AN APPROXIMATE FORMULA FOR PRICING AMERICAN OPTIONS

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Abstract

An approximate formula for pricing American options along the lines of MacMillan [1986] and Barone-Adesi and Whaley [1987] is presented. This analytical approximation is as efficient as the existing ones, but it is remarkably more accurate. In particular, it yields good results for long maturity options for which the existing analytical ones fare poorly. It is also demonstrated that this approximation is more accurate than the less efficient methods such as the four-point extrapolation schemes of Geske and Johnson [1984] and Huang, Subrahmanyam and Yu [1996].

There have been many attempts at pricing American options. Numerical methods such as the finite difference method of Brennan and Schwartz [1977] and the binomial tree model of Cox, Ross and Rubinstein [1979] are among the earliest and still widely used ones. Even though these methods are quite flexible, they are also among the most time consuming ones. A rare exception among the numerical methods is a recent paper by Figlewski and Gao [1999]. They show that efficiency and accuracy of the binomial method can be improved tremendously by fine tuning the tree in the regions where discretization induces the most serious pricing errors.

The second group of methods includes approximate schemes based on exact representations of the free boundary problem of the American options or the partial differential equation satisfied by the option prices. This group includes Geske and Johnson [1984], Bunch and Johnson [1992], Huang, Subrahmanyam and Yu [1996], Carr [1998] and Ju [1998]. These methods are essentially analytic approximations and they are convergent in the sense that as more and more terms are included, they become more and more accurate. However these methods become inefficient very rapidly.

Another category of methods uses regression techniques to fit an analytical approximation based on a lower bound and an upper bound of an American option. These methods include Johnson [1983], and Broadie and Detemple [1996]. These methods can be quite fast, but they all need regression coefficients which in turn require computing a large number of options accurately. Another drawback is that these methods are not convergent.

A fourth category of potential methods includes analytical approximations. MacMillan [1986] and Barone-Adesi and Whaley [1987] are among these methods. A common feature of these methods is that they are many times faster than most of the aforementioned ones. A drawback is that they are not very accurate, especially for long maturity options, such as the exchange-traded long-term equity anticipation securities (LEAPS).

In the absence of any closed form formula for American options, a reliable analytical approximation is obviously highly desirable. First, an analytical approximation will likely be very efficient computationally. Second, such an approximation will not involve regression coefficients which need to be calibrated and recalibrated. In this article such an analytical formula is proposed. Even though it can not attain an arbitrary accuracy, for most practical applications it is accurate enough to be a useful, reliable and efficient method. Another useful feature of the method is that it is extremely easy to program. Therefore in cases where execution time is less important than the programming time the present method offers an appealing choice.

I. DERIVATION OF THE APPROXIMATE FORMULA

Under the usual assumptions, Merton [1973] has shown that the price F of any contingent claim, whether it is American or European, written on a stock satisfies the following partial differential equation (PDE):

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + (r - \delta) S F_S - rF - F_\tau = 0. \tag{1}$$

The riskless interest rate r, volatility σ, and dividend yield δ are all assumed to be constants. The value of any particular contingent claim is determined by the terminal condition and boundary conditions. It should be pointed out that the above PDE only holds for an American option in the continuation region. Otherwise the option should be exercised immediately.

Because both American and European options satisfy the same PDE, so does the early exercise premium $V = V_A - V_E$, where V_A and V_E are the prices of an American option and its corresponding European counterpart, respectively. Following MacMillan [1986] and

Barone-Adesi and Whaley [1987], we introduce the following notations:

$$\tau = T - t, \qquad h(\tau) = 1 - e^{-r\tau},$$
$$\alpha = \frac{2r}{\sigma^2}, \qquad \beta = \frac{2(r-\delta)}{\sigma^2}, \qquad V = h(\tau)g(S,h).$$

Then g satisfies

$$S^2\frac{\partial^2 g}{\partial S^2} + \beta S\frac{\partial g}{\partial S} - \frac{\alpha}{h}g - (1-h)\alpha\frac{\partial g}{\partial h} = 0. \tag{2}$$

The MacMillan [1986] and Barone-Adesi and Whaley [1987] approximations amount to the assumption that the last term in (2) is zero. Their approximations are very good for very short maturities since then $(1-h)$ is close to zero, and good for very long maturities since then $\partial g/\partial h$ is close to zero. For intermediate cases like for maturities ranging from one year to five years, serious mispricing could result. The approximation that we are about to introduce gives better results for very short and very long maturity options and substantially reduces the pricing errors for intermediate maturity options.

