins (spins by which the structures are connected when forming the structure in the next stage), and open circles indicate the *interior* spins (spins which are not at vortices, but are vertex spins of the constituent structures of the previous stage).

Z_2, Z_3, and Z_4, corresponding to $\{+++\}$, $\{+-+\}$, $\{-+-\}$, and $\{---\}$ configurations of the vertex spins, respectively. This fact can be easily verified to be true for all members of the SG fractal family. If the recursive relations between the sets of partial partition functions, corresponding to any two consecutive stages of construction of the lattice, are established, using formula

$$Z = Z_1 + 3Z_2 + 3Z_3 + Z_4, \tag{2}$$

one can obtain the partition function at arbitrary stage of construction of the lattice, and thereby the thermodynamic response functions (as temperature and field derivatives of Z). The recursive relations for the partial partition functions can also be used to obtain the real-space RG relations for the interaction parameters.

1. Recursive relations for the partial partition functions

Recursive relations between the partial partition functions $\{Z_i', i = 1, \ldots, 4\}$ in the $(n+1)$th stage and the partial partition functions $\{Z_i, i = 1, \ldots, 4\}$ in the nth stage of construction, are obtained by associating to each recursive relation a series of diagrams representing the $(n+1)$th stage of construction with the substructure on the level of the nth stage. On all the diagrams corresponding to the recursive relation for one particular partial partition function Z_i', the vertex spins are fixed in the corresponding particular spin configuration, while the configurations of interior spins (which simultaneously represent vertex spins of the nth-stage structures) vary from diagram to diagram. Total number of diagrams for one recursive relation is eight, representing the number of possible configurations of the three interior spins. To each diagram ($k = 1, \ldots, 8$) that corresponds to Z_i' ($i = 1, \ldots, 4$), we associate a product of three partial partition functions of the nth stage of construction, multiplied by the term $e^{b(k,i)\beta H}$, where

$$b(k,i) = -\sum_{j=1}^{N_B} S_j(r_j - 1), \tag{3}$$

r_j is the number of nth stage structures joined in the $(n+1)$th stage by the jth interior site, and $\beta = 1/k_BT$ is the standard notation for the reciprocal of the product of the Boltzmann constant and temperature. This exponential multiplicative factor comes from the fact that the energy of the $(n+1)$th-stage structure is not a simple sum of energies of the nth-stage structures. That is, when the nth-stage partial partition functions are multiplied, the field dependent term in the Hamiltonian is taken into account as many times as there are nth-stage structures connected by the jth site. The recursive relation is now represented as the sum of eight terms, each corresponding to a different diagram.

In Fig. 2 we present the diagrams associated with different terms of the recursive relations. Each row represents a set of eight diagrams corresponding to the recursive relation of one partial partition function. It follows from Fig. 2 that the recursive relations for partial partition functions, at any two consecutive stages of construction of the lattice, have the form

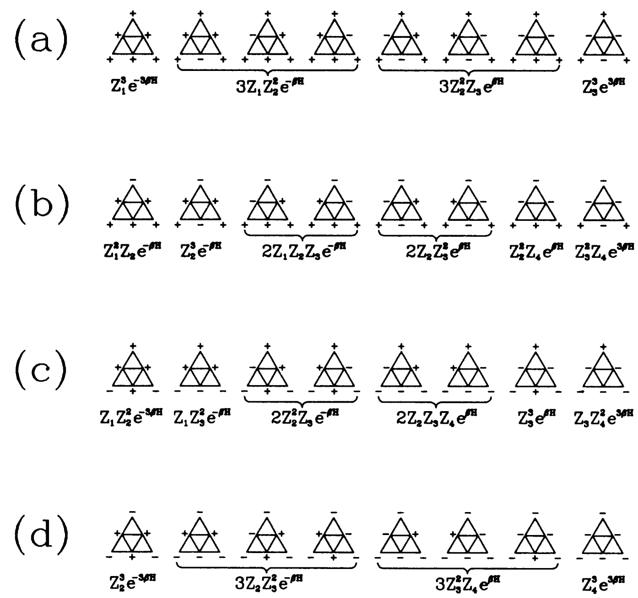

FIG. 2. Diagrams associated with different terms of the recursive relations (4) for the partial partition functions of the Sierpiński gasket ($b = 2$). Each row represents a set of eight diagrams (corresponding to different configurations of the interior spins) with vertex spins fixed in one particular configuration (corresponding to the recursive relation of one particular partial partition function). Partition functions obtained by recursive relations (4) are exact in the first stages of construction of the lattice as well as in the thermodynamic limit $n \to \infty$ (see the text).

$$Z_1' = Z_1^3 e^{-3\beta H} + 3Z_1Z_2^2 e^{-\beta H} + 3Z_2^2Z_3 e^{\beta H} + Z_3^3 e^{3\beta H}, \tag{4a}$$

$$Z_2' = Z_1^2Z_2 e^{-3\beta H} + Z_2^3 e^{-\beta H} + 2Z_1Z_2Z_3 e^{-\beta H} + 2Z_2Z_3^2 e^{\beta H} + Z_2^2Z_4 e^{\beta H} + Z_3^2Z_4 e^{3\beta H}, \tag{4b}$$

$$Z_3' = Z_1Z_2^2 e^{-3\beta H} + Z_1Z_3^2 e^{-\beta H} + 2Z_2^2Z_3 e^{-\beta H} + 2Z_2Z_3Z_4 e^{\beta H} + Z_3^3 e^{\beta H} + Z_3Z_4^2 e^{3\beta H}, \tag{4c}$$

$$Z_4' = Z_2^3 e^{-3\beta H} + 3Z_2Z_3^2 e^{-\beta H} + 3Z_2^2Z_4 e^{\beta H} + Z_4^3 e^{3\beta H}. \tag{4d}$$

In fact it is sufficient to have recursive relations for the first two partition functions Z_1 and Z_2, since the symmetry of the system implies that Z_3 is obtained from Z_2, and Z_4 from Z_1, by substituting $Z_1 \longleftrightarrow Z_4$, $Z_2 \longleftrightarrow Z_3$, and $H \longleftrightarrow -H$. However, in Fig. 2 we present all four recursive relations for the sake of completeness.

