CHAOS **15**, 026104 (2005)

The subtle nature of financial random walks

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(Received 17 December 2004; accepted 19 February 2005; published online 17 June 2005)

We first review the most important "stylized facts" of financial time series, that turn out to be, to a large extent, universal. We then recall how the multifractal random walk of Bacry, Muzy, and Delour generalizes the standard model of financial price changes and accounts in an elegant way for many of their empirical properties. In a second part, we provide empirical evidence for a very subtle compensation mechanism that underlies the random nature of price changes. This compensation drives the market close to a critical point, that may explain the sensitivity of financial markets to small perturbations, and their propensity to enter bubbles and crashes. We argue that the resulting unpredictability of price changes is very far from the neoclassical view that markets are informationally efficient. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1889265]

It is known since Bachelier (1900) that price changes are nearly uncorrelated, leading to a random-walk-like behavior of prices. However, compared to the simplest Brownian motion, price statistics reveal a large number of anomalies, such as fat tails and long memory in the volatility. Understanding and modelling these statistical anomalies is of crucial importance both from the point of view of theoretical economics and of financial engineering. Tools from physics, in particular from turbulence theory, seem to be particularly relevant to describe the self-similar statistics of volatility clustering, in strong analogy with the statistics of turbulent patches. Trade by trade analysis, now possible thanks to the availability of high frequency data, allows one to uncover the intimate mechanisms of price formation. Quite remarkably, the unpredictable nature of price changes results from a very subtle compensation mechanism, between liquidity takers and liquidity providers, that drives markets close to a dynamical critical point. This may explain naturally the sensitivity of financial markets to small perturbations, the presence of power laws and scaling, and the propensity to enter bubbles and crashes. The mechanism that we observe is very far from the neoclassical view that markets are informationally efficient; quite on the contrary, we believe that markets are by construction prone to systematic mispricing.

I. INTRODUCTION

Financial time series represent an extremely rich and fascinating source of questions. A quantitative trace of human activity is recorded and stored, in some cases every second. Some of these records span two centuries. These time series, perhaps surprisingly, turn out to reveal a very rich and nontrivial statistical structure, that is to some degree universal, across different assets (stocks, commodities, currencies, rates, etc.), regions (U.S., European, Asian) and epochs. Statistical models that describe these fluctuations have a long history, which dates back to Bachelier's "Brownian walk" model for speculative prices first published in 1900.[1] Bachelier's thesis is the first attempt to describe the endless fluctuations of stock markets in a scientific way. Many concepts of modern theoretical finance were largely anticipated by Bachelier, as was Einstein's celebrated theory of Brownian motion, that only came five years after Bachelier's remarkable insights.

Much more sophisticated models are however needed to describe faithfully the empirical data. For example, the tails of the distribution of price changes are, as is now well known, much fatter than Gaussian. Many recent empirical studies have shown that financial data share many statistical properties with turbulent flows, where the velocity reveals strong, intermittent fluctuations, much as the volatility of financial markets. In spite of its shortcomings, the model of Bachelier gets one important fact right: price changes are to a first approximation uncorrelated, which makes the prediction of stock markets difficult. However, the mechanism that converts a rather predictable human behavior into a sequence of (nearly) unpredictable price changes, has not been investigated in details until recently. The availability of high-frequency, trade by trade data, and the shift of paradigm from efficient markets by *fiat* to agent based, bounded rationality models, have motivated a series of exciting studies, that most probably anticipate an important revolution of ideas in economics. The aim of this paper is (a) to review the basic properties of financial time series (Sec. II), (b) to insist on the need to construct phenomenological models that go much beyond the Bachelier–Einstein random walk (Sec. III), and (c) to provide evidence for a very subtle compensation mechanism that underlies the random nature of price changes (Sec. IV). This compensation drives the market close to a critical point, a possibility conjectured in different contexts to explain the presence of power laws and scale invariance in the statistics of financial time series. The proximity of a critical point might also explain the enhanced sensitivity of financial markets to small perturbations, and its propensity to enter bubbles and crashes. We argue that the resulting unpredictability of price changes is very far from the neoclassical view that markets are informationally efficient.

