Triple point in correlated interdependent networks

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Many real-world networks depend on other networks, often in nontrivial ways, to maintain their functionality. These interdependent "networks of networks" are often extremely fragile. When a fraction $1 - p$ of nodes in one network randomly fails, the damage propagates to nodes in networks that are interdependent and a dynamic failure cascade occurs that affects the entire system. We present dynamic equations for two interdependent networks that allow us to reproduce the failure cascade for an arbitrary pattern of interdependency. We study the "rich club" effect found in many real interdependent network systems in which the high-degree nodes are extremely interdependent, correlating a fraction α of the higher-degree nodes on each network. We find a rich phase diagram in the plane p-α, with a triple point reminiscent of the triple point of liquids that separates a nonfunctional phase from two functional phases.

Real-world infrastructures that provide essential services such as energy supply, transportation, and communications [1] can be understood as interdependent networks. Although this interdependency enhances the functionality of each network, it also increases the vulnerability of the entire system to attack or random failure [2]. In these interdependent infrastructures, the disruption of a small fraction of nodes in one network can generate a failure cascade that disconnects the entire system.

Failure cascades in real-world interdependent systems, such as the 2003 electrical blackout in Italy caused by failures in the telecommunications network [3], are physically explainable as abrupt percolating transitions [4–6]. In Ref. [4], the authors study the simplest case of two networks A and B of the same size N with random interdependent nodes. Within each network the nodes are randomly connected through connectivity links, and pairs of nodes of different networks are randomly connected via one-to-one bidirectional interdependent links, enabling the failures to propagate through the links in either direction. The random failure of a fraction $1 - p$ of nodes in one network produces a failure cascade in both networks. As a consequence, the size of the giant component (GC) of each network, i.e., the still-functioning network within each network, dynamically decreases until the system reaches a steady state. Reference [4] describes the existence of a critical threshold p_c, which is a measure of the robustness of the entire network, below which the size of the functioning network within each network abruptly collapses as a first-order percolating transition and above which these functioning networks are preserved.

In many real systems, however, this interdependency is not fully random [7,8]. Instead, nodes of different networks connect to form a "rich club" in which a portion of high-degree nodes in one network depends on corresponding high-degree nodes in other networks. This occurs in trading and finance networks in which a well-integrated country in the global trade market is also well integrated in the financial system.

Another example of the nontrivial patterns of interdependency can be found in telecommunication networks in which important nodes often acquire a battery backup system in order to decrease their dependence on the electrical supply network. To understand the effect of these realistic features on failure cascades, some studies have focused separately on the correlation between the degrees of interdependent nodes [5,7] and the random or targeted autonomization [9–12]. In these studies, the original theoretical formalism [4] is reformulated to take into account these features.

In this Rapid Communication, we present a simple, unified theoretical framework that allows us to describe the dynamics of failure cascades in interdependent networks for an arbitrary interdependency between networks. We apply our framework to interdependent heterogeneous networks when a fraction α of the higher-degree nodes is interdependent, and a fraction $1 - \alpha$ is randomly dependent. Here α is a parameter that controls the level of correlation and allows us to explore its effect on system robustness.

We consider for simplicity, but without loss of generality, two networks A and B in which the degree distribution of the connectivity links is given by $P[k_A]$ and $P[k_B]$, where k_A and k_B are the connectivity links of nodes in A and B, respectively. We define $q_A[k_A, k_B]$ ($q_B[k_A, k_B]$) as the fraction of nodes in network A (B) that depends on network B (A). When $q_i[k_A, k_B] = 1$ (with $i = A, B$) the system is one to one and all the interdependent links are bidirectional, and for $q_i[k_A, k_B] < 1$ a node in network A (B) with degree k_A (k_B) is independent of the other network with a probability $1 - q_i[k_A, k_B]$, i.e., the link cannot transmit the failure to that node. After a random failure of $1 - p$ nodes in network A that triggers the process, at each stage n of the failure cascade that goes from A to B, a node is considered functional if it belongs to the GC of its own network and the others become dysfunctional because they lose support. As f_{An} (f_{Bn}) is the probability that, transversing a link, a node of the giant connected component is reached in network A (B) at stage n [13–15], a node on network A with degree k_A is functional if it can be reached on its own network with a probability $p[1 - (1 - pf_{An})^{k_A}]$. This node will not be affected by the failure cascade (a) if it is independent