For the initial conditions of the recursive relations (4) we can choose the partial partition functions of a simple equilateral triangle (zeroth stage of construction of the fractal), given by

$$Z_1 = e^{3\beta J + 3\beta H}, \tag{5a}$$
$$Z_2 = e^{-\beta J + \beta H}, \tag{5b}$$
$$Z_3 = e^{-\beta J - \beta H}, \tag{5c}$$
$$Z_4 = e^{3\beta J - 3\beta H}. \tag{5d}$$

Partial partition functions at the arbitrary nth stage of construction are now obtained from the initial conditions (5) by successive application (n times) of recursive relations (4), and the total partition function is finally given by (2).

2. RG relations for the interaction parameters. The $H \neq 0$ case

To obtain the standard RG recursive relations for the set of interaction parameters, it is necessary to take into consideration terms of the Hamiltonian with further than nearest neighbor interactions. Namely, it turns out [2] that if one starts with the Hamiltonian (1) and applies the exact RG spin decimation technique, two new parameters need to be included in the renormalized Hamiltonian. Consequently, the Hamiltonian in the nth stage of construction is given by

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j - J_3 \sum_{\langle i,j,k \rangle} S_i S_j S_k - H \sum_i S_i + A, \tag{6a}$$

where J_3 is three-spin interaction, A is an additive constant, and $\langle i, j, k \rangle$ denotes summation over three-spin sets belonging to upward oriented triangles. In terms of renormalized parameters, the Hamiltonian at nth stage is simply given by

$$\begin{aligned} \mathcal{H} = & -J^n(\sigma_1^n \sigma_2^n + \sigma_2^n \sigma_3^n + \sigma_1^n \sigma_3^n) - J_3^n \sigma_1^n \sigma_2^n \sigma_3^n \\ & - H^n(\sigma_1^n + \sigma_2^n + \sigma_3^n) + A^n, \end{aligned} \tag{6b}$$

where σ_i is the renormalized spin variable, and superscript n indicates that the corresponding variable has been renormalized n times. In order to simplify the notation, henceforth we will omit the superscript n, while quantities corresponding to the following step $n+1$ of construction will be denoted by a prime.

The fact that the four-dimensional parameter space stays closed under exact RG transformations (no new parameters are generated) corresponds to the number of independent partial partition functions (which is in turn determined by the topology of the lattice). Relations between the sets $\{Z_1, Z_2, Z_3, Z_4\}$ and $\{J, J_3, H, A\}$ are obtained by explicitly writing the expressions for the partial partition functions at the nth stage of construction of the fractal lattice, in terms of the renormalized parameters. Explicitly, we have

$$Z_1 = e^{3\beta J + 3\beta H + \beta J_3 + \beta A}, \tag{7a}$$
$$Z_2 = e^{-\beta J + \beta H - \beta J_3 + \beta A}, \tag{7b}$$
$$Z_3 \equiv Z_2(J, -H, -J_3, A) = e^{-\beta J - \beta H + \beta J_3 + \beta A}, \tag{7c}$$
$$Z_4 \equiv Z_1(J, -H, -J_3, A) = e^{3\beta J - 3\beta H - \beta J_3 + \beta A}. \tag{7d}$$

Solving these equations for the interaction parameters and substituting $\beta J = K$, $\beta J_3 = K_3$, $\beta H = B$, and $\beta A = C$ for the stage $n+1$ we can first write

$$e^{8K'} = \frac{Z_1' Z_4'}{Z_2' Z_3'}, \tag{8a}$$
$$e^{8B'} = \frac{Z_1' Z_2'}{Z_3' Z_4'}, \tag{8b}$$
$$e^{8K_3'} = \frac{Z_1'}{Z_4'} \left(\frac{Z_3'}{Z_2'} \right)^3, \tag{8c}$$
$$e^{8C'} = Z_1' Z_4' (Z_2' Z_3')^3. \tag{8d}$$

Expressions (7) and (8) are actually valid for *all* members of the SG fractal family. Now, inserting (7) into (4) we finally obtain

$$\begin{aligned} Z_1' = 2e^{3B+3C} \{ & e^{3K+3K_3} \cosh(6K + 3B) \\ & + e^{-K-K_3} \cosh(2K + B) \}, \end{aligned} \tag{9a}$$
$$\begin{aligned} Z_2' = 2e^{B+3C} \{ & e^{3K+K_3} \cosh(2K + 3B) \\ & + 2e^{-K+K_3} \cosh(2K + B) \\ & + e^{-K-3K_3} \cosh(2K - B) \}, \end{aligned} \tag{9b}$$
$$Z_3' = Z_2'(K, -B, -K_3, C), \tag{9c}$$
$$Z_4' = Z_1'(K, -B, -K_3, C). \tag{9d}$$

Relations (8) and (9) represent the exact real-space RG relations for the renormalized interaction parameter sets $\{K, K_3, B, C\}$ in two consecutive stages of construction of the lattice. Except for the field dependent multiplicative terms in front of braces in (9), these equations are equivalent to the recursive RG relations obtained by Luscombe and Desai [2], whose relations are correct only in the limit $n \to \infty$.

