Optimization of the robustness of multimodal networks

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We investigate the robustness against both random and targeted node removal of networks in which $P(k)$, the distribution of nodes with degree k, is a multimodal distribution, $P(k) \propto \sum_{i=1}^{m} a^{-(i-1)}\delta(k-k_i)$ with $k_i \propto b^{-(i-1)}$ and Dirac's delta function $\delta(x)$. We refer to this type of network as a scale-free multimodal network. For $m=2$, the network is a bimodal network; in the limit m approaches infinity, the network models a scale-free network. We calculate and optimize the robustness for given values of the number of modes m, the total number of nodes N, and the average degree $\langle k\rangle$, using analytical formulas for the random and targeted node removal thresholds for network collapse. We find, when $N \gg 1$, that (i) the robustness against random and targeted node removal for this multimodal network is controlled by a single combination of variables, $N^{1/(m-1)}$, (ii) the robustness of the multimodal network against targeted node removal decreases rapidly when the number of modes becomes larger than a critical value that is of the order of $\ln N$, and (iii) the values of exponent λ^{opt} that characterizes the scale-free degree distribution of the multimodal network that maximize the robustness against both random and targeted node removal fall between 2.5 and 3.

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I. INTRODUCTION

Many real world networks have been found to be scale-free networks, for which $P(k)$, the distribution of nodes of degree k, have the form $P(k) \sim k^{-\lambda}$ [1–11]. Degree distribution is only one characteristic of networks; in real networks such features as clustering and degree-degree correlations play an important role. Here we consider random graphs with a particular degree distribution and our results apply to such networks.

One of the properties of scale-free networks is their robustness against random failure of nodes. At the same time, however, the scale-free network is easy to collapse when a small number of highly connected nodes ("hubs") are selectively removed. Formally, we define the thresholds, f_r and f_t, as the fraction of nodes necessary to be removed in order to destroy the giant component of a given network by random and targeted node removal, respectively [12–17]. Then $f_r \approx 1$ and $f_t \ll 1$ for scale-free networks [14–22].

We define the optimal degree distribution for robustness against both random and targeted node removal as the degree distribution that maximizes the sum of the two critical fractions (thresholds), which is denoted by $f_T \equiv f_r + f_t$, for given values of the parameters that specify the network under consideration [20,23]. Since f_r and f_t are bounded by $[0,1]$, f_T is bounded by $[0,2]$. In previous work, the authors determined that the network that is maximally robust against both random and targeted node removal [20,21] is a network characterized by the bimodal degree distribution

$$P(k) = \begin{cases} r_1 = 1 - r_2 & k = k_1 \le \langle k\rangle, \\ r_2 \equiv \left(\dfrac{A^2}{\langle k\rangle N}\right)^{3/4} & k = k_2 = \sqrt{\langle k\rangle N}, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where

$$A \equiv \left\{\frac{2\langle k\rangle^2(\langle k\rangle - 1)^2}{2\langle k\rangle - 1}\right\}^{1/3} \tag{2}$$

with the average degree $\langle k\rangle$ (see also Ref. [22]). For this optimal bimodal network, $f_r \approx 1$ and $f_t \approx 1 - 1/(\langle k\rangle - 1)$ with each of these thresholds approaching its theoretical maximum asymptotically as $N \to \infty$.

This bimodal network can be considered to be a network that appears to be consistent with a power-law degree distribution, $P(k) \propto k^{-\lambda}$, with

$$\lambda = -\frac{\ln r_2}{\ln k_2} + 1 \underset{N\to\infty}{\longrightarrow} \frac{5}{2} = 2.5. \tag{3}$$

Considering that the scale-free network does not have the robustness against targeted removal of nodes that the bimodal network does have, while both of them are equally robust against random removal of nodes, we study how the robustness against targeted node removal is lost as the number of degrees contained in the degree distribution increases. An important question is whether the robustness against targeted node removal is lost gradually or abruptly as we increase the number of degrees contained in the degree distribution.

Motivated by this consideration, we investigate the robustness of the networks with a degree distribution specified by

$$P(k) = \sum_{i=1}^{m} r_i \delta(k - k_i) = \sum_{i=1}^{m} r_1 a^{-(i-1)} \delta(k - k_i) \tag{4}$$

with $k_i = k_1 b^{-(i-1)}$, where $\delta(x)$ is Dirac's delta function. A specific realization of graphs subject to this degree distribution is generated by the classical algorithm in the so-called configuration model [11–13]. We refer to this type of network as a scale-free multimodal network. This degree distribution is characterized by three parameters: a that controls the fraction of nodes having different degrees and is larger than 1, b that controls the values of the degrees and takes a value in the range between 0 and 1, and k_1 that is the smallest possible degree. The normalization constant r_1 is the fraction of nodes that has the smallest possible degree, k_1. The network consists of nodes with m distinct degrees, and we refer to m as the number of "modes." By varying m, we are able to study the robustness of power-law networks from the bimodal network ($m=2$) to the scale-free network ($m\to\infty$) in a unified way. By changing two parameters, a and b, we can determine the values for the optimal network that maximizes the sum of two thresholds, f_r and f_t, against random and targeted node removal. All calculations for optimization are performed for given values of the total number of nodes N, and for a fixed average degree $\langle k\rangle$.

