Universality of Potts models with two- and three-site interactions

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The q-state Potts model on the triangular lattice with both two- and three-spin interactions K and L (the Schick-Griffiths model) is shown to be equivalent to the q-state Potts model on a 3-12 lattice with *only* two-spin interactions J_1 and J_2. In particular, the $J_1>0$ and $J_2>0$ model is mapped into a Schick-Griffiths model for a range of the coupling parameters K and L, so that the two models are in the same universality class. We also show that a recent conjecture by Tsallis on the critical point of the 3-12 model is incorrect.

Consideration of spin models with multispin interactions has proved to be fruitful in many fields of physics, ranging from the determination of phase diagrams in metallic alloys[1,2] and exhibition of new types of phase transition,[3] to site percolation.[4,5] Most of these considerations are carried out for one- and two-component systems.

Several years ago, Schick and Griffiths[6] introduced a three-state Potts model with both two- and three-site interactions to describe the antiferromagnetic orderings on the triangular lattice. Since then this problem has received considerable attention. Analyses of the model have been carried out by renormalization group,[6] series analysis,[7] and by Monte Carlo simulation[8]; the generalization of the model to q states has also been proposed.[9] In particular, the $q=1$ limit leads to site-bond percolation on the honeycomb lattice,[10,11] and a special $q=\infty$ limit generates the hard hexagon problem[7] solved by Baxter.[12] The general q problem also admits some graph-theoretical formulations.[7] Despite these efforts, however, the critical properties of the q-state Schick-Griffiths model have remained unknown; the location of the critical point is also undetermined.[13]

In this Communication we report on some exact results on the q-state Schick-Griffiths model. First we establish an equivalence of the Schick-Griffiths model with a Potts model on the 3-12 lattice with pure two-site interactions. An immediate consequence of this equivalence is that the two models are in the same universality class, thus having the same set of critical exponents. This result, while plausible on intuitive grounds,[14] does not appear to have been proved rigorously. The equivalence of the two models also permits us to test the validity of a recent conjecture by Tsallis[15] on the critical trajectory of the Potts model on the 3-12 lattice. We shall see that in terms of variables in the Schick-Griffiths model the Tsallis conjecture is incorrect.

We begin by showing in Fig. 1 the triangular lattice, on which the q-state Schick-Griffiths model is defined, and the related 3-12 lattice; the two lattices have N and $6N$ sites, respectively. The Schick-Griffiths model has nearest-neighbor pair interactions K and a triplet interaction L between the three sites surrounding each triangular face. The reduced Hamiltonian is

$$\frac{-\mathcal{H}}{kT}=K\sum_{\langle ij\rangle}\delta_{\mathrm{Kr}}(\sigma_i,\sigma_j)+L\sum_{\langle ijk\rangle}\delta_{\mathrm{Kr}}(\sigma_i,\sigma_j)\delta_{\mathrm{Kr}}(\sigma_j,\sigma_k)\quad . \tag{1}$$

Here the first summation is over all neighboring sites i and j and the second summation is over all triangular faces surrounded by sites i, j, and k. The spin state at the ith site is specified by $\sigma_i=1,2,\ldots,q$. The 3-12 lattice has a similar reduced Hamiltonian, but with only pair interactions J_1 and J_2.

The equivalence of the two Potts models is summarized as follows:

$$e^K=s(J_1)[1+q/(u^2+3u)]^2\ , \tag{2}$$

$$e^L=1+q^2(u^3+3u^2-q)/(u^2+3u+q)^3\ , \tag{3}$$

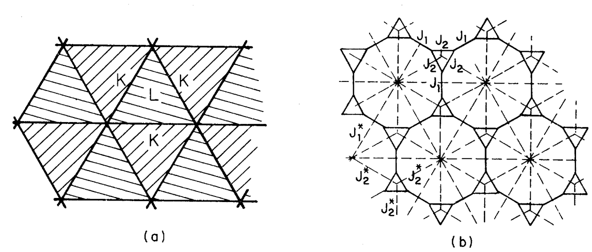

FIG. 1. (a) The triangular lattice showing the Schick-Griffiths model with two-site interactions K and three-site interactions L. (b) The 3-12 lattice (solid lines) with interactions J_1 and J_2 and its dual, the Asanoha lattice (broken lines) with interactions J_1^* and J_2^*.

where

$$u = e^{J_2} - 1 \tag{4}$$

and

$$s(x) = 1 + q/(e^x - 1) \ . \tag{5}$$

For continuity in reading, we reserve the derivation of (2) and (3) to the end of this paper, and focus our attention now on the implications of this equivalence.