3. RG relations for the interaction parameters. The $H = 0$ case

To recover the exact RG recursive relations for the zero field case [1], one should first note that it is sufficient to consider only two terms in the Hamiltonian (6), that is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j - A. \tag{10}$$

Also, we have now only two independent partial partition functions Z_1 and Z_2, which correspond to $\{+++\}$ and $\{+-+\}$ configurations of the vertex spins, respectively. Substituting $Z_1 = Z_4$, $Z_2 = Z_3$, and $H = 0$ into (4), we obtain

$$Z_1' = Z_1^3 + 3Z_1 Z_2^2 + 4Z_2^3, \tag{11a}$$
$$Z_2' = Z_1^2 Z_2 + 4Z_1 Z_2^2 + 3Z_2^3. \tag{11b}$$

Using (7) and (10) to introduce

$$t \equiv e^{4\beta J} = \frac{Z_1}{Z_2}, \tag{12}$$

and using (11), we finally obtain

$$t' = \frac{t^2 - t + 4}{t + 3}. \tag{13}$$

Recursion relation (13) was first found by Gefen *et al.* via a different approach [1].

B. The $b = 3$ member of the SG fractal family

From Fig. 1 it can be observed that in the case $b = 3$, at each stage n, there are $n_c = 6$ constituent $(n-1)$th-stage structures, and $N_B = 7$ interior spins. Recursion relation for each of the four partial partition functions contains $2^{N_B} = 128$ terms. As in the $b = 2$ case, each term is a product of $n_c = 6$ partial partition functions of the previous stage of construction, multiplied by the term $e^{b(k,i)\beta H}$, where $b(k,i)$ is given by 3. The terms are determined by considering the corresponding diagrams presented in Fig. 3 (instead of all 128 diagrams for each recursive relation, we present in this figure only one member of each distinct symmetry class). We obtain the following recursive relations for Z_1' and Z_2':

$$\begin{aligned} Z_1' = {} & Z_1^6 e^{-8\beta H} + 6Z_1^4 Z_2^2 e^{-6\beta H} + Z_1^3 Z_2^3 e^{-4\beta H} + 6Z_1^3 Z_2^2 Z_3 e^{-4\beta H} + 9Z_1^2 Z_2^4 e^{-4\beta H} \\ & +6Z_1^2 Z_2^3 Z_3 e^{-2\beta H} + 6Z_1^2 Z_2^2 Z_3^2 e^{-2\beta H} + 12 Z_1 Z_2^4 Z_3 e^{-2\beta H} + 2Z_2^6 e^{-2\beta H} \\ & +3Z_1 Z_2^4 Z_4 + 3Z_1^2 Z_2 Z_3^3 + 9Z_1 Z_2^3 Z_3^2 + 6Z_1 Z_2^2 Z_3^3 + 9Z_2^4 Z_3^2 + 6Z_1 Z_2^2 Z_3^2 Z_4 e^{2\beta H} \\ & +6Z_2^4 Z_3 Z_4 e^{2\beta H} + 6Z_1 Z_2 Z_3^4 e^{2\beta H} + 2Z_2^3 Z_3^3 e^{2\beta H} + 6Z_2^2 Z_3^4 e^{2\beta H} + 3Z_1 Z_3^4 Z_4 e^{4\beta H} \\ & +3Z_2^3 Z_3 Z_4^2 e^{4\beta H} + 9Z_2^2 Z_3^3 Z_4 e^{4\beta H} + Z_3^6 e^{4\beta H} + 6Z_2 Z_3^3 Z_4^2 e^{6\beta H} + Z_3^3 Z_4^3 e^{8\beta H}, \end{aligned} \tag{14a}$$

$$\begin{aligned} Z_2' = {} & Z_1^5 Z_2 e^{-8\beta H} + 2Z_1^4 Z_2 Z_3 e^{-6\beta H} + 4Z_1^3 Z_2^3 e^{-6\beta H} + Z_1^2 Z_2^4 e^{-4\beta H} \\ & +Z_1^3 Z_2^2 Z_4 e^{-4\beta H} + 2Z_1^3 Z_2 Z_3^2 e^{-4\beta H} + 9Z_1^2 Z_2^3 Z_3 e^{-4\beta H} + 3Z_1 Z_2^5 e^{-4\beta H} \\ & +2Z_1^2 Z_2^2 Z_3^2 e^{-2\beta H} + 4Z_1 Z_2^4 Z_3 e^{-2\beta H} + 2Z_1^2 Z_2 Z_3^3 e^{-2\beta H} + 2Z_1^2 Z_2^2 Z_3 Z_4 e^{-2\beta H} \\ & +2Z_1 Z_2^4 Z_4 e^{-2\beta H} + 4Z_2^5 Z_3 e^{-2\beta H} + 10 Z_1 Z_2^3 Z_3^2 e^{-2\beta H} + Z_1^2 Z_2 Z_3^2 Z_4 \\ & +2Z_1 Z_2^3 Z_3 Z_4 + 8Z_1 Z_2^2 Z_3^3 + Z_2^5 Z_4 + 3Z_2^4 Z_3^2 + 2Z_1 Z_2 Z_3^4 + 3Z_1 Z_2^2 Z_3^2 Z_4 \\ & +7Z_2^3 Z_3^3 + 3Z_2^4 Z_3 Z_4 + 2Z_1 Z_2^2 Z_3 Z_4^2 e^{2\beta H} + 4Z_1 Z_2 Z_3^3 Z_4 e^{2\beta H} + 8Z_2^3 Z_3^2 Z_4 e^{2\beta H} \\ & +2Z_1 Z_3^5 e^{2\beta H} + 4Z_2^2 Z_3^4 e^{2\beta H} + 4Z_2^2 Z_3^3 Z_4 e^{2\beta H} + 2Z_2 Z_3^5 e^{2\beta H} + Z_2^3 Z_4^3 e^{4\beta H} \\ & +2Z_1 Z_3^3 Z_4^2 e^{4\beta H} + 5Z_2^2 Z_3^2 Z_4^2 e^{4\beta H} + 7Z_2 Z_3^4 Z_4 e^{4\beta H} + Z_3^5 Z_4 e^{4\beta H} \\ & +4Z_2 Z_3^2 Z_4^3 e^{6\beta H} + 2Z_3^4 Z_4^2 e^{6\beta H} + Z_3^2 Z_4^4 e^{8\beta H}. \end{aligned} \tag{14b}$$