In the following, hg_1 will be the early exercise premium of MacMillan [1986] and Barone-Adesi and Whaley [1987], hg_2 will be a correction to hg_1. Let $g = g_1 + g_2$, then (2) becomes

$$S^2\frac{\partial^2 g_1}{\partial S^2} + \beta S\frac{\partial g_1}{\partial S} - \frac{\alpha}{h}g_1 + S^2\frac{\partial^2 g_2}{\partial S^2} + \beta S\frac{\partial g_2}{\partial S} - \frac{\alpha}{h}g_2 - (1-h)\alpha(\frac{\partial g_1}{\partial h} + \frac{\partial g_2}{\partial h}) = 0. \tag{3}$$

Now let

$$S^2\frac{\partial^2 g_1}{\partial S^2} + \beta S\frac{\partial g_1}{\partial S} - \frac{\alpha}{h}g_1 = 0. \tag{4}$$

A proper solution of g_1 for an American option is

$$g_1 = A(h)(S/S^*)^\lambda, \tag{5}$$

where λ is given by

$$\lambda = \frac{-(\beta-1) + \phi\sqrt{(\beta-1)^2 + \frac{4\alpha}{h}}}{2},$$

S^* is the critical early exercise stock price for maturity τ, $A(h)$ is a function of h and $\phi = 1$ for calls and $\phi = -1$ for puts. The resulting equation for g_2 is given by

$$S^2 \frac{\partial^2 g_2}{\partial S^2} + \beta S \frac{\partial g_2}{\partial S} - \frac{\alpha}{h} g_2 - (1-h)\alpha(\frac{\partial g_1}{\partial h} + \frac{\partial g_2}{\partial h}) = 0. \tag{6}$$

Now let $g_2 = \epsilon g_1$. The intention is that hg_1 will catch most of the early exercise premium and thus ϵ will be small compared with one. Plugging in and making use of the fact that g_1 satisfies (4), ϵ satisfies the following PDE:

$$S^2 \frac{\partial^2 \epsilon}{\partial S^2} + (2S^2 \frac{1}{g_1} \frac{\partial g_1}{\partial S} + \beta S) \frac{\partial \epsilon}{\partial S} - (1-h)\alpha((1+\epsilon)\frac{1}{g_1}\frac{\partial g_1}{\partial h} + \frac{\partial \epsilon}{\partial h}) = 0. \tag{7}$$

Note that no approximations have been made so far. The first approximation we introduce now involving assuming that $\partial \epsilon / \partial h$ is zero. Using (5) we have

$$S^2 \frac{\partial^2 \epsilon}{\partial S^2} + (2\lambda + \beta) S \frac{\partial \epsilon}{\partial S} - (1-h)\alpha(1+\epsilon)\left(\frac{A'(h)}{A(h)} + \lambda'(h)\log(S/S^*) - \lambda(h)\frac{1}{S^*}\frac{\partial S^*}{\partial h}\right) = 0. \tag{8}$$

Now we introduce our second approximation. For the purpose of solving the above ordinary differential equation (ODE), $(1+\epsilon)$ is treated as a constant. If $(1+\epsilon)$ is treated as a constant, the above ODE can be solved easily. Let $X = \log(S/S^*)$ and $\epsilon = B(h)X^2 + C(h)X$. Plugging in and matching the coefficients we have

$$\begin{aligned} B &= \frac{(1-h)\alpha\lambda'(h)(1+\epsilon)}{2(2\lambda+\beta-1)} = b(1+\epsilon), \\ C &= \frac{(1-h)\alpha(1+\epsilon)}{2\lambda+\beta-1}\left(\frac{A'(h)}{A(h)} - \lambda(h)\frac{1}{S^*}\frac{\partial S^*}{\partial h} - \frac{\lambda'(h)}{2\lambda+\beta-1}\right) = c(1+\epsilon), \end{aligned}$$

with

$$\begin{aligned} b &= \frac{(1-h)\alpha\lambda'(h)}{2(2\lambda+\beta-1)}, \\ c &= \frac{(1-h)\alpha}{2\lambda+\beta-1}\left(\frac{A'(h)}{A(h)} - \lambda(h)\frac{1}{S^*}\frac{\partial S^*}{\partial h} - \frac{\lambda'(h)}{2\lambda+\beta-1}\right). \end{aligned}$$

Therefore

$$\epsilon = b(1+\epsilon)(\log(S/S^*))^2 + c(1+\epsilon)\log(S/S^*). \tag{9}$$

It follows that

$$\epsilon = \frac{\mathcal{X}}{1-\mathcal{X}}, \tag{10}$$

where

$$\mathcal{X} = b(\log(S/S^*))^2 + c\log(S/S^*).$$

Putting everything together we have that the price of an American option is approximated by

$$V_A(S) = \begin{cases} V_E(S) + \frac{hA(h)(S/S^*)^\lambda}{1-b(\log(S/S^*))^2 - c\log(S/S^*)} & \text{if } \phi(S^*-S) > 0, \\ \phi(S-K) \quad \text{if } \phi(S^*-S) \le 0, \end{cases} \tag{11}$$

where $V_E(S)$ is the Black-Scholes [1973] European option formula.