This paper is organized as follows. In Sec. II, we mathematically define the scale-free multimodal network model. In Secs. III and IV, we derive the analytical expressions for the critical removal fractions (thresholds), f_r and f_t, necessary to destroy the giant component of the multimodal network by random and targeted node removal. In Sec. V, we identify the controlling combination of parameters, $N^{1/(m-1)}$. In Sec. VI, we calculate the optimal values of parameters that maximize the sum of the thresholds, f_r+f_t, and determine the corresponding optimal values of the exponent in the scale-free multimodal degree distribution. In particular, we focus on the behavior of the robustness against targeted node removal as the number of modes m increases in the optimization. Section VII summarizes our results.

II. MODEL

The multimodal network of m modes is defined by the degree distribution

$$P(k)=\sum_{i=i}^{m} r_i\delta(k-k_i), \tag{5}$$

where $\delta(x)$ is Dirac's delta function and

$$r_i \equiv r_1 a^{-(i-1)} \quad (a>1;\ i=1,2,\ldots,m) \tag{6}$$

is the fraction of nodes that have

$$k_i \equiv k_1 b^{-(i-1)} \quad (0<b<1;\ i=1,2,\ldots,m) \tag{7}$$

links. The parameter a is a real number larger than 1, and the parameter b is a real number between zero and one. Hence $r_1>r_2>\cdots>r_m$ and $k_1<k_2<\cdots<k_m$.

The degree distribution of the multimodal network obeys a power law

$$P(k_i)=r_i \propto k_i^{-\lambda}, \tag{8}$$

where, as shown in Appendix A,

$$\lambda=-\frac{\ln a}{\ln b}+1. \tag{9}$$

From the normalization condition

$$\sum_{i=1}^{m} r_i = r_1\sum_{i=1}^{m} a^{-(i-1)}=1, \tag{10}$$

we have

$$r_1=\frac{1-a^{-1}}{1-a^{-m}}, \quad \text{or} \quad r_m=\frac{a-1}{a^m-1}, \tag{11}$$

since $a\neq 1$. When the total number of nodes is N and the number of the nodes that have the highest degree k_m is q, the value of a is determined by the equation

$$\frac{a-1}{a^m-1}=\frac{q}{N}. \tag{12}$$

The average degree $\langle k\rangle$ is calculated by

$$\langle k\rangle=\sum_{i=1}^{m} k_i r_i = k_1 r_1 \sum_{i=1}^{m}(ab)^{-(i-1)}. \tag{13}$$

By fixing the values of $\langle k\rangle$ and k_1, this equation determines a relation between a and b. Since the value of a is determined by Eq. (12), we have two free parameters, q and k_1, for the optimization of the scale-free multimodal network for given values of m, N, and $\langle k\rangle$. Notice that Eqs. (12) and (13) are both mth order algebraic equations, which we can solve only numerically for $m\geq 5$.

III. THRESHOLD f_r AGAINST RANDOM FAILURES OF NODES

We consider networks that are *simple*, i.e., the probability of two edges linking the same pairs of nodes in constructing a network satisfying a given degree distribution is negligible. There is no loss of generality in restricting the networks we consider to be simple because multiple edges linking the same pairs of nodes add nothing to robustness against node removal. The requirement that the network be simple is reflected in the constraint that the largest degree with nonzero probability k_m must obey

$$k_{\max}\equiv\sqrt{\langle k\rangle N}. \tag{14}$$

This constraint is valid asymptotically for graphs with a specified degree distribution created randomly using the configuration model [24–27]. The threshold against random failures is calculated from the formula suitable for simple graphs [16,28],

$$f_r=1-\frac{1}{\kappa-1}, \tag{15}$$

where $\kappa=\langle k^2\rangle/\langle k\rangle$, which can be calculated exactly in our model as

$$\kappa=r_1\frac{k_1^2}{\langle k\rangle}\frac{1-(ab^2)^{-m}}{1-(ab^2)^{-1}}. \tag{16}$$

The averages are taken over the degree distribution before node removal. The optimal configuration for random

