Eradicating catastrophic collapse in interdependent networks via reinforced nodes

Xin Yuan[a,b], Yanqing Hu[c,d,e,1], H. Eugene Stanley[a,b,1], and Shlomo Havlin[f,g]

[a]Center for Polymer Studies, Boston University, Boston, MA 02215; [b]Department of Physics, Boston University, Boston, MA 02215; [c]School of Data and Computer Science, Sun Yat-sen University, Guangzhou 510006, China; [d]School of Mathematics, Southwest Jiaotong University, Chengdu 610031, China; [e]Big Data Research Center, University of Electronic Science and Technology of China, Chengdu 611731, China; [f]Minerva Center, Bar-Ilan University, Ramat-Gan 52900, Israel; and [g]Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

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In interdependent networks, it is usually assumed, based on percolation theory, that nodes become nonfunctional if they lose connection to the network giant component. However, in reality, some nodes, equipped with alternative resources, together with their connected neighbors can still be functioning after disconnected from the giant component. Here, we propose and study a generalized percolation model that introduces a fraction of reinforced nodes in the interdependent networks that can function and support their neighborhood. We analyze, both analytically and via simulations, the order parameter—the functioning component—comprising both the giant component and smaller components that include at least one reinforced node. Remarkably, it is found that, for interdependent networks, we need to reinforce only a small fraction of nodes to prevent abrupt catastrophic collapses. Moreover, we find that the universal upper bound of this fraction is 0.1756 for two interdependent Erdős–Rényi (ER) networks: regular random (RR) networks and scale-free (SF) networks with large average degrees. We also generalize our theory to interdependent networks of networks (NONs). These findings might yield insight for designing resilient interdependent infrastructure networks.

percolation | interdependent networks | phase transition | collapse

Complex networks often interact and depend on each other to function properly (1–8). Because of interdependencies, these interacting networks may easily suffer abrupt failures and face catastrophic consequences, such as the blackouts of Italy in 2003 and North America in 2008 (3, 4, 6). Thus, a major open challenge arises as how to tackle the vulnerability of interdependent networks. Virtually, many existing theories on the resilience of interacting networks have centered on the formation of the largest cluster (called the giant component) (4, 6, 9–15) and consider only the nodes in the giant component as functional, because all of the small clusters do not have a connection to the majority of nodes, which are in the giant component.

However, in many realistic networks, in case of network component failures, some nodes (which we call here reinforced nodes) and even clusters containing reinforced nodes outside of the giant component can resort to contingency mechanisms or backup facilities to keep themselves functioning normally (16–18). For example, small neighborhoods in a city, when facing a sudden power outage, could use alternative facilities to sustain themselves. Consider also the case where some important internet ports, after their fiber links are cutoff from the giant component, could use satellites (19) or high-altitude platforms (20) to exchange vital information. These possibilities strongly motivate us to generalize the percolation theory (9, 21) to include a fraction of reinforced nodes that are capable of securing the functioning of the finite clusters in which they are located. We apply this framework to study a system of interdependent networks and find that a small fraction of reinforced nodes can avoid the catastrophic abrupt collapse.

In this paper, we develop a mathematical framework based on percolation (4, 6, 12, 13, 22) for studying interdependent networks with reinforced nodes and find exact solutions to the minimal fraction of reinforced nodes needed to eradicate catastrophic collapses. In particular, we apply our framework to study and compare three types of random networks: (*i*) Erdős–Rényi (ER) networks with a Poisson degree distribution $[P(k)=e^{-\langle k\rangle}\langle k\rangle^k/k!]$ (23), (*ii*) scale-free (SF) networks with a power law degree distribution $[P(k)\sim k^{-\lambda}]$ (24), and (*iii*) regular random (RR) networks with a Kronecker delta degree distribution $[P(k)=\delta_{k,k_0}]$. Here, k stands for the number of connections of a single node. We find the universal upper bound for this minimal fraction to be 0.1756 for two interdependent ER networks with any average degree and SF and RR networks with a large average degree.