The equivalence (2) and (3) is valid for any value of the parameters, real or complex, but in practice one is interested only in real values of interactions. Therefore, it is necessary to consider $J_1 > 0$ and $J_2 > 0$, since e^L has the sign of u or J_2 and, for $q > 1$, e^K has the sign of J_1. It follows from (2) that $K > 0$. However, L can take on any value in $(-\infty, I_q)$, where the maximum value $I_q = \ln[1 + q^{3/2}(2+\sqrt{q})/(3+2\sqrt{q})^3] > 0$ is attained at $u = \sqrt{q}$.

Schick and Griffiths[6] introduce the variable

$$M = 3K + 2L \tag{6}$$

in place of L for reasons of symmetry. Using (2) and (3), we find

$$e^M = s^3(J_1)[1 + q(3u+q)/(u^3+3u^2)]^2 \ . \tag{7}$$

For a given $J_1 > 0$ and $J_2 > 0$, one can always use (2) and (7) to find the equivalent K and M. In fact, the transformation (2) and (7) maps the quadrant $J_1 > 0$ and $J_2 > 0$ onto a region R bounded by the line segments

$$M = 3K + 2I_q > 2\ln(1+\sqrt{q}) \tag{8}$$

and

$$M = 3K < 2\ln(3u_0 + q) \ , \tag{9}$$

where $u_0^3 + 3u_0^2 = q$, and a curve C given parametrically by (2) and (7) with $s(J_1) = 1$. A plot of the region R for $q = 100$ is shown in Fig. 2. For smaller values of q the plot is similar, with the area between $M = 3K$ and $M = 3K + 2I_q$ less pronounced. We note that R is always in the ferromagnetic regime $\{M > K, M > 0\}$ of the Schick-Griffiths model.

The critical point of the Schick-Griffiths model is exactly known for $L = 0$,[16,17] leading to an exact critical point at $(M,K) = (3a, a)$ along the line $M = 3K$, where a is the root of the cubic equation

$$x^3 = 3x + q - 2 \ . \tag{10}$$

It is significant that this point, indicated by × in Fig. 2, lies below the curve C for all q. It is also clear that the exact critical trajectory of the Schick-Griffiths model will always intersect R. Then along this portion of the critical trajectory at least, the Schick-Griffiths model is in the same universality class as the Potts model with pure pairwise interactions.[18] In particular, the exponents will have the den Nijs[19] values for all q. For fixed q, the exponents take on the same (universal) values along this section of the critical trajectory. Of course, we cannot rule out the possible existence of a multicritical point outside the range R. For $q = 3$, this multicritical point occurs in the second quadrant in Fig. 2 ($K < 0$, $M > 0$).[7,8] Our results suggest that this multicritical point moves to the right along the critical trajectory as q increases, and it would be of interest to test this proposal by numerical calculations.

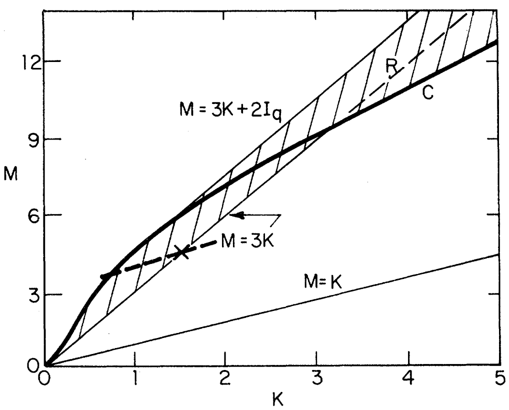

FIG. 2. The region R (shaded area) derived from the transformation (2) and (3) for $J_1 > 0$ and $J_2 > 0$, and $q = 100 (I_{100} = 0.686)$. The plots for smaller values of q are similar, but the region $3K < M < 3K + 2I_q$ is less pronounced. The curve C is obtained from (2) and (7) with $s(J_1) = 1$. The cross denotes the exact critical point at $M = 3K = 4.723$ and the heavy broken line denotes the locus of the (exact) critical trajectory.

It must be pointed out that the universality class argument breaks down at any critical point which happens to lie on the line segment (8) where the derivative dL/du vanishes. It seems, however, that this will not happen, as borne out by the exact result for $q = 2$, and numerical results for $q = 1, 3$.