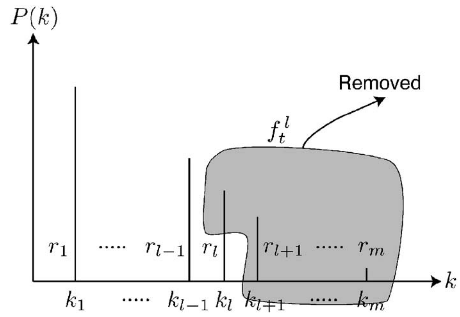

FIG. 1. Schematic representation of the degree distribution for the scale-free multimodal network and the part of the degree distribution removed by a targeted attack on the higher degree modes between k_l and k_m.

removal only is obtained with a bimodal distribution in which the highest degree nodes have degree $k_{\max}$ [29].

IV. THRESHOLD f_t AGAINST TARGETED NODE REMOVAL

Here we calculate the threshold f_t, that is the fraction of highest degree nodes which must be removed before the network loses its global connectivity. We employ the approach of Refs. [18,19] as follows: In addition to changing the maximum degree of the distribution, the removal of high degree nodes causes another effect—since the links that lead to removed nodes are eliminated, the degree distribution also changes. This effect is equivalent to the random removal of a fraction $\tilde{p}$ of nodes, where $\tilde{p}$ is the ratio of the number of links removed divided by the total number of links before the removal. Since the effect is equivalent to a random removal of nodes, Eq. (15) with f_r replaced by $\tilde{p}$ and with κ calculated after the removal of nodes can then be used to calculate the effect of the link removal. That is, we calculate the threshold against targeted node removal by solving the equation

$$1-\tilde{p}_l=\frac{1}{\kappa'-1}, \tag{17}$$

where the prime in κ' represents that the average is taken over the degree distribution after the removal of nodes. In addition, we use l to denote the index of the lowest degree nodes removed as shown in Fig. 1. Thus f_t^l is the removal fraction of nodes necessary to destroy the giant component of the network under consideration when the lowest degree necessary to be removed is k_l.

Since the calculation of f_t^l is straightforward but tedious, we describe the detail in Appendix B and only show the outline in the following.

By defining $A_l \equiv \Sigma_{i=1}^{l} a^{-(i-1)}$ and $B_l \equiv \Sigma_{i=1}^{l}(ab)^{-(i-1)}$, the left hand side of Eq. (17) becomes

$$1-\tilde{p}_l=\frac{k_l}{\langle k\rangle}\{(1-r_1A_l+b^{l-1}r_1B_l)-f_t^l\}=\frac{k_l}{\langle k\rangle}(\alpha_l-f_t^l) \tag{18}$$

with

$$\alpha_l \equiv 1-r_1A_l+b^{l-1}r_1B_l. \tag{19}$$

Similarly, by defining $C_l \equiv \Sigma_{i=1}^{l}(ab^2)^{-(i-1)}$, we have

$$\kappa' \equiv \frac{\langle k^2\rangle'}{\langle k\rangle'}=k_l\frac{\gamma_l-f_t^l}{\beta_l-f_t^l} \tag{20}$$

with

$$\beta_l \equiv 1-r_1A_{l-1}+b^{l-1}r_1B_{l-1} \tag{21}$$

and

$$\gamma_l=1-r_1A_{l-1}+b^{2(l-1)}r_1C_{l-1}. \tag{22}$$

Substituting Eqs. (18) and (20) into Eq. (17), we find that Eq. (17) is just a quadratic equation in terms of f_t^l:

$$u_l(f_t^l)^2-v_lf_t^l+w_l=0 \quad (l=1,2,\ldots,m), \tag{23}$$

with

$$u_l \equiv k_l-1, \tag{24}$$

$$v_l \equiv k_l\gamma_l-\beta_l+\alpha_l(k_l-1)-\frac{\langle k\rangle}{k_l}, \tag{25}$$

$$w_l \equiv \alpha_l(k_l\gamma_l-\beta_l)-\frac{\langle k\rangle}{k_l}\beta_l. \tag{26}$$

The threshold f_t^1 can be easily calculated as

$$f_t^1=1-\frac{\langle k\rangle}{k_1(k_1-1)} \tag{27}$$

but the calculation of the threshold f_t^l for $2 \le l \le m$ is performed numerically because of the complexity of the coefficients (24)–(26).