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of network B with a probability $1 - q_A[k_A,k_B]$, or (b) if it depends on network B, but its interdependent node in B is connected to the GC at the previous stage with a probability $q_A[k_A,k_B][1 - (1 - f_{Bn-1})^{k_B}]$. The relative size Ψ_n of the GC of network A at stage n is then given by

$$\Psi_n = p \left(\sum_{k_A=k_{\min}}^{k_{\max}} \sum_{k_B=k_{\min}}^{k_{\max}} P[k_A,k_B](1 - q_A[k_A,k_B])[1 - (1 - pf_{An})^{k_A}] + \sum_{k_A=k_{\min}}^{k_{\max}} \sum_{k_B=k_{\min}}^{k_{\max}} P[k_A,k_B] \right.$$

$$\left. \times \ q_A[k_A,k_B][1 - (1 - pf_{An})^{k_A}][1 - (1 - f_{Bn-1})^{k_B}] \right), \tag{1}$$

where $P[k_A,k_B]$ is the joint degree distribution for the interdependent links. The first term in Eq. (1) takes into account the functional nodes in A with degree k_A which do not depend on network B, and the second term corresponds to the case where functional nodes in network A with degree k_A depend on functional nodes of network B with degree k_B at step $n - 1$. Here f_{An} fulfills the self-consistent equation

$$f_{An} = \sum_{k_A=k_{\min}}^{k_{\max}} \sum_{k_B=k_{\min}}^{k_{\max}} \frac{k_A P[k_A,k_B]}{\langle k_A \rangle} (1 - q_A[k_A,k_B]) [1 - (1 - pf_{An})^{k_A-1}]$$

$$+ \sum_{k_A=k_{\min}}^{k_{\max}} \sum_{k_B=k_{\min}}^{k_{\max}} \frac{k_A P[k_A,k_B]}{\langle k_A \rangle} q_A[k_A,k_B][1 - (1 - pf_{An})^{k_A-1}][1 - (1 - f_{Bn-1})^{k_B}]. \tag{2}$$

Similarly, at stage n the relative size ϕ_n of the GC of network B is given by

$$\phi_n = \sum_{k_A=k_{\min}}^{k_{\max}} \sum_{k_B=k_{\min}}^{k_{\max}} P[k_A,k_B](1 - q_B[k_A,k_B])[1 - (1 - f_{Bn})^{k_B}]$$

$$+ p \sum_{k_A=k_{\min}}^{k_{\max}} \sum_{k_B=k_{\min}}^{k_{\max}} P[k_A,k_B]q_B[k_A,k_B][1 - (1 - pf_{An})^{k_A}][1 - (1 - f_{Bn})^{k_B}], \tag{3}$$

where f_{Bn} satisfies the self-consistent equation

$$f_{Bn} = \sum_{k_A=k_{\min}}^{k_{\max}} \sum_{k_B=k_{\min}}^{k_{\max}} \frac{k_B P[k_A,k_B]}{\langle k_B \rangle}(1 - q_B[k_A,k_B])[1 - (1 - f_{Bn})^{k_B-1}]$$

$$+ p \sum_{k_A=k_{\min}}^{k_{\max}} \sum_{k_B=k_{\min}}^{k_{\max}} \frac{k_B P[k_A,k_B]}{\langle k_B \rangle}q_B[k_A,k_B][1 - (1 - pf_{An})^{k_A}][1 - (1 - f_{Bn})^{k_B-1}]. \tag{4}$$

Note that in the right-hand side of Eq. (4) f_{Bn} is not multiplied by p, since we assume that the initial failure of $1 - p$ nodes occurs only in network A.

In the steady state, i.e., for $n \to \infty$, $\Psi_n \approx \Psi_{n-1}$ and $\phi_n \approx \phi_{n-1}$, thus Ψ_n and ϕ_n converge to Ψ_∞ and ϕ_∞, respectively. Our equations for the steady state were obtained by Son *et al.* [16] for uncorrelated interdependent networks and used by Baxer *et al.* [17] to explain the origin of the avalanche collapse.