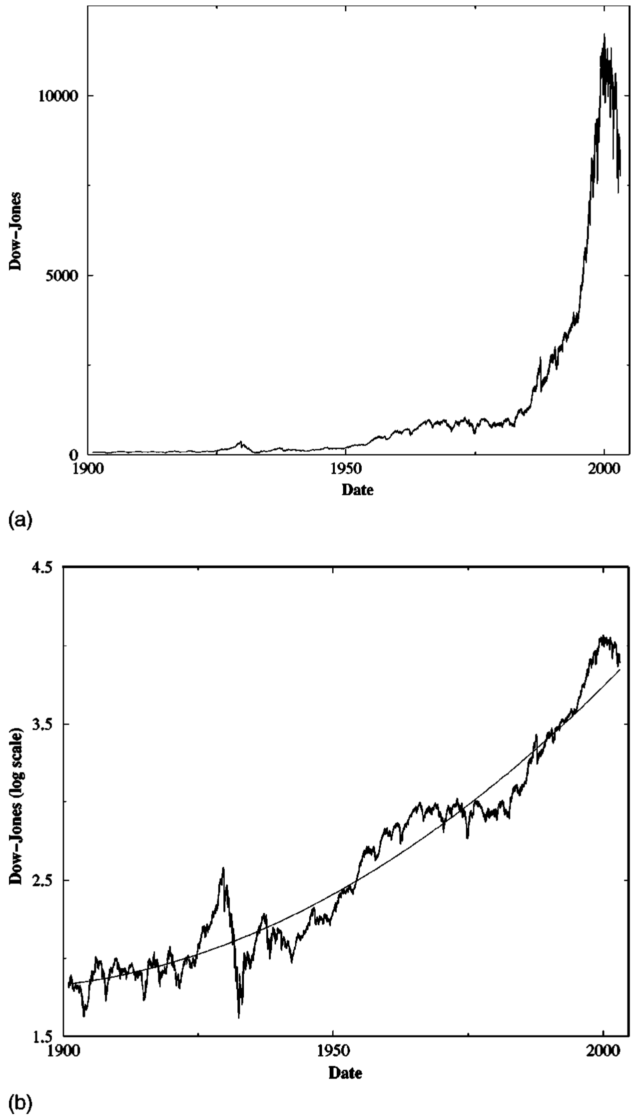

FIG. 1. (a) Evolution of the Dow-Jones index price over the last century (1900–2003). (b) Same data in a linear-log (base 10) representation. The full line is a parabolic fit, which shows that the average annual return has actually increased with time.

II. UNIVERSAL FEATURES OF RETURN TIME SERIES

The modeling of random fluctuations of asset prices is of primary importance in finance, with many applications to risk control, derivative pricing, and systematic trading. During the last decade, the availability of huge data sets of high frequency time series has promoted intensive statistical studies that invalidate the classic and popular "Brownian walk" model, and to uncover many new and robust features. In this section, we briefly review the main statistical properties of asset prices that can be considered as universal, in the sense that they are common across most markets and epochs.[2–4]

Let us first define some basic notions. If one denotes $p(t)$ the price of an asset at time t, the *return* $r_\tau(t)$, at time t and scale τ is simply the relative variation of the price from t to $t+\tau$, $r_\tau(t)=[p(t+\tau)-p(t)]/p(t)$. If τ is small enough, one has approximately $r_\tau(t)\simeq \ln p(t+\tau)-\ln p(t)$.

We show in Fig. 1 the level of the Dow-Jones index over the last century. One can see that price fluctuations organize around a mean (super-)exponential trend. In Fig. 1(b), we have plotted the logarithmic price time series $x(t)=\ln p(t)$, in that case, the fluctuations are seen to be stationary around a mean return where the drift m is around 5% per year, but has slowly increased during the whole century. Note that the current level of the Dow-Jones (after the Internet crash) is, perhaps anecdotally, very close to its historical extrapolation.