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Model

Formally, for simplicity and without loss of generality, our model consists of two networks, A and B, with N nodes in each network (Fig. 1). Within network A, the nodes are randomly connected by A links with degree distribution $P_A(k)$, whereas in network B, the nodes are randomly connected by B links with degree distribution $P_B(k)$. In addition, a fraction q_A of nodes in A is randomly dependent (through dependency links) on nodes in network B, and a fraction q_B of nodes in network B is randomly dependent on nodes in network A (25). We also assume that a node from one network depends on no more than one node from the other network, and if a node i in network A is dependent on a node j in network B and j depends on a node l in network A, then $l=i$ [a no-feedback condition (4, 6, 26, 27)]. We denote ρ_A and ρ_B as the fractions of nodes that are randomly chosen as reinforced nodes in network A and network B, respectively. In each network, together with the giant component,

Significance

Percolation theory assumes that only the largest connected component is functional. However, in reality, some components that are not connected to the largest component can also function. Here, we generalize the percolation theory by assuming a fraction of reinforced nodes that can function and support their components, although they are disconnected from the largest connected component. We find that the reinforced nodes reduce significantly the cascading failures in interdependent networks system. Moreover, including a small critical fraction of reinforced nodes can avoid abrupt catastrophic failures in such systems.

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[1]To whom correspondence may be addressed. Email: hes@bu.edu or yanqing.hu.sc@gmail.com.

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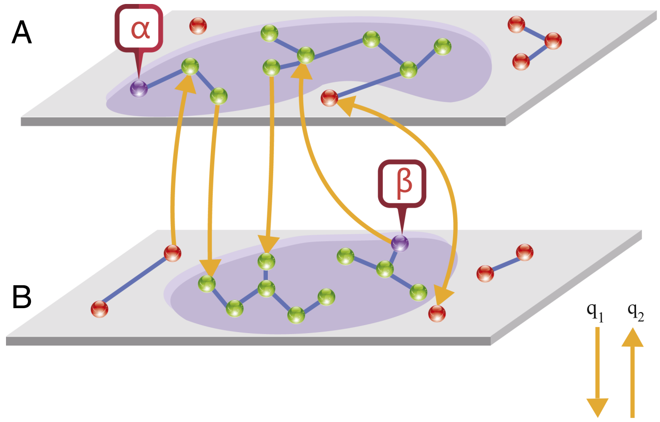

Fig. 1. Demonstration of the model studied here, where two interdependent networks A and B have gone through cascading failures and reached a steady state. The yellow arrows represent a fraction $q_{A(B)}$ of nodes from network A(B) depending on nodes from network B(A) for critical support. Reinforced nodes α and β (purple circles) are nodes that survive and also support their clusters, even if the clusters are not connected to the largest component. Some regular nodes (green circles) survive the cascading failures, whereas some other regular nodes (red circles) fail. Note that the clusters of circles in the shaded purple areas constitute the functioning component studied in our model.

those smaller clusters containing at least one reinforced node make up the functioning component, as shown in Fig. 1. The failure process is initiated by removing randomly a fraction $1-p$ of nodes from each network. Therefore, when nodes from one network fail, their dependent counterparts from the other network must also fail. In this case, an autonomous node (a node that does not need support from the other network) (25) survives if it is connected to a functioning component of its own network; a dependent node n_0 survives if both n_0 and the node that it depends on are connected to their own networks' functioning components.

We introduce the generating function of the degree distribution $G_{A0}(x)=\sum_k P_A(k)x^k$ and the associated branching processes $G_{A1}(x)=G'_{A0}(x)/G'_{A0}(1)$ (14), where $G'_{A0}(x)=\sum_k kP_A(k)x^{k-1}$; similar equations exist to describe network B.

At the steady state, using the probabilistic framework (28–34), we denote x (y) as the probability that a randomly chosen link in network A (B) reaches the functioning component of network A (B) at one of its nodes. Thus, x and y satisfy the following self-consistent equations (*SI Text*, section 2):

$$x = p\,[1-(1-\rho_A)G_{A1}(1-x)] \times \{1-q_A+pq_A[1-(1-\rho_B)G_{B0}(1-y)]\} \qquad [1]$$

and

$$y = p\,[1-(1-\rho_B)G_{B1}(1-y)] \times \{1-q_B+pq_B[1-(1-\rho_A)G_{A0}(1-x)]\}. \qquad [2]$$

These two equations can be transformed into $x=F_1(p,y)$ and $y=F_2(p,x)$, respectively, which can be solved numerically by iteration with the proper initial values of x and y.