Because of the lattice symmetry, the relations for Z_3' and Z_4' are obtained respectively from Z_2' and Z_1', by substituting $Z_1 \longleftrightarrow Z_4$, $Z_2 \longleftrightarrow Z_3$, and $H \longleftrightarrow -H$. Similarly to the $b = 2$ case, these relations are exact for *all* stages of construction of the lattice. While relations (8) hold for all members of the SG family, finding the analogs of (9) would present a straightforward but rather tedious job. On the other hand, symbolically writing Eq. (7) as $\mathcal{Z} = \mathcal{F}(\mathcal{K})$, Eq. (8) as $\mathcal{K}' = \mathcal{F}^{-1}(\mathcal{Z}')$, and Eq. (14) as $\mathcal{Z}' = \mathcal{G}(\mathcal{Z}, \mathcal{K})$, we can write $\mathcal{K}' = \mathcal{F}^{-1}(\mathcal{G}(\mathcal{F}(\mathcal{K}), \mathcal{K}))$. In this sense Eqs. (7), (8), and (14) implicitly contain the real-space RG recursive relations for interaction parameters.

To recover the exact RG recursive relations [3] for $H = 0$, we first substitute (as in the $b = 2$ case) $Z_1 = Z_4$, $Z_2 = Z_3$, and $H = 0$ into (14), to obtain expressions

$$\begin{aligned} Z_1' = {} & Z_1^6 + 6Z_1^4 Z_2^2 + 8Z_1^3 Z_2^3 \\ & +45 Z_1^2 Z_2^4 + 48 Z_1 Z_2^5 + 20 Z_2^6, \end{aligned} \tag{15a}$$

$$\begin{aligned} Z_2' = {} & Z_1^5 Z_2 + 4Z_1^4 Z_2^2 + 18 Z_1^3 Z_2^3 \\ & +32 Z_1^2 Z_2^4 + 53 Z_1 Z_2^5 + 20 Z_2^6. \end{aligned} \tag{15b}$$

Using (7) and (10) to introduce

$$u \equiv e^{-4\beta J} = \frac{Z_2}{Z_1}, \tag{16}$$

and using (15) we find

$$u' = u \frac{1 + 4u + 18u^2 + 32u^3 + 53u^4 + 20u^5}{1 + 6u^2 + 8u^3 + 45u^4 + 48u^5 + 20u^6}. \tag{17}$$

Equation (17) was first obtained by Bhattacharya [3], by application of the standard real-space RG decimation technique.

C. General case of finitely ramified fractals

In this section we present an overview of the introduced approach for determining the exact RG recursive relations, and formally generalize it to the case of an arbitrary finitely ramified fractal.

A common feature of deterministic finitely ramified fractals is that they can be constructed iteratively, by joining at each stage a constant number n_c of previous stage lattices connected *exclusively* by a *constant* number of vertices. In contrast, the number of spins shared by constituent parts of infinitely ramified fractals increases with the stage of construction. At a given stage of construction of a ramified fractal, one can define a set of 2^{N_A} partial partition functions $\{Z_i\}$, each corresponding to a particular fixed configuration of the N_A vertex spins, while the summation is performed over all the possible configurations of all the other spins. Because of the symmetry some of the partial partition functions can be equal to each other so that the total number of independent functions is $n_z \leq 2^{N_A}$. Recursive relations between the partial partition functions $\{Z_i', i = 1, \ldots, n_z\}$