The simplest universal feature of financial time series, uncovered by Bachelier in 1900, is the linear growth of the variance of the return fluctuations with time scale. More precisely, if $m\tau$ is the mean return on scale τ, the following property holds, to a very good approximation:

$$\langle [r_\tau(t)-m\tau]^2\rangle_e \simeq \sigma^2\tau, \tag{1}$$

where $\langle \cdots \rangle_e$ denotes the sample average. This behavior typically holds for τ between a few minutes to a few years, and is equivalent to the statement that relative price changes are, in a first approximation, *uncorrelated*. Very long time scales (beyond a few years) are difficult to investigate, in particular because the average drift m becomes itself time dependent, but there are systematic studies suggesting some degree of mean reversion on these long time scales[5] (see the conclusion for a discussion of this particular point). The absence of linear correlations (Bachelier's first law) in financial time series is often related to the so-called market efficiency according to which one cannot make anomalous profits by predicting future price values.

The root mean square per unit time, σ in the above equation, is called the *volatility*. Volatility is the simplest quantity that measures the amplitude of price fluctuations and therefore quantifies the risk associated with some given asset. A linear growth of the variance of the fluctuations with time is typical of the Brownian motion for the log-price x, and, as mentioned above, was proposed as a model of market fluctuations by Bachelier. (In Bachelier's model, absolute price changes, rather than relative returns, were considered; there are however only minor differences between the two at short time scales, $\tau < 1$ month, see, e.g., the detailed discussion of that point in Ref. 4.) In this model, that became the standard of modern finance after the work of Black and Scholes, returns are not only uncorrelated but in fact *independent* and identical Gaussian random variables. However, this model completely fails to capture many other statistical features of financial markets that even a rough analysis of empirical data allows one to identify, at least qualitatively.

(i) The distribution of returns is in fact strongly non-Gaussian and its shape continuously depends on the return period τ: for τ large enough (around few months), one observes quasi-Gaussian distributions while for small τ values, the return distributions have fat tails (see Fig.3). Several studies actually suggest that these distributions can be characterized by Pareto (power-law) tails $|r_\tau|^{-1-\mu}$ with an exponent μ close to 3 even for liquid markets such as the US stock index, major currencies, or interest rates.[6,7,2,8] Emerging markets have even more extreme tails, with an exponent μ that can be less than 2—in which case the volatility is formally infinite.