Accordingly, the sizes of the functioning components are determined by (*SI Text*, section 2)

$$P^A_\infty = p[1-(1-\rho_A)G_{A0}(1-x)] \times \{1-q_A+pq_A[1-(1-\rho_B)G_{B0}(1-y)]\} \qquad [3]$$

and

$$P^B_\infty = p[1-(1-\rho_B)G_{B0}(1-y)] \times \{1-q_B+pq_B[1-(1-\rho_A)G_{A0}(1-x)]\}. \qquad [4]$$

If the system has an abrupt phase transition at $p=p^I_c$, the functions $x=F_1(p,y)$ and $y=F_2(p,x)$ satisfy the condition

$$\frac{\partial F_1(p^I_c,y^I)}{\partial y^I}\cdot\frac{\partial F_2(p^I_c,x^I)}{\partial x^I}=1, \qquad [5]$$

namely the curves $x=F_1(p^I_c,y)$ and $y=F_2(p^I_c,x)$ touch each other tangentially at (x^I,y^I) (32, 35).

Results

For a general system of interdependent networks A and B, P^A_∞, P^B_∞, and the existence of p^I_c can be easily determined numerically using Eqs. **1–5**. As an example, Fig. 2 shows the excellent agreement between simulation and theory.

However, it is important to find analytic expressions for P^A_∞, P^B_∞, and p^I_c, at least for simpler cases, that can serve as a

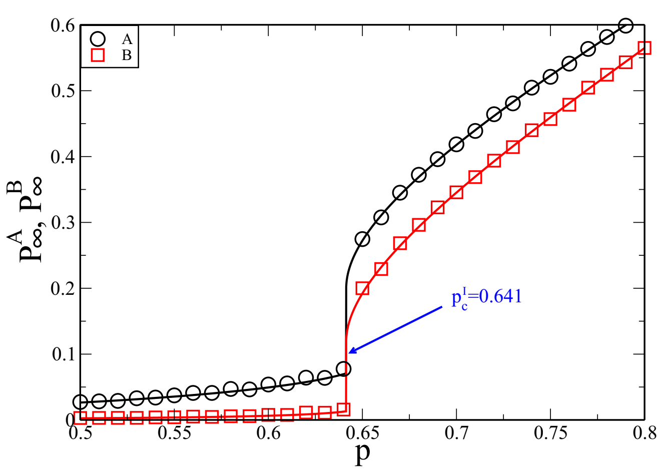

Fig. 2. The sizes of functioning components as a function of p for ER networks with $\rho_A=0.05$, $\rho_B=0.03$, $q_A=0.65$, $q_B=0.95$, $\langle k\rangle_A=4$, and $\langle k\rangle_B=5$. The simulation results (symbols) are obtained from two networks of 10^5 nodes and in good agreement with the theoretical results (solid lines) (Eqs. **3** and **4**). Note that, for $\rho_A\neq 0$ and $\rho_B\neq 0$, network A(B) always has at least a fraction $p^2\rho_A\rho_B q_A$ ($p^2\rho_A\rho_B q_B$) of nodes functioning after fractions $1-p$ of nodes are removed from both networks.

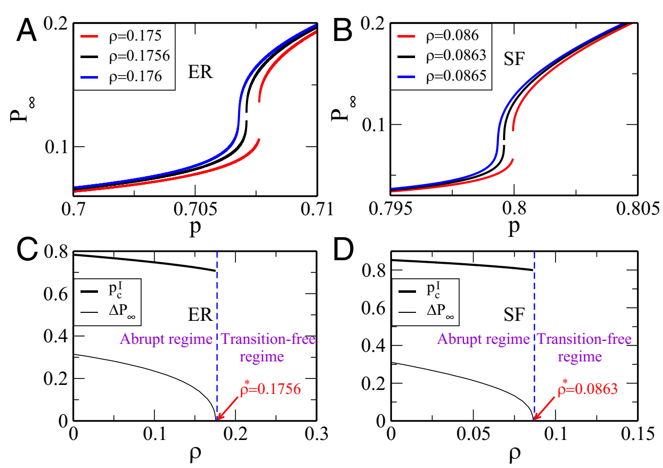

Fig. 3. Percolation properties of symmetric interdependent ER and SF networks. (*A* and *B*) Demonstration of the behavior of P_∞ around ρ^* for (*A*) ER networks with $\langle k\rangle=4$ and $q=1$ and (*B*) SF networks with $P(k)\sim k^{-\lambda}$, $\lambda=2.7$, $k_{\min}=2$, $k_{\max}=2048$, and $q=1$. (*C* and *D*) The abrupt collapse point p^I_c (thick black line) and the jump of the functioning component $\triangle P_\infty$ (thin black lines) at p^I_c as a function of ρ for (*C*) ER and (*D*) SF networks. We find ρ^* for both cases as highlighted in the graphs.