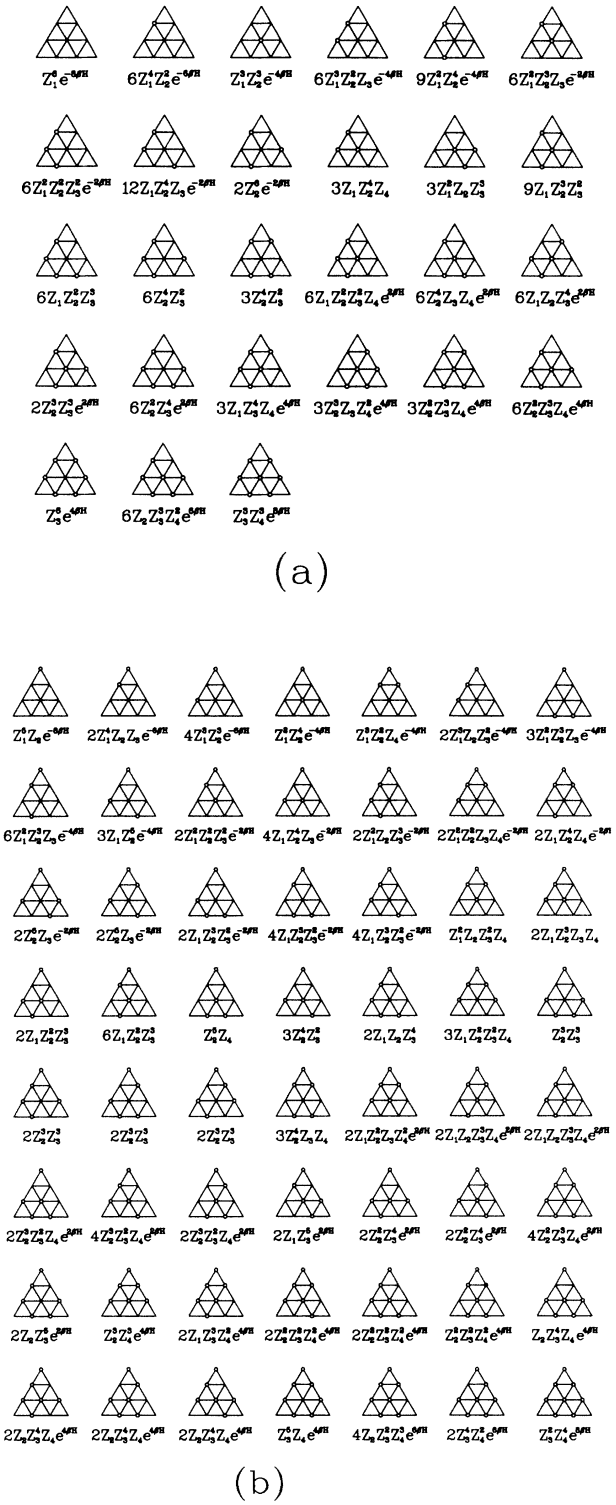

FIG. 3. Diagrams associated with different terms of the recursive relations (14) for the partial partition functions of the $b=3$ member of the Sierpiński-gasket family. Instead of all the 128 terms for each of the four recursion relations we depict only one representative of each of the (a) 27 distinct symmetry classes for the recursive relation (14a), and (b) 56 classes for the relation (14b). The open circles here represent the down oriented spins. Recursive relations for Z_3' and Z_4' are obtained from (14b) and (14a), respectively, by substituting $Z_1 \longleftrightarrow Z_4$, $Z_2 \longleftrightarrow Z_3$, and $H \longleftrightarrow -H$.

in the $(n+1)$th stage and the partial partition functions $\{Z_i, i=1,\ldots,n_z\}$ in the nth stage of construction, are obtained by associating to each recursive relation a series of diagrams representing the $(n+1)$th stage of construction with the substructure on the level of the nth stage. On all the diagrams corresponding to the recursive relation for one particular partial partition function Z_i', the vertex spins are fixed in the corresponding particular spin configuration, while the configurations of interior spins (which simultaneously represent vertex spins of the nth-stage structures) vary from diagram to diagram. Total number of diagrams for one recursive relation is 2^{N_B}, representing the number of possible configurations of the N_B interior spins. To each diagram ($k=1,\ldots,2^{N_B}$) we associate a product of n_c partial partition functions of the nth stage of construction, multiplied by the term $e^{b(k,i)\beta H}$, where $b(k,i)$ is given by 3. The recursive relation for a given partial partition function Z_i' is now represented as the sum of 2^{N_B} terms corresponding to different diagrams, that is

$$Z_i' = \sum_{k=1}^{2^{N_B}} e^{b(k,i)\beta H} \prod_{\ell=1}^{n_z} Z_\ell^{a(\ell,k,i)}, \tag{18}$$

where $\sum_{\ell=1}^{n_z} a(\ell,k,i) = n_c$, and $i=1,\ldots,n_z$. Because of the symmetry some of the terms in the sum in (18) can be equivalent, and we can write

$$Z_i' = \sum_{k=1}^{n_i} d(k,i) e^{b(k,i)\beta H} \prod_{\ell=1}^{n_z} Z_\ell^{a(\ell,k,i)}, \tag{19}$$

where $n_i \le 2^{N_B}$. These relations are exact for *all* stages of construction of the lattice, that is, they are valid for small systems as well as in the thermodynamic limit. The total partition function of the system at each stage is determined by

$$Z = \sum_{i=1}^{n_z} c(i) Z_i, \tag{20}$$

where $c(i)$ is the number of times the partial partition function Z_i is found in the set $\{Z_i,\ i=1,\ldots,2^{N_A}\}$.

The recursive relations for the partial partition functions, together with the relations between these functions and the set of interaction parameters $\{K_i\}$ (which represent the parameters of the standard real-space RG procedure), implicitly contain the recursive relations for the set of interactions $\{K_i, i=1,\ldots,n_z\}$. We have demonstrated this fact by recovering the exact RG formulas for SG fractals with $b=2$ and $b=3$, for $H=0$ [2,3], and for $b=2$, when $H \neq 0$ [2], that were obtained by the standard RG decimation technique.

We conclude this section by noting that the set of all independent partial partition functions can be regarded as a basis in the parameter space of real-space RG transformations, obtained by nonlinear transformations of the original basis (represented by the interaction parameters). In this new basis the RG recursive relations acquire a very simple (polynomial) form. We now proceed to the higher members of the SG fractal family, and to the question of fractal-to-Euclidean crossover of thermodynamic response functions.

III. FRACTAL-TO-EUCLIDEAN CROSSOVER OF THE THERMODYNAMIC RESPONSE FUNCTIONS

In this section we will first address the question of calculating the response functions from the exact recursive relations for the partial partition functions, and then we will present our data calculated for $2 \le b \le 15$, for $H = 0$, and $2 \le b \le 8$, for $H \neq 0$.

In the case of members of SG fractal family with $b > 3$ the number of diagrams corresponding to *one* recursive relation is 2^{N_B}, where $N_B = \frac{(b+1)(b+2)}{2} - 3$. Already for $b = 4$ the number of interior spins is $N_B = 12$, so each of the recursive relations for the four partial partition functions is determined by 4096 diagrams. The diagrammatic technique can, however, be computerized, and we have developed a computer algorithm for determining coefficients $d(k,i)$ and exponents $b(k,i)$ and $a(\ell,k,i)$ ($k = 1,\dots n_i; i = 1,\dots 4; \ell = 1,\dots n_z$) in recursive relations (19). In this way we have obtained results for $b \le 8$, in the case $H \neq 0$, and for $b \le 15$ when $H = 0$ [11]. The actual obtained data are too massive to be presented here in the form of tables (each recursive relation for $b = 8$, $H \neq 0$ consists of approximately 35000 terms), but can be obtained from the authors upon request in the form of data files.

As it was mentioned in the preceding section, relations (7) and (8) between sets $\{Z_1, Z_2, Z_3, Z_4\}$ and $\{K, K_3, B, C\}$ are valid for all members of the SG fractal family (for $2 \le b < \infty$), so that the RG recursive relations between parameters $\{K, K_3, B, C\}$ in two consecutive stages of construction can, in principle, be obtained from the corresponding recursive relations for the partial partition functions. Such explicit analytic expressions, however, become too lengthy, already for $b = 3$ in nonzero field. Nevertheless, the thermodynamic response functions at each stage of construction can be obtained directly from the matrices $\hat{a}$, $\hat{b}$, and $\hat{d}$ that determine the recursive relations for the partition functions.