The threshold against targeted node removal f_t is the only solution of Eq. (23) for all $l(=1,2,\ldots,m)$ that satisfies the inequality

$$\begin{cases} 1-r_1 \le f_t^l \le 1 & (l=1), \\ 1-\sum_{i=1}^{l} r_i \le f_t^l \le 1-\sum_{i=1}^{l-1} r_i & (2 \le l \le m). \end{cases} \tag{28}$$

V. SCALING RELATION BETWEEN n AND m

The fraction of the highest degree nodes, r_m, is related to the parameter a by Eq. (11), which is restated as

$$a^m-r_m^{-1}a+r_m^{-1}-1=(a-1)(a^{m-1}+a^{m-2}+\cdots+1-r_m^{-1})=0. \tag{29}$$

The trivial solution, $a=1$, is excluded because the parameter a should take a value larger than 1 in our model.

If we distribute q highest degree nodes in N total nodes,

$$r_m=\frac{q}{N}\equiv N^{\alpha-1}, \tag{30}$$

where we introduce a new exponent α and put $q=N^{\alpha}$. For the bimodal network that is the most robust against both random and targeted node removal, $\alpha \approx 0.25$ as seen from Eq. (1) that states $r_2 \sim N^{-3/4}$. The smallest possible value of α is zero that corresponds to $q=1$. Thus we expect that $0 < \alpha \lesssim 0.25$.

When N is large, Eqs. (29) and (30) yield the asymptotic relation

$$a \sim N^{(1-\alpha)/(m-1)}, \tag{31}$$

or $\ln_N a \approx (1-\alpha)/(m-1)$.

The asymptotic form for b is derived as follows. The average degree is calculated by Eq. (13), which is restated as

$$\langle k \rangle = k_1 r_1 \sum_{i=1}^{m} (ab)^{-(i-1)}. \tag{32}$$

When $ab=1$, Eq. (32) becomes $\langle k \rangle = m k_1 r_1$. From Eqs. (11) and (31), the fraction of the lowest degree nodes r_1 for large N becomes $r_1 \approx 1 - N^{-(1-\alpha)/(m-1)}$. Thus we have an asymptotic relation $\langle k \rangle = m k_1$ in this case. Since the lowest degree k_1 must satisfy the inequality $1 \le k_1 < \langle k \rangle$, this asymptotic relation implies the inequality $1 < m \le \langle k \rangle$, which is incompatible to our model where we choose both $\langle k \rangle$ and m as free parameters. Thus we exclude the solution $ab=1$ in the following.

Equation (32) can be written also in the form

$$\langle k \rangle = k_m r_m \sum_{i=1}^{m} (ab)^{i-1}, \tag{33}$$

where $r_m = N^{\alpha-1}$ as stated previously. For the optimal bimodal network, the highest degree k_m takes the maximum allowable value for the network to be simple, $k_{\max} \equiv \sqrt{\langle k \rangle N}$. We assume that k_m also takes the values of the order of $k_{\max}$ for the scale-free multimodal network with $m > 2$. Thus by setting $k_m = k_{\max}$, the product ab is determined using Eq. (33) to be

$$(ab)^{m-1} + (ab)^{m-2} + \cdots + 1 = \frac{\langle k \rangle}{k_m r_m} = \frac{\langle k \rangle}{\sqrt{\langle k \rangle N} \cdot N^{\alpha-1}} = \sqrt{\langle k \rangle} N^{1/2-\alpha}. \tag{34}$$

Assuming $ab \gg 1$, we can solve Eq. (34) by considering only the leading term of the left hand side and find

$$ab \sim N^{(1/2-\alpha)/(m-1)}, \tag{35}$$

which is consistent with the assumption, $ab \gg 1$. Using the asymptotic relation for a, Eq. (31), we find the asymptotic relation for b,

$$b \sim N^{(1/2-\alpha)/(m-1)-(1-\alpha)/(m-1)} \sim N^{-1/2(m-1)}. \tag{36}$$

or $\ln_N b \approx -1/[2(m-1)]$. It is interesting that the exponent α is not present in the asymptotic relation for b.

From the two asymptotic relations (31) and (36), we see that the degree distribution for this multimodal network is controlled by a single combination of the total number of nodes N and the mode number m which is $N^{1/(m-1)}$, when $N \gg 1$. Since we fix the total number of nodes N and observe how the response of the multimodal network to random and targeted node removal varies with mode number m we introduce the "scaled" mode number,

$$m^* \equiv \frac{m-1}{\log_{10} N}, \tag{37}$$

which is the inverse of the logarithm of $N^{1/(m-1)}$ for the analysis of the results in the following sections.

VI. OPTIMIZATION OF ROBUSTNESS

A. Optimal configuration and thresholds

As is described in Sec. I, we take the sum of the two thresholds, $f_T \equiv f_r + f_t$, as a measure for the robustness against both random and targeted node removal and define the optimal degree distribution for robustness as the degree distribution which maximizes the measure f_T for given values of the number of modes m, the total node number N, and the average degree $\langle k \rangle$. We adjust two parameters for the optimization; the one is the number of the highest degree nodes, $q \equiv N^{\alpha}$, for which the exponent α should take the value between 0 and 0.25, and the other is the lowest degree, k_1, which should take the value between 1 and $\langle k \rangle$.