We introduce here a correlated interdependency model, in which interdependent links are connected bidirectionally and one to one ($q_A[k_A,k_B] = q_B[k_A,k_B] = 1$), and a fraction α of the higher-degree nodes are fully correlated. This extends the "rich club" concept [18,19] to interdependent networks. Assuming that the degree distribution of both networks is the same, the joint degree distribution $P[k_A,k_B]$ is given by

$$P[k_A,k_B] = \begin{cases} P[k_A]P[k_B]/(1-\alpha), & k_A < k_S, \ k_B < k_S, \\ (1-w)P[k_S]P[k_B]/(1-\alpha), & k_A = k_S, \ k_B < k_S, \\ (1-w)P[k_A]P[k_S]/(1-\alpha), & k_B = k_S, \ k_A < k_S, \\ (1-w)^2 P[k_S]P[k_S]/(1-\alpha) + w\,P[k_S], & k_A = k_B = k_S, \\ P[k_A]\delta_{k_A,k_B}, & k_S < k_A, \ k_S < k_B. \end{cases} \tag{5}$$

Here k_S is the degree above which a fraction α of interdependent nodes are correlated, and w is the fraction of correlated nodes with degree k_S such that $wP[k_s] + \sum_{k=k_S+1}^{k_{\max}} P[k] = \alpha$.
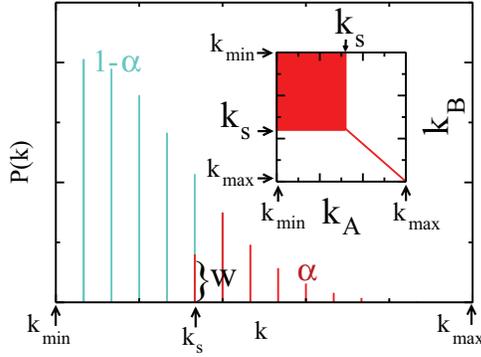
FIG. 1. (Color online) Schematic of the degree distribution used to correlate the interdependent networks. If k_s is the minimal degree for which the nodes are correlated, α represents the fraction of correlated interdependent nodes denoted by red, and w is the fraction of interdependent correlated nodes with degree k_s. The light blue region represents the fraction $1 - \alpha$ of uncorrelated nodes. In the inset we show with red color the pairs of interdependent nodes with degree $k_A - k_B$ present in this model.

In Eq. (5), the factor $1 - \alpha$ takes into account that a fraction of nodes in two different networks with degrees at and below k_S are randomly connected. In Fig. 1 we show schematically the model used to correlate the degrees between interdependent nodes and in the inset we show the pairs of interdependent nodes with degree $k_A - k_B$.

As $P[k_A, k_B] = P[k_B, k_A]$ and by the symmetry of Eqs. (2) and (4) in the steady state ($n \to \infty$), $pf_{A\infty} = f_{B\infty} \equiv f_\infty$, and the self-consistent equations reduce to

$$f_\infty = p \sum_{k_A=k_{\min}}^{k_{\max}} \sum_{k_B=k_{\min}}^{k_{\max}} \frac{k_B P[k_A, k_B]}{\langle k \rangle} [1 - (1 - f_\infty)^{k_A}]$$
$$\times [1 - (1 - f_\infty)^{k_B - 1}]. \quad (6)$$

We apply this model to pure scale-free (SF) networks with $\lambda = 2.5$, $k_{\min} = 2$, and maximal degree cutoff $k_{\max} = N^{1/2}$, with $N = 10^6$ [20]. Here the finite cutoff mimics real networks in which resources and energy are limited and nodes cannot have an unbounded number of links [21].

In Fig. 2 we show the solution of the theoretical equations (1)–(4) and the simulation results for the size of the GC of network A, Ψ_n, as a function of the stage number n [Fig. 2(a)] and Ψ_∞ as a function of the p for different values of α [Fig. 2(b)] [22].