A. Calculation of the response functions

Specific heat, magnetization, and susceptibility per spin at the nth stage of fractal construction are given by the general formulas

$$C_H = \frac{1}{N}\left[k_B T^2 \frac{1}{Z}\frac{\partial^2 Z}{\partial T^2} + 2k_B T \frac{1}{Z}\frac{\partial Z}{\partial T} - k_B T^2 \left(\frac{1}{Z}\frac{\partial Z}{\partial T}\right)^2\right], \tag{21a}$$

$$\langle m \rangle = \frac{k_B T}{N}\frac{1}{Z}\frac{\partial Z}{\partial H}, \tag{21b}$$

$$\chi = \frac{k_B T}{N}\left[\frac{1}{Z}\frac{\partial^2 Z}{\partial H^2} - \left(\frac{1}{Z}\frac{\partial Z}{\partial H}\right)^2\right], \tag{21c}$$

where N is the total number of spins. From here one can proceed to calculate the partition function at a given stage (for different values of temperature and field), and then perform numerical differentiation to calculate the response functions. It turns out, however, that requirements for computational precision rapidly increase with system size (higher stages of construction and/or higher members of the fractal family). In fact, quadruple FORTRAN precision REAL*16 is not sufficient to analyze our data. This problem can be solved by finding the appropriate recursive relations for the field and temperature derivatives of the partial partition functions. Formally, differentiating (19) with respect to temperature we obtain

$$\frac{\partial Z_i'}{\partial T} = \sum_{k=1}^{n_i} d(k,i) e^{\beta H b(k,i)} \left(\frac{-Hb(k,i)}{k_b T^2} + \sum_{\ell=1}^{n_z} a(\ell,k,i)\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial T}\right) \prod_{\ell'=1}^{n_z} Z_{\ell'}^{a(\ell',k,i)}, \tag{22a}$$

and

$$\begin{aligned}\frac{\partial^2 Z_i'}{\partial T^2} = \sum_{k=1}^{n_i} d(k,i) e^{\beta H b(k,i)} \Bigg\{ & \left(\frac{Hb(k,i)}{k_b T^2}\right)^2 + \frac{2Hb(k,i)}{k_b T^3} \\ & + \sum_{\ell=1}^{n_z} a(\ell,k,i)\left[-\frac{2Hb(k,i)}{k_b T^2}\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial T} + [a(\ell,k,i)-1]\left(\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial T}\right)^2 + \frac{1}{Z_\ell}\frac{\partial^2 Z_\ell}{\partial T^2}\right] \\ & + \sum_{\ell=1}^{n_z} a(\ell,k,i)\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial T}\sum_{\ell'\neq\ell} a(\ell',k,i)\frac{1}{Z_{\ell'}}\frac{\partial Z_{\ell'}}{\partial T}\Bigg\} \prod_{\ell''=1}^{n_z} Z_{\ell''}^{a(\ell'',k,i)},\end{aligned} \tag{22b}$$

and differentiating (19) with respect to field we find

$$\frac{\partial Z_i'}{\partial H} = \sum_{k=1}^{n_i} d(k,i) e^{\beta H b(k,i)} \left(\beta b(k,i) + \sum_{\ell=1}^{n_z} a(\ell,k,i)\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial H}\right) \prod_{\ell'=1}^{n_z} Z_{\ell'}^{a(\ell',k,i)}, \tag{22c}$$

and

$$\frac{\partial^2 Z_i'}{\partial H^2} = \sum_{k=1}^{n_i} d(k,i) e^{\beta H b(k,i)} \left\{ \beta^2 b(k,i)^2 + \sum_{\ell=1}^{n_z} a(\ell,k,i) \left[2\beta b(k,i) \frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial H} + [a(\ell,k,i) - 1] \left(\frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial H} \right)^2 + \frac{1}{Z_\ell}\frac{\partial^2 Z_\ell}{\partial H^2} \right] \right.$$
$$\left. + \sum_{\ell=1}^{n_z} a(\ell,k,i) \frac{1}{Z_\ell}\frac{\partial Z_\ell}{\partial H} \sum_{\ell' \neq \ell} a(\ell',k,i) \frac{1}{Z_{\ell'}}\frac{\partial Z_{\ell'}}{\partial H} \right\} \prod_{\ell''=1}^{n_z} Z_{\ell''}^{a(\ell'',k,i)}. \tag{22d}$$

Equations (22) are the recursive relations for the *derivatives* of the partial partition functions. Starting from initial values for the partial partition functions and their derivatives at some particular stage (e.g., zeroth stage) of construction, for given temperature and field, by successive application of (22) the derivatives, can be obtained for all higher stages of construction. The specific heat, magnetization, and susceptibility can be now obtained for arbitrary stage using (2) and (21) *without using numerical differentiation.*

B. Results and discussion

In Fig. 4 we present the specific heat and susceptibility of the Sierpiński gasket ($b = 2$) for several initial stages