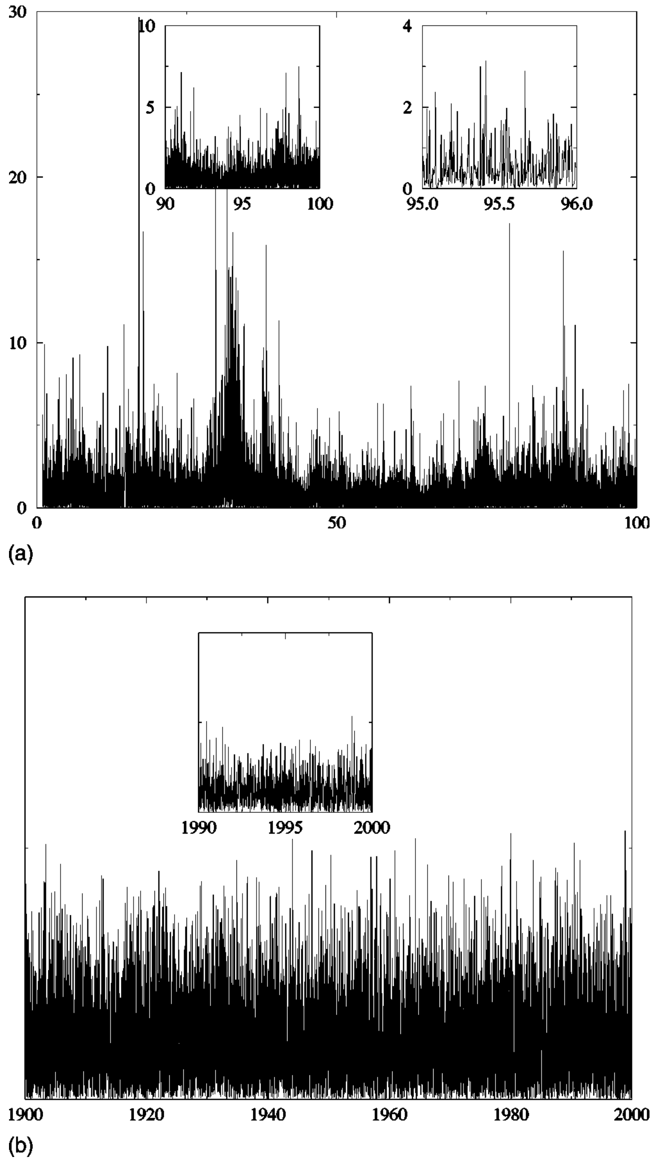

FIG. 2. (a) Absolute value of the daily returns r_τ for the Dow-Jones index over a century (1900–2000), and zoom on different scales (1990–2000, left inset and 1995, right inset). Note that the volatility can remain high for a few years (like in the early 1930s) or for a few days. This volatility clustering can also be observed on high frequency (intraday) data. (b) Same plot for a Brownian random walk, which shows a featureless pattern in this case.

(ii) Another striking feature is the intermittent and correlated nature of return amplitudes. At some given time period τ, a proxy for the local volatility can be defined in various ways, the simplest one being the absolute return $|r_\tau(t)|$. The volatility signal is characterized by self-similar outbursts (see Fig. 2) that are reminiscent of intermittent variations of dissipation rate in fully developed turbulence.[9] The occurrence of such bursts are strongly correlated and high volatility periods tend to persist in time. This feature is known as *volatility clustering*.[10–12,4] This effect can be analyzed more quantitatively, the temporal correlation function of the (e.g., daily) volatility can be fitted by an inverse power $\Delta^{-\nu}$ of the lag Δ, with a rather small exponent ν in the range 0.1–0.3.[11,13–15]

(iii) One observes a nontrivial "multifractal" scaling,[16,17,15,18] in the sense that higher moments of price changes scale anomalously with time,

$$M_q(\tau) = \langle |r_\tau(t) - m\tau|^q \rangle_e \simeq A_q \tau^{\zeta_q}, \tag{2}$$

with $\zeta_q \neq q/2$, as one would have expected for a Brownian random walk. As will be discussed more precisely below, this behavior is intimately related to the intermittent nature of the volatility process.

(iv) Past price changes and future volatilities are negatively correlated—this is the so-called *leverage effect*, which reflects the fact that markets become more active after a price drop, and tend to calm down when the price rises. This correlation is most visible on stock indices.[19] This leverage effect leads to an anomalous negative skew in the distribution of price changes.[4]

The most important message of these empirical studies is that price changes behave very differently from the simple geometric Brownian motion description. Extreme events are much more probable, and interesting nonlinear correlations (volatility–volatility and price–volatility) are observed. These "statistical anomalies" are very important for a reliable estimation of financial risk and for quantitative option pricing and hedging (see, e.g., Ref. 4), for which one often requires an accurate model that captures the statistics of returns on different time horizons τ. It is rather striking that empirical properties (i)–(iv) are, to some extent, also observed on experimental velocity data in fully developed turbulent flows (see Fig. 3). The framework of scaling theory and multifractal analysis, initially proposed to characterize turbulent signals,[9] may therefore be well suited to further characterize statistical properties of price changes on different time periods.[16,15,17,18] In particular, a beautiful multifractal random walk model, that we review next, was constructed by Bacry, Muzy, and Delour[15,20] to account for the statistical anomalies of price changes (see also Refs. 21 and 22 for alternative formulations).

III. THE MULTIFRACTAL RANDOM WALK

Mandelbrot cascades[23] are considered to be the paradigm of multifractal processes and have been extensively used for modeling scale-invariance properties in many fields, in particular statistical finance.[17,18] However, as discussed in Refs. 15 and 20, this class of model has several drawbacks; in particular they violate causality. In that respect, it is difficult to see how such models could arise from a realistic (agent based) description of financial markets.