To determine the price of an option we need to apply the boundary conditions. Applying the value match condition we have

$$\phi(S^*-K) = V_E(S^*) + hA(h). \tag{12}$$

The high contact condition yields

$$\frac{\partial V_A(S)}{\partial S}|_{S\to S^*} = \phi = \phi e^{-\delta\tau} N(\phi d_1(S^*)) + \frac{\lambda(h)hA(h)}{S^*} + \frac{chA(h)}{S^*}, \tag{13}$$

where $d_1(S^*)$ is given by

$$d_1(S^*) = \frac{\log(S^*/K) + (r-\delta+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

Differentiating the value match condition (12) with respect to h, we have

$$
\begin{aligned}
h\frac{\partial A(h)}{\partial h} &= \phi\frac{\partial S^*(h)}{\partial h} - \frac{\partial V_E(S^*, h)}{\partial h} - \frac{\partial V_E(S^*, h)}{\partial S^*}\frac{\partial S^*(h)}{\partial h} - A(h) \\
&= \phi(1 - e^{-\delta\tau}N(\phi d_1(S^*)))\frac{\partial S^*(h)}{\partial h} - \frac{\partial V_E(S^*, h)}{\partial h} - A(h). \qquad (14)
\end{aligned}
$$

Using (14) we have

$$
\begin{aligned}
c = \quad & \frac{\phi(1-h)\alpha}{2\lambda+\beta-1}\left(\frac{1 - e^{-\delta\tau}N(\phi d_1(S^*))}{hA(h)} - \frac{\phi\lambda(h)}{S^*}\right)\frac{\partial S^*(h)}{\partial h} - \\
& \frac{(1-h)\alpha}{2\lambda+\beta-1}\left(\frac{1}{hA(h)}\frac{\partial V_E(S^*, h)}{\partial h} + \frac{1}{h} + \frac{\lambda'(h)}{2\lambda+\beta-1}\right), \qquad (15)
\end{aligned}
$$

where

$$
\begin{aligned}
& \frac{\partial V_E(S^*, h)}{\partial h} = \frac{\partial V_E(S^*, \tau)}{\partial \tau}\frac{\partial \tau}{\partial h} = \frac{S^* n(d_1(S^*))\sigma e^{(r-\delta)\tau}}{2r\sqrt{\tau}} - \\
& \qquad \phi\delta S^* N(\phi d_1(S^*))e^{(r-\delta)\tau}/r + \phi K N(\phi d_2(S^*)), \\
& d_2 = d_1 - \sigma\sqrt{\tau}.
\end{aligned}
$$

To find S^* we ignore the term involving c in (13), which is consistent with our intention that we treat g_2 as a correction to g_1. It follows that S^* solves the following equation:

$$
\phi = \phi e^{-\delta\tau}N(\phi d_1(S^*)) + \frac{\lambda(h)(\phi(S^* - K) - V_E(S^*))}{S^*}. \qquad (16)
$$

An unexpected benefit of doing so is that the term involving $\partial S^*/\partial h$ in (15) is now zero and c is then approximated by

$$
c = -\frac{(1-h)\alpha}{2\lambda+\beta-1}\left(\frac{1}{hA(h)}\frac{\partial V_E(S^*, h)}{\partial h} + \frac{1}{h} + \frac{\lambda'(h)}{2\lambda+\beta-1}\right). \qquad (17)
$$

Equations (12), (16) and (17) determine $hA(h)$, S^* and c respectively. $hA(h)$, S^*, b and c jointly determine the option price from (11). For easy reference we collect the relevant formulas in Exhibit 1.

The case $r = 0$ needs to be considered separately. The above formulas still apply if the limit $r \to 0$ is taken appropriately. This involves setting $\alpha/h = 2/(\sigma^2\tau)$ wherever this combination appears and $r = 0$, $h = 0$ in other places. For easy reference the resulting formulas for $r = 0$ are also listed in Exhibit 1.

In most cases not only are the prices of interest but the hedging parameters. They can be obtained easily using the analytical formulas. For easy reference they are collected in Exhibit 2.

II. NUMERICAL RESULTS AND COMPARISONS

In this section we present extensive numerical results to compare the accuracy and efficiency of our new analytical approximation and several other widely used methods. We choose a binomial tree model with $N = 10,000$ time steps as our benchmark for the true values. Even though better benchmark can be obtained using the state of the art implementation of the binomial tree of Figlewski and Gao [1999], we have checked that our benchmark values are accurate to the digit (3rd) reported and are accurate enough for our comparison. We compare our method (hereafter Mquad) with the four point extrapolation scheme of Geske and Johnson [1984] (hereafter GJ4), the modified two point Geske-Johnson method of Bunch and Johnson [1992] (hereafter MGJ2), the four point extrapolation recursive method of Huang, Subrahmanyam and Yu [1996] (hereafter HSY4), the lower and upper bound approximation of Broadie and Detemple [1996] (hereafter LUBA), the binomial tree model with $N = 150$ time steps (hereafter BT150), the accelerated binomial tree method of Breen [1991] with $N = 150$ time steps (hereafter ABT150), and the quadratic analytical approximation of MacMillan [1986] and Barone-Adesi and Whaley [1987] (hereafter Quad).