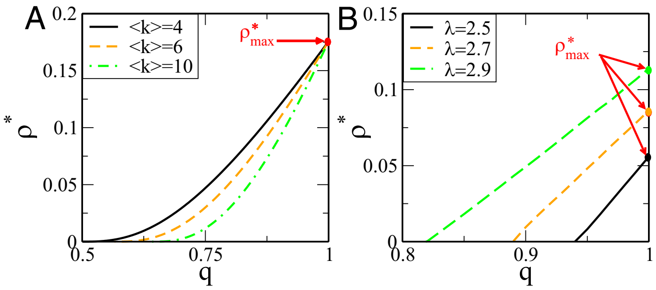

Fig. 4. (*A*) ρ^* As a function of q for symmetric ER networks with different values of $\langle k \rangle$. The results are obtained using Eq. **7**, and these curves converge at the point (1, 0.1756). (*B*) ρ^* As a function of q for symmetric SF networks with $k_{\min} = 2$ and different values of λ. The results are obtained from numerical calculations (Eq. **S30** in *SI Text*, section 3). We always have $\rho^*_{\max}$ at $q = 1$ corresponding to the fully interdependent scenario.

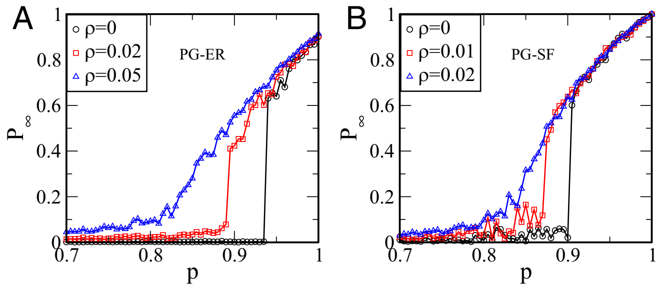

Fig. 6. Percolation transition in real world systems with the introduction of reinforced nodes. (*A*) The circles, squares, and triangles represent simulation results of a system composed of the US PG ($N = 4941$ and $\langle k \rangle = 2.699$) and an ER network ($N = 4941$ and $\langle k \rangle = 2.699$) with interdependence strength $q = 0.65$ and $\rho = 0$, 0.02, 0.05, respectively. (*B*) The circles, squares, and triangles represent simulation results of a system composed of the same PG and an SF network ($N = 4941$, $\lambda = 2.7$, and $k_{\min} = 2$) with interdependence $q = 0.65$ and $\rho = 0$, 0.01, 0.02, respectively. The symbols are results obtained from a single realization.

benchmark to better understand simulated solutions of more realistic cases. Thus, here, for simplicity, we consider the symmetric case, where $P_A(k) = P_B(k)$, $\rho_A = \rho_B = \rho$, and $q_A = q_B = q$. This symmetry readily implies that $x = y \equiv F(p, x)$, reducing Eqs. **1** and **2** to a single equation. Similarly, it renders $P^A_\infty = P^B_\infty \equiv P_\infty$ and transforms Eq. **5** to $\partial F(p^I_c, x^I)/\partial x^I \cdot dx^I/dx^I = 1$ [i.e., $\partial F(p^I_c, x^I)/\partial x^I = 1$]. Using Eqs. **1–5**, we derive p^I_c and P_∞ rigorously (*SI Text*, section 3).

Surprisingly, we find that, even for a system built with a relatively high dependency coupling, there exists a specific value ρ^* that divides the phase diagram into two regimes. Specifically, if $\rho \le \rho^*$, the system is subject to abrupt transitions; however, if $\rho > \rho^*$, the abrupt percolation transition is absent in the system, because the giant component changes from a first-order phase transition behavior to a second-order phase transition behavior (*SI Text*, section 3). Therefore, ρ^* is the minimum fraction of nodes in each network that needs to be reinforced to make the interdependent system less risky and free from abrupt transitions. Moreover, ρ^* satisfies the condition (*SI Text*, section 3)

$$\frac{dp^I_c}{dx^I}|_{\rho=\rho^*} = 0. \qquad [6]$$

Fig. 3 shows the existence of ρ^* for systems of fully interdependent ER networks ($\rho^* \approx 0.1756$) and SF networks ($\rho^* \approx 0.0863$). Fig. 3 *A* and *B* depicts the dramatic behavior change of the functioning components as ρ increases slightly from under ρ^* to above ρ^*. Fig. 3*C* shows that p^I_c slowly decreases as ρ approaches ρ^* and ceases to exist for $\rho > \rho^*$. We can also see in Fig. 3*D* that the jump of the functioning component $\triangle P_\infty$ at p^I_c decreases to zero as ρ increases from zero to ρ^*.