In Fig. 2 we plot the values of $\log_N a^{\text{opt}}$ and $\log_N b^{\text{opt}}$ that optimize the total threshold versus the mode number m. The plots are consistent with the asymptotic relations, Eqs. (31) and (36), for large values of m and N. We also see that the asymptotic relation for a, Eq. (31), with $\alpha=0$, provides the upper bound of $\log_N a^{\text{opt}}$ for all values of m and that the asymptotic relation for b, Eq. (36), provides the lower bound of $\log_N b^{\text{opt}}$ for all values of m.

In Fig. 3(a), we plot the optimal values of the measure f_T^{opt} for network collapse in terms of the bare mode numbers m for $\langle k \rangle = 2.8$. The data are re-plotted in terms of the scaled mode number, $m^* \equiv (m-1)/\log_{10} N$ in Fig. 3(b). The collapse of the data on a single curve confirms the validity of the argument in the previous section that m^* is the controlling parameter. The results for other values of $\langle k \rangle$ are similar to this plot.

The optimal values of the measure, f_T^{opt}, decreases from the maxima at $m=2$ for every value of N as the number of modes increases. This is because the robustness against targeted node removal rapidly decreases from the maximum value $1 - 1/(\langle k \rangle - 1)$ at $m=2$ as the number of high degree nodes increases due to the increase in mode number, while the robustness against random failure remains the values approximately equal to 1 under increase in mode number, as we can see in Fig. 4. It should be emphasized that the decrease in the sum of thresholds due to this loss of robustness against targeted node removal takes place at rather small values of m of the order of $\log_{10} N$.

We define the critical scaled mode number m_c^* for a given value of $\langle k \rangle$, as the average of the values of the scaled mode number at which the profiles of the optimal total thresholds change for given values of the total number of nodes N. For example, the critical scaled mode number m_c^* for $\langle k \rangle = 2.8$ is approximately 0.7 [see Fig. 3(b)]. When the scaled mode number is larger than m_c^*, the scale-free multimodal network becomes very fragile against targeted attacks and thus behaves essentially like the scale-free network. In the following, we concentrate on the behavior of important quantities

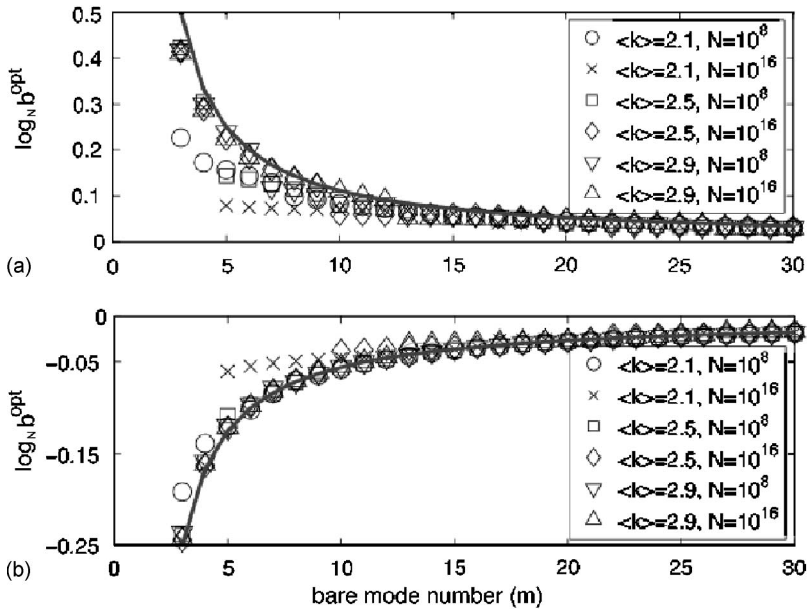

FIG. 2. (a) The dependence of the values of $\ln_N a^{\rm opt}$ for the values of a that maximize the total thresholds on the total number of modes m. The thick curve stands for $1/(m-1)$. (b) The dependence of the values of $\ln_N b^{\rm opt}$ for the values of b that maximize the total thresholds on the total number of modes m. The thick curve stands for $-1/2(m-1)$.

for the scaled mode number smaller than the critical value, m_c^*.