We use the root of mean squared errors (RMSE) to measure the overall accuracy of a set of options and maximum absolute error (MAE) to measure the maximum possible error. The computational efficiency is measured using the total CPU time (in seconds) required to compute the whole set of the options. The computation is done on a Sparc-20 in FORTRAN. Because there are different techniques to compute the multivariate cumulative normal functions, no attempt is made to optimize the computations for GJ4. But it is reasonable to assume that it is much less efficient than MGJ2. We follow Bunch and Johnson [1992] in the implementation of MGJ2. A more sophisticated optimization routine is not likely to affect the efficiency and accuracy significantly. For more extensive studies concerning the computational efficiency of various methods, we leave the reader to other sources, for example, Broadie and Detemple [1996].

Exhibit 3 reports results for the 27 options considered in Huang, Subrahmanyam and Yu [1996] and Geske and Johnson [1984]. From the exhibit it is clear that GJ4, HSY4, ABT150, and Mquad have essentially the same accuracy for these 27 options. BT150 and LUBA are very accurate and have about the same accuracy for this set of short and moderate maturity options. MGJ2 is clearly the least accurate method. It has a RMSE of 2.0 cents, MAE of 8.6 cents. It appears that MGJ2 is not a very useful method in terms of efficiency and accuracy. Even though Quad is not as accurate as GJ4, HSY4, LUBA, BT150, ABT150, and Mquad, for these short and moderate maturity options, it is still an efficient and reliable method.

The 20 call options considered in Exhibit 4 are adopted from Table 1 in Broadie and Detemple [1996]. The general observation concerning Exhibit 3 also holds here. It is reasonable to conclude that for options with maturities less than 0.5 years, except perhaps MGJ2 and Quad, all other methods should give similar and reliable results.

For the options considered in Exhibit 3 and Exhibit 4, the improvement of Mquad over Quad is that the RMSE and MAE of the former are about only half or less of those of the latter. The improvement of Mquad over MGJ2 is more drastic. Mquad is also extremely efficient. It is about 20 times or more faster than MGJ2 and GJ4. It is also about 7 times faster than HSY4 and 15 times faster than LUBA. For short and moderate maturity options, Mquad could be the choice of methods in many applications.

To test the accuracies of these methods for longer maturity options, we consider the 20 put options in Table V of Barone-Adesi and Whaley [1987] in Exhibit 5 and the 20 call options in Table 2 of Broadie and Detemple [1996] in Exhibit 6. For these long maturity options, except LUBA and BT150, Mquad outperforms all the other methods substantially. The MAE and RMSE of GJ4 are about as twice as those of Mquad. The MAE and RMSE of Mquad are about 1.6 and 3 times smaller than those of HSY4. The improvement of Mquad over Quad is more drastic. Its RMSE and MAE are more than five times smaller than those of Quad. The improvement of Mquad over MGJ2 is even more drastic. Its RMSE and MAE are about ten or more times smaller than those of MGJ2. The improvement of Mquad over ABT150 is also substantial. It is clear from Exhibit 5 and Exhibit 6 that for options with long maturities, Quad, MGJ2 and ABT150 cease to be reliable methods and the reliabilities of GJ4 and HSY4 are greatly reduced. On the other hand, considering the efficiency and accuracy, Mquad could still be the choice of methods for long maturity options if the requirement for accuracy is not too stringent.

III. Summary and Discussions

We have proposed a new approximate analytical formula for pricing American options. Our analytical approximation is as efficient as the existing ones such as that of MacMillan [1986] and Barone-Adesi and Whaley [1987] and much more efficient than other methods such as the four point extrapolation schemes of Geske and Johnson [1984] and Huang, Subrahmanyam and Yu [1996], the modified two point Geske-Johnson of Bunch and Johnson [1992] and the accelerated binomial model of Breen [1991] with 150 time steps. Our analytical approximation is markedly more accurate than the equally efficient analytical approximation of MacMillan [1986] and Barone-Adesi and Whaley [1987] and the modified two point Geske-Johnson of Bunch and Johnson [1992] for both short and long maturity options. For short maturity options, our approximation and those of Geske and Johnson [1984], Huang, Subrahmanyam and Yu [1996] yield similar results. For long maturity options, our approximation is more accurate than those of Geske and Johnson [1984], and Huang, Subrahmanyam and Yu [1996]. A drawback of our approximation is that it is not convergent. Nevertheless, considering the efficiency and accuracy of the present approximation, it should be a useful and reliable tool for pricing American options.

Even though we have only discussed standard American stock options, our approximation obviously applies to other American options such as futures options, quanto options, index options and currency options.

ENDNOTES

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EXHIBIT 1

FORMULAS FOR OPTION PRICE

$$V_A(S) = \begin{cases} V_E(S) + \frac{hA(h)(S/S^*)^{\lambda(h)}}{1-\mathcal{X}} & \text{if } \phi(S^* - S) > 0, \\ \phi(S-K) \quad \text{if } \phi(S^* - S) \leq 0, \end{cases}$$

where $V_E(S)$ is the Black-Scholes [1973] European option formula ($\phi = 1$ for calls and $\phi = -1$ for puts), $hA(h) = \phi(S^* - K) - V_E(S^*)$ and S^* solves the following equation:

$$\phi = \phi e^{-\delta\tau} N(\phi d_1(S^*)) + \frac{\lambda(h)(\phi(S^* - K) - V_E(S^*))}{S^*},$$

