Current (spot) price of European call (if the stock does not pay dividends) is higher than its intrinsic value.
Options

• Put option (European):
  - Tied to a specific asset, for example a stock.

K- “strike price”
S- stock value
T-time to maturity
The option gives the owner the **right** to **SELL**
the stock for price K at some specified time T.
The owner does not need to execute this right.
American options

• Same as European, but can be executed at any moment prior the maturity

• American call

• American put
Example of a derivative

• Does it make sense to buy both call and a put?

Its price is positive! This derivative is called STRADDLE
TRADING ON THE MARKET

Limit order book

Offer

Price $p$

$\text{p}_0$, best ask

Demand

Trade at price $\text{p}_0$

New best bid

New spread

LAST MATCH

Price $384.9000$

Time 15:18:56

GOOGL

TODAY'S ACTIVITY

Orders 1,295,622

Volume 2,791,809

BUY ORDERS

SHARES PRICE

50 384.8200

100 384.8200

100 384.8100

300 384.8100

100 384.8000

500 384.7900

200 384.7700

500 384.7600

100 384.7100

100 384.6900

200 384.6800

300 384.5900

100 384.5000

50 384.0000

100 384.0000

SELL ORDERS

SHARES PRICE

93 384.9500

100 385.0300

100 385.0600

100 385.0700

200 385.0900

100 385.1800

100 385.2400

25 385.2500

100 385.3500

15 385.5000

200 385.5500

200 385.6000

360 385.6300

100 385.6800

100 385.7100
TRADING GAME

Each player initially has: **100 “dollars”**

Player 1: $100
Player 2: $100
Player 3: $100
Player 4: $100
Player 5: $100
Player 6: $100
Player 7: $100
Player 8: $100
...

Round 1

Asset: The result of rolling 6 dice.

Value = $X_1 + X_2 + X_3 + X_4 + X_5 + X_6$

Rules:
1. Each player bids.
2. I accept 3 best bids.
3. The Value is revealed, the 3 players receive it and sell it on the market.
4. They earn the following profit: Profit = Value - Bid
Analysis of round 1:
This situation was an aggressive “market order”.

Expected Value = 6 * 3.5 = 21

When you buy something, you want to bid below the expected value.

If there are many players, bid-ask spread narrows (in real markets the bid-ask spread is very narrow)
Round 2

• Same game one more time.

Asset: The result of rolling 6 dice.

\[ \text{Value} = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \]

Profit = Value - Bid

However, this time I accept an unknown number of bids.
Round 3

• A big player comes in: an investor who has a large amount of Dice Technologies LTD stock. He wants to get rid of this stock quickly so he can raise money for other investments.

• He/she and announces that he/she is selling a huge amount of shares of Dice Technologies LTD.

• He is willing to take 7 best bids!!!!

• [you must not communicate between each others]

• Again, the game is the same.

Value= $X_1 + X_2 + X_3 + X_4 + X_5 + X_6$

Profit= \text{Value-Bid}
• Analysis of the *limit order book* for this round:

• Large offer decreases the price of an asset.

• Large demand increases the price.

• When you are on the market, you should not disclose (tell) what are your intentions.

• The information about how badly you want something is valuable to other players, and they will use it against you. When you want to buy or sell a huge amount of stocks, you should do it little by little.

• Still, there are computer programs on the market trying to detect your true intentions!

• If you are a broker, *front running* (using information against your client) is illegal.
Time to sell some derivatives!!!

Auction:

Derivative 1:
Receiving 3 points every round until someone reaches the score of 150.
Time to sell some derivatives!!!

Auction:

Derivative 2:
$Profit = 10 - t + (\text{number of people above 130})*4$

Can be executed at any moment. (American style)

→ Initial value: $7 + \text{variable part}$
→ One component slowly depreciating in time, but the third term has a potential for a significant profit.
Round 4

- The result of rolling 2 dice.

- Value = $X_1 * X_2$

- 4 best bids are accepted
Round 5 – one more roll?

• The result of rolling 2 dice.

• Value= $X_1 * X_2$

• 4 best bids are accepted
Round 6

• Asset: $10 + (rolling 3 dice) – (rolling 3 dice one more time)

• Value = $10 + X1 + X2 + X3 - (X4 + X5 + X6)

• 5 highest bids accepted
Round 7

• Value = [Population of Nigeria in millions]/10

• One best bid for the Value
Round 8

(NA / secret)
Game theory

• On the market, many players are smart and they try to predict other people’s actions

• what other player will do if you do something?

• What they think that you think that they think?

• See on Wikipedia: Nash Equilibrium
Round 9

• Choose a secret integer X between 0 and 15.
• Write it on the paper and do not show to anybody.
• If there are N players, there are N points on this line (some possibly overlapping).

• Prize= X + 8*|distance to your closest neighbor|
Round 10

• Choose a secret integer $X$ between 0 and 15.

Prize= $-20 + 8*|\text{distance to your closest neighbor}|$

• You can choose to fold (not play this game).
• In that case, write “FOLD” on your paper.
How do we price options?

• We need a special math for that
Introduction to STOCHASTIC CALCULUS

- Brownian process
• We already know that if $B(0)=0$, then

$$E[B(t)^2] = \sigma \, t$$

Or for the unit standard deviation we have:

$$E[B(t)^2] = t$$

Let's observe an infinitesimal interval of time:

$$E[dB(t)^2] = dt$$
• Now, we claim:

\[ dB(t)^2 = dt \]

**Sketch of a proof:**
Observe the following stochastic “variable”: \( dB(t)^2 - dt \)
Its expectation value is: 0
Using \( E(X^4) = 3\sigma^4 \) for normal stochastic variables, it can be shown that its variance is equal to \( 2(dt)^2 \). When \( dt \) is small, squared \( dt \) is even smaller, so we can informally write \( dB(t)^2 - dt = 0 \), or

\[ dB(t)^2 = dt \]
• This is a very simple but powerful equation, and it produces a totally different calculus - **Ito calculus**

• Ito calculus may give strange results

• For example, if \( y = B(t) \), where \( B(t) \) is a Brownian process:

\[
\int y \, dy \neq \frac{1}{2}y^2 + C
\]

Here \( C \) cannot be any numerical constant, like for standard integrals.
You need to forget most of the things that you learnt in the ordinary calculus.
• In Ito calculus, second order differentials cannot always be neglected. For example, if \( f = f(x, y) \) then sometimes

\[
df \neq \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy
\]

• We need to keep higher order terms, if \( x \) or \( y \) are stochastic processes
• Model for stock prices: geometric Brownian motion

\[ dS = \mu S \, dt + \sigma S \, dB \]

• Now we finally come to options (or, general derivatives).

Assume a variable (derivative price \( f \)) is some function of \( S \) and \( t \):

\[ f = f(S, t) \]

Then

\[ df = \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial S} \, dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \, (dS)^2 \]
\[ dS = \mu S \, dt + \sigma S \, dB \]

\[ df = \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial S} \, dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \, (dS)^2 \]

If we plug in the process for the stock into the total differential for \( f \):

\[ df = \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial S} \left[ \mu S \, dt + \sigma S \, dB \right] + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \left[ \mu S \, dt + \sigma S \, dB \right]^2 \]

Using \( dB(t)^2 = dt \) and neglecting \((dt)^2\), we get:

\[ df = \left[ \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] \, dt + \frac{\partial f}{\partial S} \sigma S \, dB \]