Recently, Bacry, Muzy, and Delour (BMD)[15,20] introduced a model that does not possess the limitations of Mandelbrot's cascades while capturing their essential ingredient, and generalizes the usual geometric random walk model of prices in a natural way.

Let us first show, from a general point of view, how volatility fluctuations and correlations can induce multiscaling. We will then discuss the BMD model where multifractality is indeed exact. As we mentioned in Sec. II, the em-

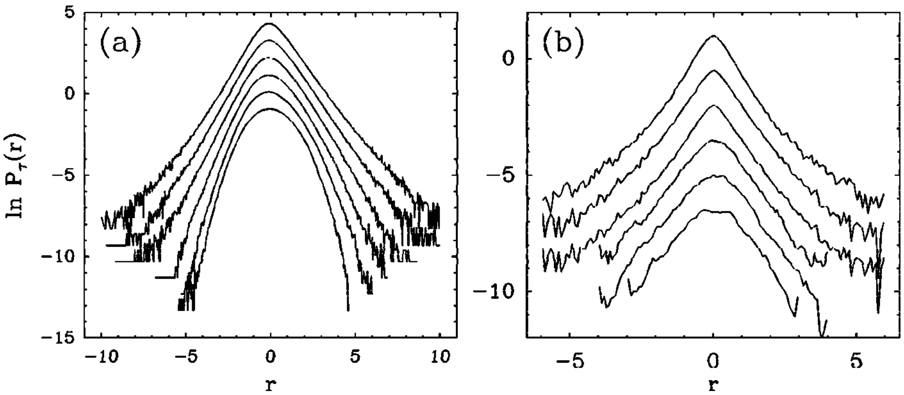

FIG. 3. Continuous deformation of turbulent velocity increments and financial returns distributions from small (top) to large (bottom) scales. (a) Standardized probability distribution functions of spatial velocity increments at different length scales in a high Reynolds number wind tunnel turbulence experiment. The distributions are plotted in logarithmic scale so that a parabola corresponds to a Gaussian distribution. (b) Standardized pdf of S&P 500 index returns at different time scales from few minutes to one month. From Ref. 15.

pirical volatility correlation function decays slowly, as a power law. More precisely, if the correlation function of the *square returns* (which serves as a proxy for the true volatility) decays as a function of the lag Δ as $\Delta^{-\nu}$ with $\nu<1$, it is quite easy to obtain explicitly the fourth moment of the price return for large τ,[4]

$$M_4(\tau) \approx \sigma^4\tau^2(1+A\tau^{-\nu}), \tag{3}$$

where A measures the amplitude of the long range part of the square volatility correlation. The fourth moment of the price difference therefore behaves as the *sum* of two power laws, not as a unique power law as for a multifractal process, for which by definition $M_q(\tau)\sim\tau^{\zeta_q}$ exactly. However, when ν is small and in a restricted range of τ, this sum of two power laws is indistinguishable from a unique power law with an effective exponent $\zeta_{4,\mathrm{eff}}$ somewhere in-between 2 and $2-\nu$; therefore $\zeta_{4,\mathrm{eff}}<2\zeta_2=2$.

In the BMD model, the key ingredient is the volatility correlation shape that mimics that arising in cascade models. Indeed, as remarked in Ref. 24, the treelike structure underlying a Mandelbrot cascade, implies that the volatility logarithm covariance decreases very slowly, as a logarithm function, i.e.,

$$\langle \ln[\sigma_\tau(t)]\ln[\sigma_\tau(t+\Delta)]\rangle - \langle \ln[\sigma_\tau(t)]\rangle^2 = C_0 - \lambda^2\ln(\Delta+\tau). \tag{4}$$