and $\mathcal{X}$, b and c are given by

$$\mathcal{X} = b(\log(S/S^*))^2 + c\log(S/S^*), \quad b = \frac{(1-h)\alpha\lambda'(h)}{2(2\lambda + \beta - 1)},$$
$$c = -\frac{(1-h)\alpha}{2\lambda + \beta - 1}\left(\frac{1}{hA(h)}\frac{\partial V_E(S^*, h)}{\partial h} + \frac{1}{h} + \frac{\lambda'(h)}{2\lambda + \beta - 1}\right),$$

where

$$\tau = T - t, \qquad h(\tau) = 1 - e^{-r\tau}, \qquad \alpha = \frac{2r}{\sigma^2}, \qquad \beta = \frac{2(r-\delta)}{\sigma^2},$$
$$\lambda(h) = \frac{-(\beta - 1) + \phi\sqrt{(\beta-1)^2 + \frac{4\alpha}{h}}}{2}, \quad \lambda'(h) = -\frac{\phi\alpha}{h^2\sqrt{(\beta-1)^2 + \frac{4\alpha}{h}}},$$
$$\frac{\partial V_E(S^*, h)}{\partial h} = \frac{S^* n(d_1(S^*))\sigma e^{(r-\delta)\tau}}{2r\sqrt{\tau}} - \phi\delta S^* N(\phi d_1(S^*))e^{(r-\delta)\tau}/r + \phi K N(\phi d_2(S^*)),$$
$$d_1(S^*) = \frac{\log(S^*/K) + (r - \delta + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau},$$

If $r = 0$ and the option is a call, we use the following limiting values,

$$\lambda = \frac{-(\beta-1) + \phi\sqrt{(\beta-1)^2 + \frac{8}{\sigma^2\tau}}}{2}, \quad b = \frac{-2}{\sigma^4\tau^2((\beta-1)^2 + \frac{8}{\sigma^2\tau})},$$
$$c = \frac{-\phi}{\sqrt{(\beta-1)^2 + \frac{8}{\sigma^2\tau}}}\left(\frac{S^* n(d_1(S^*))e^{-\delta\tau}}{hA(h)\sigma\sqrt{\tau}} - \frac{\phi 2\delta S^* N(\phi d_1(S^*))e^{-\delta\tau}}{hA(h)\sigma^2} + \right.$$
$$\left.\frac{2}{\sigma^2\tau} - \frac{4}{\sigma^4\tau^2((\beta-1)^2 + \frac{8}{\sigma^2\tau})}\right).$$

EXHIBIT 2

FORMULAS FOR HEDGING PARAMETERS DELTA, GAMMA and THETA

$$\begin{aligned}
\Delta &= \phi e^{-\delta\tau} N(\phi d_1(S^*)) + \left(\frac{\lambda(h)}{S(1-\mathcal{X})} + \frac{\mathcal{X}'(S)}{(1-\mathcal{X})^2}\right)(\phi(S^*-K) - V_E(S^*))(S/S^*)^{\lambda(h)}, \\
\Gamma &= \phi e^{-\delta\tau}\frac{n(\phi d_1(S^*))}{S\sigma\sqrt{\tau}} + \left(\frac{2\lambda(h)\mathcal{X}'(S)}{S(1-\mathcal{X})^2} + \frac{2\mathcal{X}'^2(S)}{(1-\mathcal{X})^3} + \frac{\mathcal{X}''(S)}{(1-\mathcal{X})^2} + \frac{\lambda^2(h)-\lambda(h)}{S^2(1-\mathcal{X})}\right) \\
&\quad (\phi(S^*-K) - V_E(S^*))(S/S^*)^{\lambda(h)},
\end{aligned}$$

where $n(\cdot)$ is the standard normal density function and

$$\begin{aligned}
\mathcal{X}'(S) &= \frac{2b}{S}\log(\frac{S}{S^*}) + \frac{c}{S}, \\
\mathcal{X}''(S) &= \frac{2b}{S^2} - \frac{2b}{S^2}\log(\frac{S}{S^*}) - \frac{c}{S^2}.
\end{aligned}$$

Theta can be easily obtained through the PDE satisfied by the option price,

$$\Theta = rV_A - \frac{1}{2}\sigma^2 S^2 \Gamma - (r-\delta)S\Delta.$$

EXHIBIT 3

VALUES OF AMERICAN PUTS ($S = \$40$, $r = 0.0488$, $\delta = 0.0$)