The figures show an excellent agreement between the theoretical results and the simulations. In the temporal evolution, Fig. 2(a) shows that a small variation in p ($\Delta p \approx 0.02$) can dramatically change the final size of the GC. The inset of Fig. 2(a) shows that the approach of Ψ_n to Ψ_∞ is exponential. This behavior is due to the fact that the number of iterations of f_n in Eqs. (2) and (4) needed to reach the steady state is the same as the number of iterations needed to find the fixed point of Eq. (6), in which the approach of Ψ_n to fixed point Ψ_∞ is exponential [4] and, as a consequence, the temporal percolating dilution slows down. We can also see that at $p \approx 0.63$ the dilution rate decreases more quickly than for other values of p, i.e., the size of the functional networks
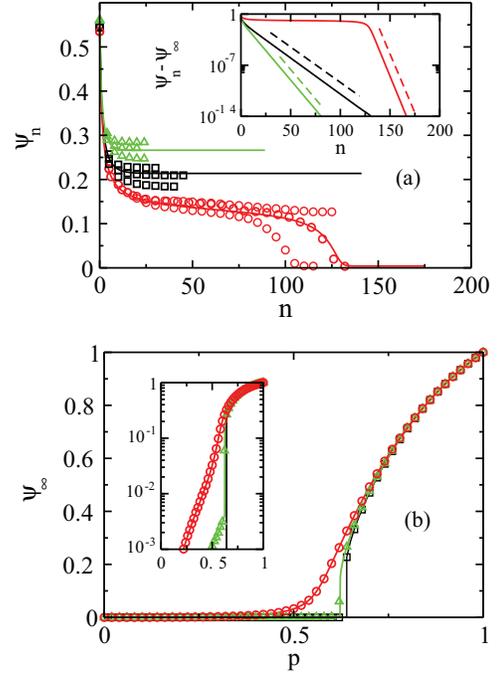

FIG. 2. (Color online) Cascade of failure on network A for different values of α and $q = 1$ on SF networks with $\lambda = 2.5$ and $2 \leqslant k \leqslant 1000$. (a) Ψ_n for $\alpha = 0.01\%$ and $p = 0.640$ (green, $\triangle$), $p = 0.630$ (black, $\square$), and $p = 0.622$ (red, $\bigcirc$) obtained from three single realizations of the simulations (symbols) and from Eqs. (1)–(4) (solid line). In the inset we show a log-linear figure of the exponential decay of Ψ_n to Ψ_∞. The dashed lines correspond to the exponential fit of the theoretical results with a characteristic time $\tau = 2.70$, 4.5, and 1.35 for $p = 0.640$, 0.630, and 0.622, from top to bottom. (b) Ψ_∞ as a function of p obtained from simulations (symbols) and from Eqs. (1)–(4) (solid lines) for $\alpha = 0.001\%$ (black, $\square$), $\alpha = 0.01\%$ (green, $\triangle$), $\alpha = 0.1\%$ (red, $\bigcirc$). In the inset we plot the main figure in log-linear scale in order to capture the abrupt collapse of the GC as explained in the text. The symbols are the average over 100 network realizations.

decays slowly, indicating that there is time to intervene and prevent the collapse of the GC. This slow behavior around critical points are shown as peaks in the number of iteration (NOI) steps needed to reach the steady state, as we will show below. Figure 2(b) shows that, as α increases, the system is still functional for high initial failure values. The critical threshold p_c at which the system is completely destroyed decreases and thus the networks are more robust.

Note that, because we are using a finite degree cutoff when $\alpha \to 1$, the threshold does not go to zero, but when $k_{\max} \to \infty$ in SF networks with $\lambda \leqslant 3$ and $\alpha = 1$, $p_c \to 0$ in this limit [5].