In Fig. 5, we plot the behavior of the exponent α, which is defined as $\log_N q^{\rm opt}$, and the behavior of the highest degree $k_m^{\rm opt}$, for the optimal configuration for $\langle k\rangle=2.8$ with respect to the scaled mode number m^* smaller than the critical value m_c^*. The values of α smoothly decrease from the values for the optimal bimodal network, which are approximately equal to 0.25, and reach zero, which is the smallest possible value corresponding to $q=1$, at $m^*=m_c^*$. For the values of the highest degree for the optimal configuration the relation $k_m^{\rm opt} \approx k_{\rm max}=\sqrt{\langle k\rangle N}$ always holds. This fact supports the assumption we made for the derivation of the asymptotic relation for b [see Eq. (36)].

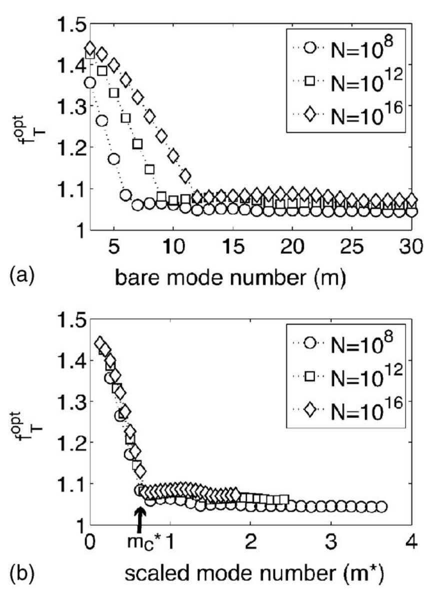

FIG. 3. (a) Optimal values for the sum of the thresholds, $f_T \equiv f_r+f_t$, versus mode number for $\langle k\rangle=2.8$ for several values of N. (b) The optimal values for the measure, $f_T^{\rm opt}$, for $\langle k\rangle=2.8$ are replotted in terms of the scaled mode number $m^* \equiv (m-1)/\log_{10}N$. The collapse of the data on a single curve is seen. The curve changes its profile at a certain value of m^*, which is about 0.7 for $\langle k\rangle=2.8$, and the value is denoted by m_c^*.

B. Critical mode number

The multimodal network loses robustness against targeted node removal at m_c^*. We perform the least square fit for the values of m_c^* and find the fit

$$m_c^* = (0.62 \pm 0.015)\langle k\rangle - (1.0 \pm 0.038) \tag{38}$$

as seen in Fig. 6. Therefore in order to make the multimodal network robust against both random failure and targeted node removal for given values of N and $\langle k\rangle$, the number of modes should be kept lower than the critical mode number m_c calculated using the formula

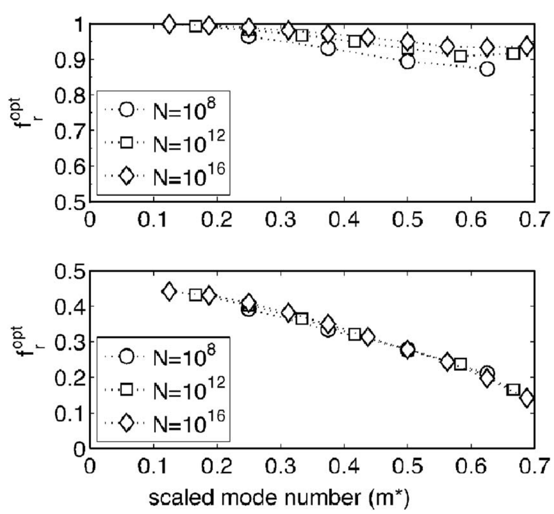

FIG. 4. The values for the threshold against random node removal f_r and the threshold against targeted node removal f_t for the optimal configuration for $\langle k\rangle=2.8$ are plotted in terms of the scaled variable $m^* \equiv (m-1)/\log_{10}N$ for $0<m^*<m_c^*\approx 0.7$.

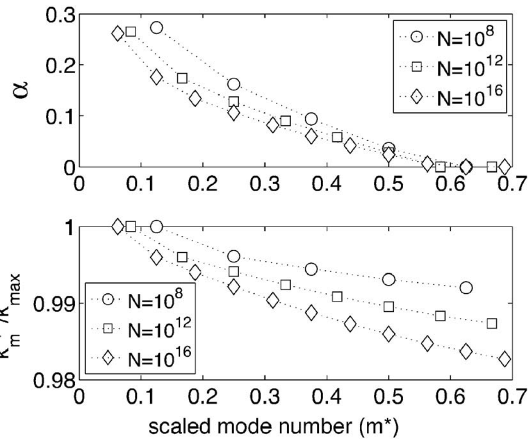

FIG. 5. The behavior of the exponent α, which is defined as $\ln_N q^{\mathrm{opt}}$, where q^{opt} is the number of the highest degree nodes, and the behavior of the highest degree k_m^{opt} for the optimal configuration for $\langle k\rangle=2.8$ in terms of the scaled mode number m^* smaller than the critical value $m_c^* \approx 0.7$. The values of α smoothly decrease from the values for the optimal bimodal network, which are approximately equal to 0.25, and reach zero, which is the smallest possible value corresponding to $q=1$, at $m^*=m_c^*$. For the values of the highest degree for the optimal configuration the relation $k_m^{\mathrm{opt}} \approx k_{\max} \equiv \sqrt{\langle k\rangle N}$ holds for $0<m^*<m_c^*$.