(1) $(K, \sigma, \tau$ (yr))	(2) TRUE	(3) GJ4	(4) MGJ2	(5) HSY4	(6) LUBA	(7) BT150	(8) ABT150	(9) Quad	(10) Mquad
(35, 0.2, 0.0833)	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.007	0.006
(35, 0.2, 0.3333)	0.200	0.200	0.200	0.200	0.200	0.199	0.199	0.204	0.201
(35, 0.2, 0.5833)	0.433	0.432	0.431	0.434	0.433	0.434	0.434	0.442	0.433
(40, 0.2, 0.0833)	0.852	0.853	0.852	0.854	0.852	0.851	0.851	0.850	0.851
(40, 0.2, 0.3333)	1.580	1.581	1.580	1.587	1.580	1.578	1.574	1.577	1.576
(40, 0.2, 0.5833)	1.990	1.991	1.992	1.999	1.990	1.989	1.984	1.989	1.984
(45, 0.2, 0.0833)	5.000	4.999	5.000	5.002	5.000	5.000	4.997	5.000	5.000
(45, 0.2, 0.3333)	5.088	5.095	5.002	5.095	5.090	5.089	5.102	5.066	5.084
(45, 0.2, 0.5833)	5.267	5.272	5.244	5.263	5.268	5.268	5.285	5.236	5.260
(35, 0.3, 0.0833)	0.077	0.077	0.077	0.078	0.077	0.078	0.077	0.078	0.078
(35, 0.3, 0.3333)	0.698	0.697	0.696	0.698	0.698	0.699	0.698	0.701	0.697
(35, 0.3, 0.5833)	1.220	1.219	1.217	1.223	1.220	1.224	1.224	1.228	1.218
(40, 0.3, 0.0833)	1.310	1.310	1.309	1.312	1.310	1.308	1.308	1.308	1.309
(40, 0.3, 0.3333)	2.483	2.482	2.481	2.492	2.483	2.480	2.476	2.478	2.477
(40, 0.3, 0.5833)	3.170	3.173	3.170	3.184	3.170	3.166	3.159	3.167	3.161
(45, 0.3, 0.0833)	5.060	5.060	5.067	5.060	5.060	5.060	5.063	5.047	5.059
(45, 0.3, 0.3333)	5.706	5.701	5.733	5.697	5.704	5.707	5.698	5.679	5.699
(45, 0.3, 0.5833)	6.244	6.237	6.281	6.230	6.243	6.245	6.239	6.215	6.231
(35, 0.4, 0.0833)	0.247	0.247	0.246	0.247	0.247	0.245	0.245	0.247	0.247
(35, 0.4, 0.3333)	1.346	1.345	1.344	1.347	1.346	1.350	1.350	1.349	1.344
(35, 0.4, 0.5833)	2.155	2.157	2.152	2.160	2.155	2.160	2.159	2.162	2.150
(40, 0.4, 0.0833)	1.768	1.768	1.768	1.769	1.768	1.766	1.766	1.766	1.767
(40, 0.4, 0.3333)	3.387	3.363	3.386	3.397	3.388	3.384	3.383	3.382	3.381
(40, 0.4, 0.5833)	4.353	4.356	4.351	4.370	4.353	4.348	4.339	4.349	4.342
(45, 0.4, 0.0833)	5.287	5.286	5.295	5.285	5.286	5.287	5.286	5.273	5.288
(45, 0.4, 0.3333)	6.510	6.509	6.520	6.513	6.509	6.510	6.505	6.487	6.501
(45, 0.4, 0.5833)	7.383	7.383	7.387	7.398	7.383	7.390	7.381	7.360	7.367
RMSE		0.005	0.020	0.007	0.002	0.003	0.007	0.013	0.006
MAE		0.024	0.086	0.017	0.006	0.007	0.018	0.031	0.016
CPU (sec)		$\gg$ 9.5e-02	9.5e-2	3.1e-02	8.0e-2	4.1e-1	3.2e-1	4.2e-3	4.8e-3

The 'TRUE' value is based on binomial with $N = 10,000$. Columns 3-10 represent the methods Geske and Johnson [1984], Bunch and Johnson [1992], Huang, Subrahmanyam and Yu [1996], Broadie and Detemple [1996], binomial and accelerated binomial with $N = 150$, MacMillan [1986] and Barone-Adesi and Whaley [1987], and the modified quadratic approximation of this article. RMSE is the root of mean squared errors. MAE is the maximum absolute error. CPU is the total computing time for the whole set of options.

EXHIBIT 4

VALUES OF AMERICAN CALLS (K = \$100, $\tau = 0.5$ years)