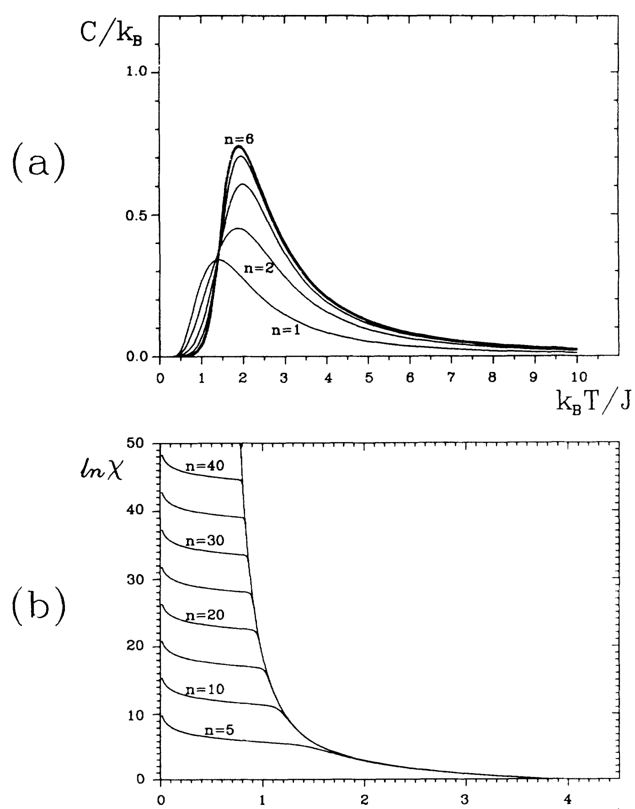

FIG. 4. (a) Specific heat and (b) susceptibility for several initial stages n of construction of the Sierpiński gasket ($b = 2$). While specific heat saturates after just a few steps (and thus well represents behavior of the infinite fractal), susceptibility increases with increasing stage of construction (the envelope of the finite stage curves corresponds to the susceptibility of the infinite fractal). Corresponding curves for higher members of the Sierpiński-gasket family show similar behavior.

of construction. While the specific-heat curves rapidly saturate into a definite curve, after just a few steps of fractal construction, the susceptibility curves gradually increase with increasing stage of construction in such a way that the envelope of the finite stage curves corresponds to the susceptibility of the infinite fractal lattice. Although in the limit $n \to \infty$ susceptibility eventually diverges at $T = 0$ [2], the fact that the absolute value of the derivative of $\ln \chi$ with respect to temperature is rather large at finite temperatures corresponds to the extremely slow decay of correlations with increasing temperature [5]. The presented results compare well with the appropriate curves of Refs. [2,3], and are shown here to illustrate the extent to which the calculations for finite stage lattices can represent the infinite system behavior. Specific-heat curves of the higher members of the SG family also saturate after just a few steps, while the susceptibility curves for infinite fractals again correspond to the envelopes of finite stage curves.

The saturated specific-heat curves (large n) for all our data ($b \leq 15$), together with the specific-heat curves for the corresponding generators, are depicted in Fig. 5. The specific-heat curve for the infinite triangular lattice [12] is also shown for comparison. Except for the first few members of the SG family, the fractals and their generators appear to approach the triangular lattice behavior in a quite similar manner. The initial difference in behavior can be attributed to the fact that for small b, fractals ($n \to \infty$) and their generators ($n = 1$) have different percentages of bulk spins (with coordination number $z = 6$), edge spins ($z = 4$), and the corner spins ($z = 2$). For large b these differences diminish. The fractal-to-Euclidean crossover of the specific heat thus can be expected to be smooth and to be equivalent to the thermodynamic limit approach through the fractal generators (that is, through the infinite set of the finite size Euclidean lattices).

Temperature dependence of susceptibility for members of the SG family with $b \leq 8$ is shown in Fig. 6. In Fig. 6(a) we present the logarithm of χ for $n \to \infty$ (envelopes of the finite n curves), while in Fig. 6(b) we present $\ln \chi$ for the corresponding fractal generators. Susceptibilities of the infinite triangular lattice (from high temperature series [13]) and of the Ising chain are shown for comparison in both cases. In contrast to the specific-heat situation, the susceptibility curves of infinite fractal lattices show quite different behavior (even when b increases) from the susceptibility curves of the corresponding generators. This difference is so pronounced that one can hardly say that in the susceptibility case the fractal-to-Euclidean crossover, as a limiting process, can become equivalent to the thermodynamic limit approach. The physical reason for such a difference lies in

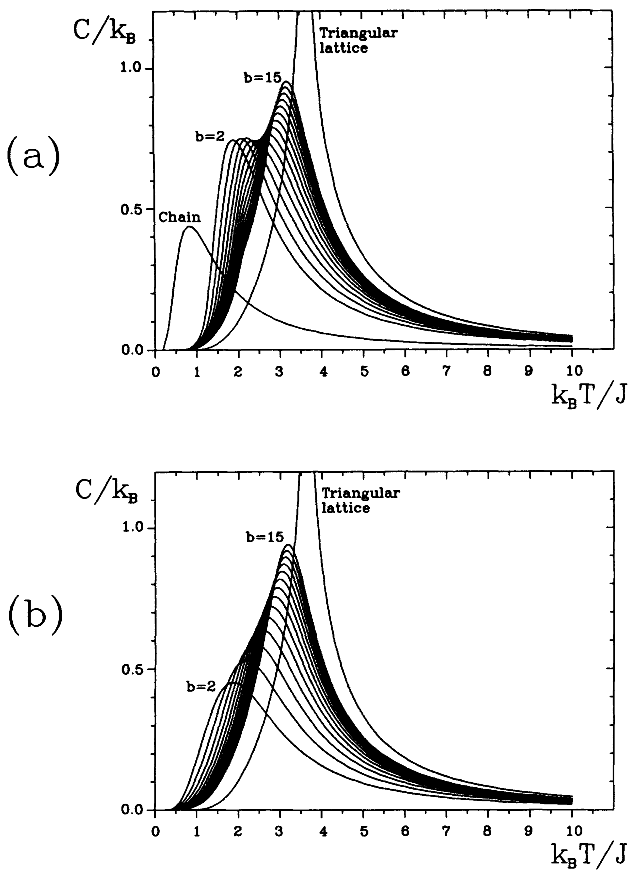

FIG. 5. (a) Saturated specific-heat curves (large n) for the members of the Sierpiński-gasket family with $2 \leq b \leq 15$, and (b) the specific-heat curves for the corresponding generators (equilateral wedges of the triangular lattice). Specific heat of the Ising chain and of the infinite triangular lattice are also shown for comparison. For large generator base b the specific-heat curves of infinite fractals become very similar to those of the corresponding generators, and are thus explained by finite size scaling. Irregular behavior of the Schottky peak maxima, for the first few members, is attributed to strong dependence of percentages of spins with different coordination numbers, on the stage of construction.