The BMD model involves Eq. (4) within the continuous time limit of a discrete stochastic volatility model. One first discretizes time in units of an elementary time step τ_0 and set $t\equiv i\tau_0$. The volatility σ_i at "time" i is a log-normal random variable such that $\sigma_i=\sigma_0\exp\xi_i$, where the Gaussian process ξ_i has the same covariance as in Eq. (4),

$$\langle\xi_i\rangle = -\lambda^2\ln\left(\frac{T}{\tau_0}\right)\equiv\mu_0,$$

$$\langle\xi_i\xi_j\rangle-\mu_0^2 = \lambda^2\ln\left(\frac{T}{\tau_0}\right)-\lambda^2\ln(|i-j|+1), \tag{5}$$

for $|i-j|\tau_0\leqslant T$. Here T is a large cutoff time scale beyond which the volatility correlation vanishes. In the above equation, the brackets stands for the mathematical expectation. The choice of the mean value μ_0 is such that $\langle\sigma^2\rangle=\sigma_0^2$. As before, the parameter λ^2 measures the intensity of volatility fluctuations (called in the finance jargon the "vol of the vol"), and corresponds to the intermittency parameter.

Now, the price returns are constructed as

$$x[(i+1)\tau_0]-x(i\tau_0)=r_{\tau_0}(i)\equiv\sigma_i\varepsilon_i=\sigma_0e^{\xi_i}\varepsilon_i, \tag{6}$$

where the ε_i are a set of independent, identically distributed random variables of zero mean and variance equal to τ_0. One also assumes that the ε_i and the ξ_i are independent (but see Ref. 25 for a generalization that accounts for the leverage effect). In original BMD model, ε_i's are Gaussian, and the continuous time limit $\tau_0=dt\to0$ is taken. Since $x=\ln p$, where p is the price, the exponential of a sample path of the BMD model is plotted in Fig. 4(a), which can be compared to the real price charts of Fig. 1(a).

The multifractal scaling properties of this model can be computed explicitly. Moreover, using the properties of multivariate Gaussian variables, one can get closed expressions for all even moments $M_q(\tau)$ $(q=2k)$. In the case $q=2$ one trivially finds

$$M_2(\tau=\ell\tau_0)=\sigma_0^2\ell\tau_0\equiv\sigma_0^2\tau, \tag{7}$$

independently of λ^2. For $q\neq2$, one must distinguish between the cases $q\lambda^2<1$ and $q\lambda^2>1$. For $q\lambda^2<1$, the corresponding moments are finite, and one finds, in the scaling region $\tau_0\ll\tau\leqslant T$, a *true* multifractal behavior,[15,20]

$$M_q(\tau)=A_q\tau^{\zeta_q}, \tag{8}$$

where $\zeta_q=q(1/2+\lambda^2)-q^2\lambda^2/2$, and A_q a prefactor that can be exactly calculated. For $q\lambda^2>1$, on the other hand, the moments diverge, suggesting that the unconditional distribution of $x(t+\tau)-x(t)$ has power-law tails with an exponent $\mu=1/\lambda^2$ (possibly multiplied by some slow function, but see Ref. 26). These multifractal scaling properties of BMD processes are numerically checked in Figs. 4(b) and 4(c) where one recovers the same features as for the S&P 500 index. Since volatility correlations are absent for $\tau\gg T$, the scaling becomes that of a standard random walk, for which $\zeta_q=q/2$. The corresponding distribution of returns thus becomes progressively Gaussian. An illustration of the progressive deformation of the distributions as τ increases, in the BMD model is reported in Fig. 4(d). This figure can be directly compared to Fig. 3.

To summarize, the BMD process is attractive for modeling financial time series since it reproduces in a parsimonious way many of the stylized facts reviewed in Sec. II and has a rich mathematical structure, in particular exact multifractal properties. Moreover, this model has stationary incre-