(1) (S, σ, r, δ)	(2) TRUE	(3) GJ4	(4) MGJ2	(5) HSY4	(6) LUBA	(7) BT150	(8) ABT150	(9) Quad	(10) Mquad
(80, 0.2, 0.03, 0.07)	0.219	0.219	0.219	0.220	0.220	0.220	0.218	0.230	0.222
(90, 0.2, 0.03, 0.07)	1.386	1.385	1.382	1.390	1.386	1.392	1.392	1.405	1.386
(100, 0.2, 0.03, 0.07)	4.783	4.785	4.786	4.804	4.782	4.778	4.758	4.782	4.768
(110, 0.2, 0.03, 0.07)	11.098	11.089	11.255	11.069	11.098	11.098	11.104	11.041	11.079
(120, 0.2, 0.03, 0.07)	20.000	20.007	20.000	20.053	20.000	20.000	20.009	20.000	20.000
(80, 0.4, 0.03, 0.07)	2.689	2.686	2.683	2.690	2.689	2.690	2.689	2.711	2.687
(90, 0.4, 0.03, 0.07)	5.722	5.721	5.716	5.736	5.723	5.731	5.735	5.742	5.711
(100, 0.4, 0.03, 0.07)	10.239	10.245	10.235	10.275	10.240	10.227	10.215	10.242	10.214
(110, 0.4, 0.03, 0.07)	16.181	16.183	16.211	16.201	16.182	16.177	16.144	16.152	16.146
(120, 0.4, 0.03, 0.07)	23.360	23.342	23.477	23.329	23.357	23.353	23.335	23.288	23.321
(80, 0.3, 0.00, 0.07)	1.037	1.035	1.032	1.037	1.037	1.041	1.038	1.062	1.040
(90, 0.3, 0.00, 0.07)	3.123	3.124	3.115	3.144	3.123	3.118	3.115	3.147	3.118
(100, 0.3, 0.00, 0.07)	7.035	7.038	7.041	7.067	7.035	7.029	7.003	7.028	7.015
(110, 0.3, 0.00, 0.07)	12.955	12.934	13.064	12.909	12.953	12.960	12.937	12.886	12.928
(120, 0.3, 0.00, 0.07)	20.717	20.742	20.438	20.727	20.721	20.719	20.833	20.607	20.695
(80, 0.3, 0.07, 0.03)	1.664	1.664	1.664	1.664	1.664	1.668	1.668	1.665	1.664
(90, 0.3, 0.07, 0.03)	4.495	4.495	4.495	4.495	4.495	4.485	4.485	4.495	4.495
(100, 0.3, 0.07, 0.03)	9.250	9.251	9.251	9.251	9.251	9.237	9.237	9.251	9.251
(110, 0.3, 0.07, 0.03)	15.798	15.797	15.798	15.798	15.798	15.803	15.803	15.799	15.798
(120, 0.3, 0.07, 0.03)	23.706	23.708	23.706	23.706	23.706	23.709	23.709	23.709	23.707
RMSE		0.009	0.080	0.023	0.001	0.006	0.031	0.038	0.017
MAE		0.025	0.279	0.053	0.004	0.013	0.116	0.110	0.039
CPU (sec)		$\gg$ 6.6e-2	6.6e-2	2.9e-2	6.2e-2	3.0e-1	2.4e-1	2.9e-3	3.4e-3

The 'TRUE' value is based on binomial with $N = 10,000$. Columns 3-10 represent the methods Geske and Johnson [1984], Bunch and Johnson [1992], Huang, Subrahmanyam and Yu [1996], Broadie and Detemple [1996], binomial and accelerated binomial with $N =$ 150, MacMillan [1986] and Barone-Adesi and Whaley [1987], and the modified quadratic approximation of this article. RMSE is the root of mean squared errors. MAE is the maximum absolute error. CPU is the total computing time for the whole set of options.

EXHIBIT 5

VALUES OF AMERICAN PUTS ($K = \$100$, $\tau = 3.0$ years, $\sigma = 0.2$, $r = 0.08$)

(1) (S, δ)	(2) TRUE	(3) GJ4	(4) MGJ2	(5) HSY4	(6) LUBA	(7) BT150	(8) ABT150	(9) Quad	(10) Mquad
(80, 0.12)	25.658	25.653	25.949	25.686	25.657	25.647	25.545	26.245	25.725
(90, 0.12)	20.083	20.109	20.201	20.128	20.083	20.078	20.030	20.641	20.185
(100, 0.12)	15.498	15.512	15.550	15.536	15.499	15.478	15.475	15.990	15.608
(110, 0.12)	11.803	11.802	11.824	11.823	11.803	11.810	11.840	12.221	11.905
(120, 0.12)	8.886	8.880	8.897	8.894	8.886	8.884	8.907	9.235	8.974
(80, 0.08)	22.205	22.208	22.711	22.245	22.199	22.199	22.384	22.395	22.148
(90, 0.08)	16.207	16.164	16.531	16.134	16.199	16.205	16.201	16.498	16.170
(100, 0.08)	11.704	11.705	11.811	11.718	11.699	11.690	11.662	12.030	11.700
(110, 0.08)	8.367	8.389	8.407	8.436	8.363	8.378	8.356	8.687	8.390
(120, 0.08)	5.930	5.944	5.931	5.988	5.926	5.933	5.932	6.222	5.968
(80, 0.04)	20.350	20.513	20.000	20.523	20.334	20.344	20.577	20.326	20.336
(90, 0.04)	13.497	13.525	14.025	13.378	13.498	13.491	13.677	13.563	13.471
(100, 0.04)	8.944	8.841	9.109	8.804	8.942	8.934	8.910	9.108	8.931
(110, 0.04)	5.912	5.890	5.931	5.919	5.912	5.919	5.872	6.123	5.920
(120, 0.04)	3.898	3.905	3.882	3.978	3.898	3.897	3.848	4.115	3.922
(80, 0.0)	20.000	19.731	20.000	19.846	20.000	20.000	19.467	20.000	20.000
(90, 0.0)	11.697	11.884	10.176	11.761	11.695	11.684	12.014	11.634	11.705
(100, 0.0)	6.932	6.927	6.939	6.786	6.935	6.921	7.081	6.962	6.956
(110, 0.0)	4.155	4.103	4.145	4.090	4.155	4.154	4.100	4.257	4.190
(120, 0.0)	2.510	2.491	2.455	2.559	2.511	2.506	2.485	2.640	2.551
RMSE		0.087	0.401	0.085	0.0048	0.009	0.166	0.297	0.053
MAE		0.269	1.521	0.173	0.017	0.020	0.533	0.587	0.110
CPU (sec)		$\gg$ 6.7e-02	6.7e-2	2.5e-02	5.4e-2	3.0e-1	2.4e-1	2.8e-3	3.2e-3