$$m_c = 1 + (0.62\langle k\rangle - 1.0)\log_{10}N$$

derived from Eqs. (37) and (38). If, e.g., we take $N=10^8$ and $\langle k\rangle=2.5$, then $m_c \approx 5$.

C. Optimal values of the exponent in the degree distribution

The exponent in the multimodal degree distribution for the optimal configuration λ^{opt}, is calculated by

$$\lambda^{\mathrm{opt}} = 1 - \frac{\ln a}{\ln b}.$$

For the lowest value of $m=2$, we have shown in Sec. I that the value is 2.5 [see Eq. (3)]. For large values of N, the asymptotic relations for a and b, Eqs. (31) and (36), holds. Since Eq. (31) gives the upper bound for a and Eq. (36) gives the lower bound for b, the (asymptotic) upper bound for λ^{opt}, which is denoted by $\bar{\lambda}^{\mathrm{opt}}$, is calculated as

$$\bar{\lambda}^{\mathrm{opt}} = 1 - \frac{1/(m-1)}{-1/2(m-1)} = 3.$$

Thus we can conclude that the exponents for the optimal multimodal network take the values within the interval 2.5 and 3. In Fig. 7, we plot the values of λ^{opt} in terms of the scaled mode number m^*. Since for $m^*>m_c^*$ the scale-free multimodal network effectively loses its discreteness, we only plot the data for $m^*<m_c^*$.

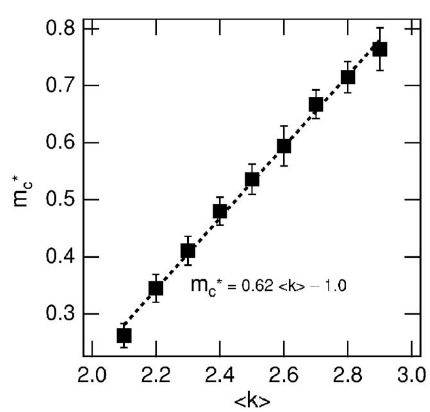

FIG. 6. The dependence of the critical values of the scaled mode number m_c^* on the average degree $\langle k\rangle$. The value of m_c^* for each $\langle k\rangle$ is the average over all m_c^* for different values of the total node number N. The critical value m_c^* is the value of the scaled mode number at which the scale-free multimodal network completely loses its robustness against targeted node removal. The broken line is the fit $m_c^*=0.62\langle k\rangle-1.0$.

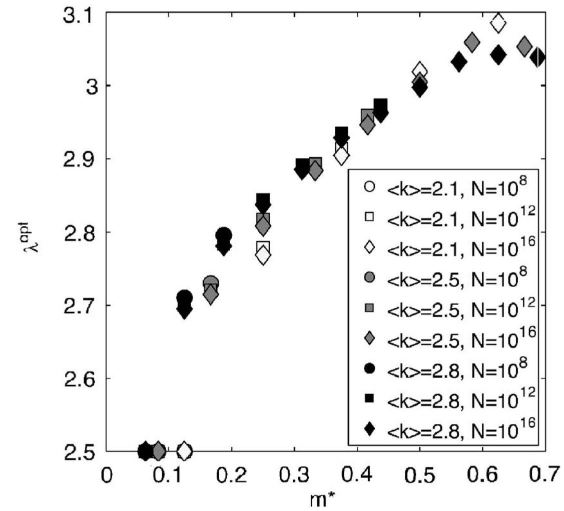

FIG. 7. The exponent in the multimodal degree distribution for the optimal configuration λ^{opt} for the scale-free multimodal degree distribution, versus the scaled variable $m^* \equiv (m-1)/\log_{10}N$ of Eq. (37). Note that, as argued in Sec. VI C, the values of λ^{opt} reside mainly in the interval between 2.5 and 3.