We next solve this critical value ρ^* as a function of q and $\langle k \rangle$ for two interdependent ER networks as (*SI Text*, section 3)

$$\rho^* = 1 - \frac{exp\left\{\frac{1}{2}\left[1 - \langle k \rangle (1-q)^2/2q\right]\right\}}{2 - \sqrt{\langle k \rangle (1-q)^2/2q}}, \qquad [7]$$

where $q_0 \le q \le 1$, and q_0 is the minimum strength of interdependence required to abruptly collapse the system (36). If we set $\rho^* = 0$ in Eq. **7**, q_0 can be obtained from $\langle k \rangle (1-q_0)^2/2q_0 = 1$ as $q_0 = \left(1 + \langle k \rangle - \sqrt{2\langle k \rangle + 1}\right)/\langle k \rangle$, as found in refs. 35 and 37. Applying Taylor expansion to Eq. **7** for $q \to q_0$, we get the critical exponent β_1 defined via $\rho^* \backsim (q - q_0)^{\beta_1}$ with $\beta_1 = 3$.

Hence, for any $q \in [q_0, 1]$, we first calculate ρ^* using Eq. **7**; then, p^I_c corresponding to this q and ρ^* can be computed as (*SI Text*, section 3)

$$p^I_c(q, \rho^*) = \left[2 - (1-q)\sqrt{\langle k \rangle /2q}\right]/\sqrt{2\langle k \rangle q}, \qquad [8]$$

and the size of the functioning component at this p^I_c is

$$P_\infty(p^I_c) = [1 - \langle k \rangle (1-q)^2/2q]/2\langle k \rangle. \qquad [9]$$

The behavior of the order parameter $P_\infty(p)$ near the critical point is defined by the critical exponent β_2, where $P_\infty(p) - P_\infty(p^I_c) \backsim (p - p^I_c)^{\beta_2}$ with $\beta_2 = 1/3$ if $\rho = \rho^*$ and $\beta_2 = 1/2$ if $\rho < \rho^*$ (*SI Text*, section 3.1.1) (25). Similar scaling behaviors have been reported in a bootstrap percolation problem (29) and a Fredrickson–Andersen model on Bethe lattice with quenched impurities (38, 39).

In Fig. 4*A*, we plot ρ^* from Eq. **7** as a function q for several different values of $\langle k \rangle$. Interestingly, at $q = 1$, namely, for two fully interdependent ER networks, we find, for all mean degrees, the maximum of ρ^* to be

$$\rho^*_{\max} = 1 - e^{1/2}/2 \approx 0.1756, \qquad [10]$$

which is independent of $\langle k \rangle$. In Fig. 4*B*, we plot ρ^* as a function of q for several degree exponents λ of SF networks. Here, ρ^* increases as λ increases and takes its maximum $\rho^*_{\max}$ at $q = 1$, corresponding to the fully interdependent case, which is the most vulnerable. Thus, if the dependency strength q is unknown, $\rho^*_{\max}$ is the minimal fraction of reinforced nodes that can prevent catastrophic collapse.

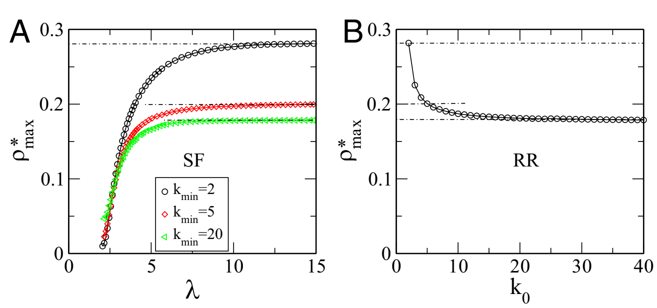

Fig. 5. (*A*) $\rho^*_{\max}$ As a function of λ for two fully interdependent SF networks with the same number of nodes and degree exponent and $k_{\min} = 2$ (circles), 5 (diamonds), and 20 (triangles); $\rho^*_{\max}$ has upper limits of 0.282 (circles), 0.201 (diamonds), and 0.181 (triangles) as $\lambda \to \infty$. (*B*) $\rho^*_{\max}$ As a function of k_0 for two fully interdependent RR networks with the same number of nodes and k_0; $\rho^*_{\max}$ approaches 0.1756 as $k_0 \to \infty$.