The 'TRUE' value is based on binomial with $N = 10,000$. Columns 3-10 represent the methods Geske and Johnson [1984], Bunch and Johnson [1992], Huang, Subrahmanyam and Yu [1996], Broadie and Detemple [1996], binomial and accelerated binomial with $N = 150$, MacMillan [1986] and Barone-Adesi and Whaley [1987], and the modified quadratic approximation of this article. RMSE is the root of mean squared errors. MAE is the maximum absolute error. CPU is the total computing time for the whole set of options.

EXHIBIT 6

VALUES OF AMERICAN CALLS (K = \$100, $\tau = 3.0$ years)

(1) (S, σ, r, δ)	(2) TRUE	(3) GJ4	(4) MGJ2	(5) HSY4	(6) LUBA	(7) BT150	(8) ABT150	(9) Quad	(10) Mquad
(80, 0.2, 0.03, 0.07)	2.580	2.588	2.560	2.644	2.580	2.576	2.531	2.711	2.605
(90, 0.2, 0.03, 0.07)	5.167	5.155	5.164	5.195	5.168	5.168	5.087	5.301	5.182
(100, 0.2, 0.03, 0.07)	9.066	9.014	9.216	8.941	9.065	9.056	9.023	9.154	9.065
(110, 0.2, 0.03, 0.07)	14.443	14.446	14.982	14.288	14.444	14.443	14.666	14.444	14.430
(120, 0.2, 0.03, 0.07)	21.414	21.578	20.000	21.488	21.412	21.405	21.757	21.336	21.398
(80, 0.4, 0.03, 0.07)	11.326	11.344	11.352	11.440	11.327	11.311	11.231	11.625	11.336
(90, 0.4, 0.03, 0.07)	15.722	15.714	15.796	15.769	15.724	15.714	15.617	16.028	15.711
(100, 0.4, 0.03, 0.07)	20.793	20.741	21.028	20.720	20.793	20.772	20.660	21.084	20.760
(110, 0.4, 0.03, 0.07)	26.494	26.401	26.938	26.297	26.489	26.492	26.347	26.749	26.440
(120, 0.4, 0.03, 0.07)	32.781	32.676	33.569	32.512	32.772	32.797	32.833	32.982	32.709
(80, 0.3, 0.00, 0.07)	5.518	5.514	5.495	5.600	5.520	5.527	5.483	5.658	5.552
(90, 0.3, 0.00, 0.07)	8.842	8.795	8.889	8.809	8.843	8.842	8.761	8.947	8.868
(100, 0.3, 0.00, 0.07)	13.142	13.064	13.345	12.948	13.142	13.128	13.095	13.177	13.158
(110, 0.3, 0.00, 0.07)	18.453	18.419	19.101	18.191	18.453	18.450	18.636	18.394	18.458
(120, 0.3, 0.00, 0.07)	24.791	24.894	25.265	24.643	24.797	24.788	25.194	24.638	24.786
(80, 0.3, 0.07, 0.03)	12.146	12.144	12.146	12.147	12.145	12.162	12.167	12.282	12.177
(90, 0.3, 0.07, 0.03)	17.368	17.366	17.372	17.371	17.368	17.377	17.385	17.553	17.411
(100, 0.3, 0.07, 0.03)	23.348	23.345	23.356	23.355	23.349	23.317	23.331	23.586	23.402
(110, 0.3, 0.07, 0.03)	29.964	29.962	29.976	29.977	29.964	29.956	29.975	30.259	30.028
(120, 0.3, 0.07, 0.03)	37.104	37.108	37.126	37.126	37.104	37.100	37.118	37.459	37.177
RMSE		0.060	0.441	0.126	0.003	0.012	0.150	0.200	0.037
MAE		0.164	1.414	0.269	0.009	0.031	0.403	0.355	0.073
CPU (sec)		$\gg$ 6.2e-2	6.2e-2	3.0e-2	6.0e-2	3.0e-2	2.4e-1	2.7e-3	3.3e-3

The 'TRUE' value is based on binomial with $N = 10,000$. Columns 3-10 represent the methods Geske and Johnson [1984], Bunch and Johnson [1992], Huang, Subrahmanyam and Yu [1996], Broadie and Detemple [1996], binomial and accelerated binomial with $N =$ 150, MacMillan [1986] and Barone-Adesi and Whaley [1987], and the modified quadratic approximation of this article. RMSE is the root of mean squared errors. MAE is the maximum absolute error. CPU is the total computing time for the whole set of options.