Very recently Tsallis[15] has conjectured that the critical trajectory of the 3-12 Potts lattice of Fig. 1(b) takes the form (10) with

$$x = \frac{s(J_1)[s^3(J_2) + q - 1]}{s^2(J_2) + s(J_2) + q - 2} \ . \tag{11}$$

Using (2) and (3), we reduce (11) to

$$x = e^{K+L} \ , \tag{12}$$

so that the Tsallis conjecture can be restated in the Schick-Griffiths variables as

$$e^{3K+3L} = 3e^{K+L} + q - 2 \ . \tag{13}$$

It is sufficient to disprove the conjecture (13) by showing that it is incorrect in one special case. Consider the hard hexagon problem limit of $K \to \infty$, $L \to -\infty$, $e^{K+L} \to [(q-1)/z]^{1/6}$, and $q \to \infty$, for which the critical point is known to be $z_c = (11 + 5\sqrt{5})/2$.[7] It is readily seen that (13) does not yield this value, and therefore must be incorrect.

The conjecture (13) is presumably restricted to the region R corresponding to $J_1 > 0$ and $J_2 > 0$. It is of interest, however, to continue outside R and compare (13) with an earlier conjecture made by one of us[9]

$$e^{3K+2L} = 3e^K + q - 2 \quad . \tag{14}$$

The two conjectures are identical for $L = 0$ and for $q = 2$, a curious coincidence which happens to lead to the exact critical point. For other values of the interactions one can only compare with numerical results. In the $q = 3$ model Enting and Wu[7] have shown that the Wu conjecture (14) gives the erroneous result $K_c = -\infty$ along the coexistence line ($M = 0$) as compared to the numerical estimate of $K_c = -2.00$.[7,8] In contrast, the Tsallis conjecture, if continued to $M = 0$, gives a value $K_c = -1.261\,89$. For site-bond percolation on the honeycomb lattice (see, e.g., Ref. 20) the conjectures (13) and (14) lead to the expressions

$$ps = 1 - 2\sin(\pi/18) = 0.652\,704\ldots \tag{15}$$

and

$$1 + p^3 s^2 = 3p^2 s^2 \quad , \tag{16}$$

where p and s are, respectively, the critical bond and site occupation probabilities. For pure bond percolation ($s = 1$) both (15) and (16) lead to the exact p_c. But for pure site percolation ($p = 1$), (15) and (16) lead to the critical probabilities $s_c = 0.652\,704$ and $s_c = 0.707\,107$. Both values differ significantly from the best numerical estimate of $s_c = 0.6962 \pm 0.0006$.[21,22]

Finally, we sketch the derivation of (2) and (3). Consider the dual of the 3-12 lattice, the Asanoha or hemp-leaf lattice, shown by the broken lines in Fig. 1(b). The interactions of the Asanoha lattice are J_1^* and J_2^* given by the duality relations

$$e^{J_1^*} = s(J_1), \quad e^{J_2^*} = s(J_2) \quad . \tag{17}$$

Consider next the three intersecting J_2^* bonds, and perform a star-triangle transformation as shown in Fig. 3. It is well-known[16] that such a transformation is valid for special values of K^* and L^* given by

$$e^{K^*} = \frac{e^{2J_2^*} + e^{J_2^*} + q - 2}{(3e^{J_2^*} + e^{J_2^*} + q - 2)^3} \quad , \tag{18}$$

$$e^{L^*} = \frac{(e^{3J_2^*} + q - 1)(3e^{J_2^*} + q - 3)^2}{(e^{2J_2^*} + e^{J_2^*} + q - 2)^3} \quad . \tag{19}$$

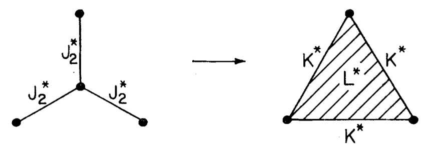

FIG. 3. The star-triangle transformation used in the derivation of (18) and (19).

Therefore, for J_2 fixed we determine J_2^*, hence K^*, L^*, from (17)–(19) to effect the star-triangle transformation. This procedure results in a Schick-Griffiths model with interactions

$$K = J_1^* + 2K^*, \quad L = L^* \quad , \tag{20}$$

which are independent of one another, thus completing the transformation. One obtains the relations (2) and (3) after combining (20) with (17)–(19).

The equivalence between the *partition functions* of the two models of Fig. 1 can also be calculated and is found to be given by

$$Z(J_1, J_2) = [(e^{J_1} - 1)^3 (e^{J_2} - 1)^4 (e^{J_2} + 2)/q]^N Z(K, L) \quad . \tag{21}$$

The above transformation can be extended to the case in which alternating triangular faces have arbitrary three-spin interactions L and L',[9] so that our results apply to the Kim-Joseph[16] model ($L' = 0$) as well as the Schick-Griffiths[6] model ($L = L'$).

In summary, we have mapped a q-state Potts model with two-spin couplings into a particular Potts model with both two- *and* three-spin couplings. This result is important because it suggests that recent experiments on systems with multisite interactions[23] may not necessarily find critical behavior different from systems with only two-site interactions.

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