VII. SUMMARY

In summary, we define and investigate the robustness of scale-free multimodal networks, built as random graphs with a particular degree distribution, against random and targeted node removal and find the optimal configuration for the degree distribution of the scale-free multimodal network for given values of the total number of nodes N and the average degree $\langle k\rangle$ for each value of the mode number m. We find the following: (i) The robustness for a fixed value of the average degree $\langle k\rangle$ depends only on the "scaled" mode number, $m^* \equiv (m-1)/\log_{10}N$. (ii) The robustness against targeted node removal rapidly decreases as the mode number increases, and effectively becomes zero at a critical value of the scaled mode number, which is $m_c^*=0.62\langle k\rangle-1.0$ for a given value of $\langle k\rangle$. (iii) As the robustness against targeted node removal decreases, the values of the exponent λ^{opt} that appear in the degree distribution of the scale-free multimodal network varies from 2.5 to 3. Our work should be of value in designing a scale-free multimodal network which is robust to both random and targeted attacks. The network designer must take care not to have the number of different degrees exceed m_c^*.

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APPENDIX A: RELATION TO THE SCALE-FREE DEGREE DISTRIBUTION

To find a relationship between the multimodal degree distribution and the scale-free degree distribution, we should consider the multimodal $P(k_i)$ as made up from the corresponding scale-free distribution $P_{\mathrm{sf}}(k)$ $(=Ck^{-\lambda})$ by adding up the number of nodes within a range whose width is Δ_i:

$$P(k_i) = \int_{k_i-\Delta_i}^{k_i+\Delta_i} P_{\mathrm{sf}}(k)dk = C\int_{k_i-\Delta_i}^{k_i+\Delta_i} k^{-\lambda}dk \tag{A1}$$

$$= \frac{C}{\lambda-1}k_i^{1-\lambda}\left\{\left(1-\frac{\Delta_i}{k_i}\right)^{1-\lambda} - \left(1+\frac{\Delta_i}{k_i}\right)^{1-\lambda}\right\}. \tag{A2}$$

Since our values of k_i are distributed in k space on an exponential scale, Δ_i has almost the same order of magnitude of k_i. In this case $[\Delta_i = O(k_i)]$,

$$\left(1-\frac{\Delta}{k_i}\right)^{1-\lambda} - \left(1+\frac{\Delta}{k_i}\right)^{1-\lambda} = O(1). \tag{A3}$$

Thus

$$P(k_i) \propto k_i^{-\lambda'} \propto k_i^{1-\lambda}. \tag{A4}$$

Therefore

$$\lambda' = \lambda - 1. \tag{A5}$$

APPENDIX B: DETAIL OF THE CALCULATION OF f_t^l

For the calculation of the ratio of the links removed by the targeted attack $\tilde{p}_l$, we have

$$\begin{aligned}\tilde{p}_l &= \frac{1}{\langle k\rangle}\left\{k_l\left[f_t^l - \left(1-\sum_{i=1}^{l} r_i\right)\right] + \sum_{i=l+1}^{m} k_i r_i\right\}\\ &= \frac{1}{\langle k\rangle}\left[k_l\left(f_t^l - 1 + \sum_{i=1}^{l} r_i\right) + \langle k\rangle - \sum_{i=1}^{l} k_i r_k\right]\\ &= 1 - \frac{k_l}{\langle k\rangle}\left\{1 - \sum_{i=1}^{l} r_i + \sum_{i=1}^{l}\frac{k_i}{k_l} r_i - f_t^l\right\}.\end{aligned} \tag{B1}$$

For the average degree after the removal of nodes, we have

$$\begin{aligned}\langle k\rangle' &\propto \sum_{i=1}^{l-1} k_i r_i + k_l\left(1-\sum_{i=1}^{l-1} r_i - f_t^l\right)\\ &= k_l\left(\sum_{i=1}^{l-1}\frac{k_i}{k_l} r_i + 1 - \sum_{i=1}^{l-1} r_i - f_t^l\right)\\ &= k_l\{(1 - r_1 A_{l-1} + b^{l-1} r_1 B_{l-1}) - f_t^l\} = k_l(\beta_l - f_t^l).\end{aligned} \tag{B2}$$

For the average of squared degree after the removal of nodes, we have

$$\begin{aligned}\langle k^2\rangle' &\propto \sum_{i=1}^{l-1} k_i^2 r_i + k_l^2\left(1-\sum_{i=1}^{l-1} r_i - f_t^l\right)\\ &= k_l^2\left\{\frac{k_1^2}{k_l^2} r_1 \sum_{i=1}^{l-1}(ab^2)^{-(i-1)} + 1 - A_{l-1} - f_t^l\right\}\\ &= k_l^2\{(1 - A_{l-1} + b^{2(l-1)} r_1 C_{l-1}) - f_t^l\} = k_l^2(\gamma_l - f_t^l).\end{aligned} \tag{B3}$$

Equations (B2) and (B3) lead to Eq. (20).

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