Similarly, we obtain $\rho^*_{\max}$ as a function of the degree exponent λ for two fully interdependent SF networks (Fig. 5*A*) and $\rho^*_{\max}$ as a function of k_0 for two fully interdependent RR networks (Fig. 5*B*). Note that, as λ increases, $\rho^*_{\max}$ initially increases but later stabilizes at a value determined by $k_{\min}$ as the degree distribution becomes more homogeneous and its network structure becomes the same as that in an RR network with $k_0 = k_{\min}$ (*SI Text*, section 3.2). For RR networks, as k_0 increases, $\rho^*_{\max}$ initially decreases but later stabilizes at a value close to 0.1756, because at very large k_0, the structure of these RR networks resembles that of ER networks with $\langle k\rangle = k_0$ (*SI Text*, section 3.2).

Next, we solve $\rho^*_{\max}$ of two fully interdependent networks as a function of α, where $\alpha = \langle k\rangle_A / \langle k\rangle_B$ (Fig. S10 in *SI Text*, section 4.1). We find that, in two ER networks, as α increases, $\rho^*_{\max}$ increases and has a maximum at $\alpha = 1$, corresponding to the symmetric case studied above. In the case of RR networks with large k_0, $\rho^*_{\max}$ behaves similarly to its counterpart in ER networks, peaking around $\alpha = 1$ at 0.1756 (Fig. 5*B*). Moreover, in the case of SF networks when $\lambda \in (2, 3]$, $\rho^*_{\max} \leq 0.11$; whereas when λ and $k_{\min}$ are relatively large, $\rho^*_{\max}$ will also peak around $\alpha = 1$, with a value close to that obtained in RR networks. Therefore, in the extreme case where λ and $k_{\min}$ are large, SF networks converge to RR networks with $k_0 = k_{\min}$, which further converge to ER networks with $\langle k\rangle = k_0$. Thus, in these extreme cases, there exists a universal $\rho^*_{\max}$ equal to 0.1756 (*SI Text*, section 4.2).

Our approach can be generalized to solve the case of tree-like networks of networks (NONs) (6, 34). For example, we study the symmetric case of an ER NON with n fully interdependent member networks and obtain

$$\rho^*_{\max} = 1 - e^{1-1/n}/n, \qquad \textbf{[11]}$$

which is independent of the average degree $\langle k\rangle$ (*SI Text*, section 3.1.2). This relationship indicates that the bigger n is, the larger $\rho^*_{\max}$ should be, which is consistent with the previous finding that the more networks an NON has, the more vulnerable it will (6).

Test on Empirical Data

We next test our mathematical framework on an empirical network, the US power grid (PG) (40), with the introduction of a small fraction of reinforced nodes. It is difficult to establish the exact structure of the network that the PG interacts with and their interdependencies because of lack of data. However, to get qualitative insight into the problem, we couple the PG with either ER or SF network, which can be regarded as approximations to many real world networks. Our motivation is to test how our model performs in the interdependent networks system with some real world network features. Note that, here, our results present cascading failures caused by structural failures and do not represent failures caused by real dynamics, such as cascading failures caused by overloads, that appear in a PG network system. Fig. 6 compares the mutual percolation of two systems of interdependent networks with the same interdependence strength: PG coupled to a same-sized ER network (Fig. 6*A*) and PG coupled to a same-sized SF network (Fig. 6*B*). As discussed above, for ρ below a certain critical value ρ^*, the systems will undergo abrupt transitions, whereas for ρ above ρ^*, the systems do not undergo any transition at all. We also find that, for the interdependence strength $q = 0.65$ shown here, the ρ^* value of the latter case is very small and close to 0.02 (Fig. 6*B*).

Summary

In summary, we have developed a general percolation framework for studying interdependent networks by introducing a fraction of reinforced nodes at random. We show that the introduction of a relatively small fraction of reinforced nodes, ρ^*, can avoid abrupt collapse and thus, enhance its robustness. By comparing ρ^* in ER, SF, and RR networks, we reveal the close relationship between these network structures in extreme cases and find the universal upper bound for ρ^* to be 0.1756. We also observe improved robustness in systems with some real world network structure features. The framework presented here might offer some useful suggestions on how to design robust interdependent networks.

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