In order to demonstrate how correlation improves the robustness of the networks, in Fig. 3(a) we show the NOI of these systems. For very low values of α there is only one peak at the critical threshold p_c that is related to a first-order percolating transition. Surprisingly, for increasing α (see the case of $\alpha = 0.01\%$ in the figure) there is another peak around the threshold $p_c^+ > p_c \equiv p_c^-$ at which the sizes of the GCs decrease abruptly but, because the hubs support each other, the functional networks are not destroyed, and the
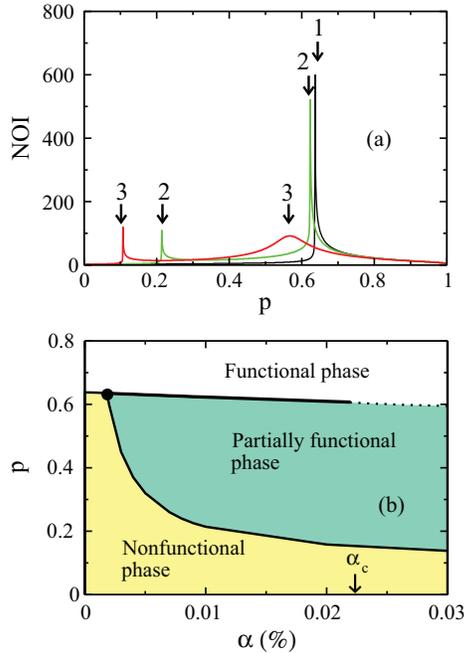
FIG. 3. (Color online) (a) The NOI as a function of p, obtained from the iterations of Eqs. (1)–(4). The network parameters and the color are the same as in Fig. 2. The labels denote the position of the peaks for $\alpha = 0.001\%$ (label 1), $\alpha = 0.01\%$ (label 2), $\alpha = 0.1\%$ (label 3). (b) Phase diagram in the plane α-p: (i) The light yellow area corresponds to the nonfunctional phase, i.e., $\Psi_\infty = \phi_\infty = 0$, (ii) the green area corresponds to a partial functional phase in which the size of the GC of both networks is $\lesssim 10^{-3}$, and (iii) the white area corresponds to a functional phase where $\Psi_\infty = \phi_\infty \gtrsim 10^{-2}$. The black point on the left corresponds to the triple point. The solid lines represent the abrupt change on the network's sizes and the dotted line, which it is defined for $\alpha > \alpha_c$, represents a fast and continuous variation of Ψ_∞ at p_c^+.

robustness of the system against failure cascades is enhanced. For higher values of α we also find that there is a sharp peak that corresponds to a first-order phase transition at $p = p_c^-$ and a rounded peak at $p = p_c^+$ around which the size of the GC decreases continuously with an increasing value of its

derivative with respect to p, $d\Psi_\infty/dp$ close to p_c^+. These findings suggest that finite correlations generate a crossover between an abrupt and a continuous, sharp decrease in the sizes of the GCs.

Figure 3(b) shows the rich phase diagram in the p-α plane. Note that as α increases, the line of the first-order transition that separates a functional GC phase from a nonfunctional phase forks into two branches, generating a new phase characterized by a small GC ($\lesssim 10^{-3}$). Around this point small fluctuations in the temporal evolution—or in the steady state—can induce an abrupt change in the size of the GC, which is reminiscent of the instability of the triple point of liquids where three phases coexist [23]. The lower branch that emerges from the triple point corresponds to the first-order transition that separates functional from nonfunctional phases. The upper one corresponds to the second threshold where the dynamics slows down and, at $\alpha = \alpha_c = 0.0218\%$ [24], the transition changes from an abrupt variation to a rapid but continuous variation of $\Psi_\infty(p)$. The small value of α_c indicates that a small correlation of the highest-degree nodes can avoid the abrupt change in the size of the GC. We found the same qualitative behavior for other SF networks with $2 < \lambda \leqslant 3$ [25], indicating that the triple point is characteristic of nontrivial patterns of interdependency.

In summary, we have used a general framework to describe the temporal behavior of failure cascades with any pattern of interdependency links, and we have found a rich phase diagram for degree-degree correlated interdependency with a triple point at which a first-order transition line splits into two first-order lines with an abrupt collapse of the sizes of the functional networks. The agreement between theory and simulations is excellent. Our framework can be extended to study the dynamics of failure cascades and the robustness of networks with degree-degree correlation in their connectivity links and in their multiple interdependent links, where we expect to find a rich phase diagram [26].

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