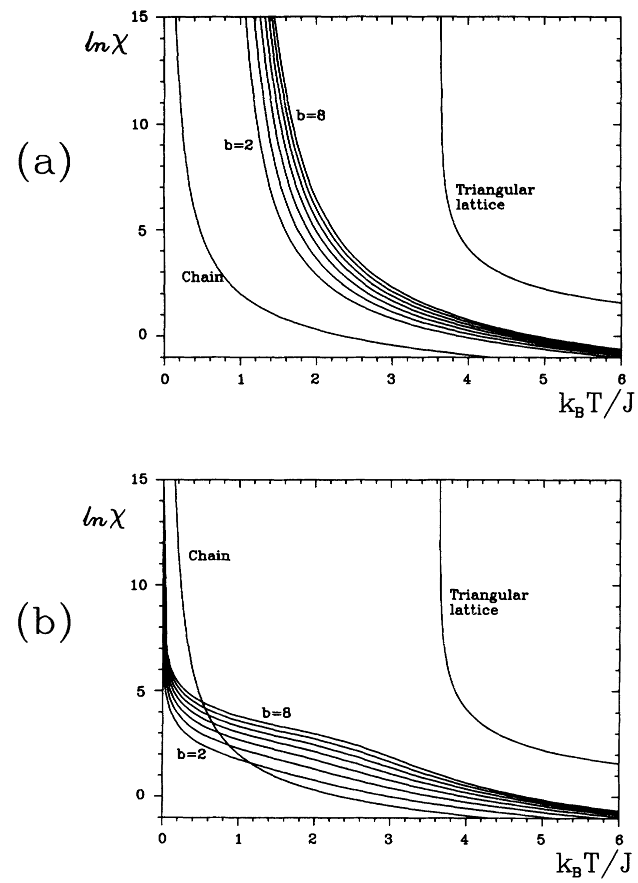

FIG. 6. Logarithm of susceptibility of the members of the Sierpiński-gasket family with $2 \leq b \leq 8$ for (a) infinite fractal lattices and (b) for the corresponding generators. Susceptibility of the Ising chain and high temperature susceptibility of the triangular lattice are also shown for comparison in both cases. While finitely ramified fractals do not exhibit a phase transition at nonzero temperatures, susceptibility appears to be surprisingly large before temperature reaches the zero value (for all b) indicating early appearance of long-range spin-spin correlations, in contrast to the case of generators ($n = 1$).

the long range spin-spin correlations that exist in the infinite fractal lattices at finite temperatures [2,4]. A better impression about the crossover behavior of the fractal susceptibilities can be gained from Fig. 7 in which $1/\chi$ is drawn versus temperature for the first seven members of the SG fractal family. It should be noted that our data (for C and χ) are obtained from *exact* recursive relations, without using numerical differentiation. In this sense all the curves may be regarded as *exact*, since any region may be magnified with arbitrary precision.

In conclusion, we can state that our results make expectation about the smooth fractal-to-Euclidean crossover for the thermodynamic response functions more plausible. Note, for instance, that no such conclusion could have been drawn if only the specific-heat curves for the first five members of the fractal family were available [see Fig. 5(a)]. In this respect, our results are closely

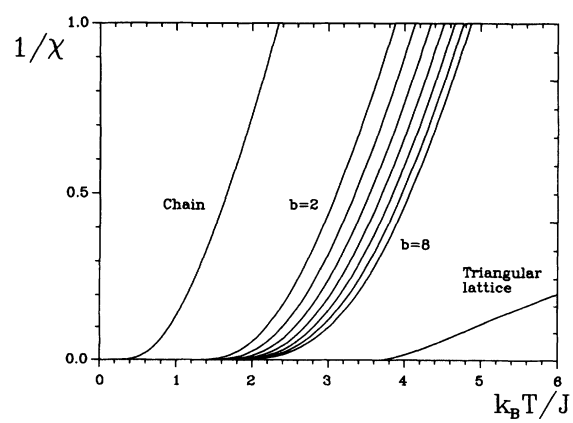

FIG. 7. Inverse susceptibility for the first seven members of the Sierpiński-gasket family, together with susceptibility of the Ising chain and the triangular lattice.

related to the rigorous proof [14,15] that the free energy per spin of the Ising model system on the fractal lattices under study converge to the free energy per spin of the same model situated on the Euclidean triangular lattice. However, when the latter proof was provided a word of caution was made concerning the possibility of a smooth crossover of the thermodynamic response functions (because of the simple fact that response functions are the second derivatives of the free energy). Now, on the grounds of presented findings, one can rightfully expect the smooth crossover for the thermodynamic response functions as well.

Independent studies have also been performed [16] on the fractal-to-Euclidean crossover of the Ising model on a family of hierarchical lattices. The main difference between these lattices and the Sierpiński gasket family is that they display critical behavior at nonzero temperatures. Evidence was found [16] that the corresponding sequence of critical exponents converges towards the exact values of the Euclidean lattice. A rigorous proof of the existence of the thermodynamic limit for the free energy of various spin models on hierarchical lattices was also given in [17].

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[11] The algorithm we use is of the transfer-matrix type, in the sense that when calculating the parameters for a given b, information acquired for a previous value of b is used. For $b = 15$ there are $2^{133} \sim 10^{40}$ diagrams, and their direct enumeration would be far out of reach of present-day computer hardware. Our longest runs required approximately 10 hours CPU time on an IBM RS6000 workstation, and 100 Mbytes of virtual memory. The algorithm can be obtained from the authors upon request.
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