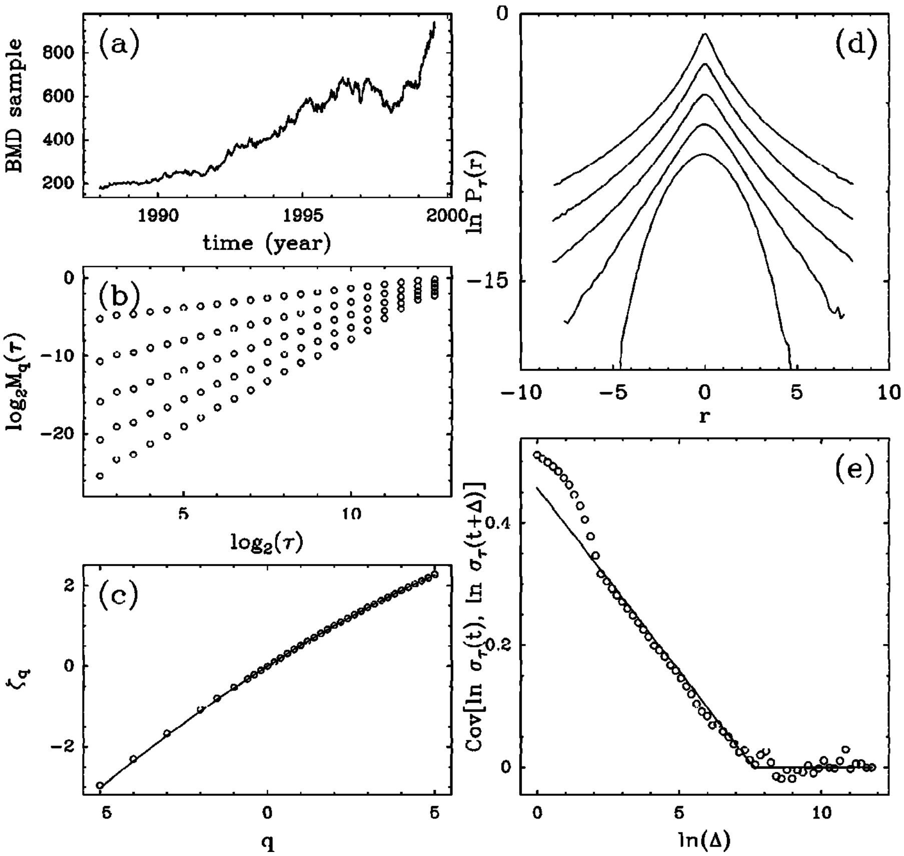

FIG. 4. Multifractal properties of BMD model. (a) Exponential of a BMD process realization, with parameters calibrated on the S&P 500. (b) Multifractal scaling of BMD return moments for $q=1,2,3,4,5$. (c) Estimated ζ_q spectrum (○) compared with the log-normal analytical expression (solid line). (d) Evolution of the return probability distributions across scales, from nearly Gaussian at coarse scale (bottom) to fat tailed law at small scales (top). (e) Log-volatility covariance as a function of the logarithm of the lag Δ. From Ref. 15.

ments, and can be formulated in a purely causal way, the log volatility ξ_i can be expressed as a sum over past random shocks, with a memory kernel that decays as the inverse square root of the time lag.[27] It would be interesting to give a precise economic justification to this causal construction.

Let us end this section with several remarks.

(i) Direct studies of the empirical distribution of the volatility is indeed compatible with the assumption of log-normality, although an inverse gamma distribution also fits the data very well.[28,4]

(ii) One of the predictions of the BMD model is the equality between the intermittency coefficient estimated from the curvature of the ζ_q function and the slope of the log-volatility covariance logarithmic decrease. The direct study of various empirical log-volatility correlation functions show that they can indeed be fitted by a logarithm of the time lag, with a slope that is roughly equal to the corresponding intermittency coefficient λ^2. These empirical studies also suggest that the integral time T is a few years.

(iii) On the other hand, the empirical tail of the distribution of price increments is described by a power law with an exponent μ in the range 3–5,[6,7,2] much smaller than the value $\mu=1/\lambda^2 \sim 10-100$ predicted by the BMD model. However, this issue is extremely subtle, as the model exhibits quasinonergodic properties,[26] and it might well be that the apparent exponent produced by the model is in the correct range.

(iv) One can extend the above symmetric multifractal model to account for a skewed distribution of returns and the return-volatility correlations mentioned in Sec. II (see also Ref. 19). In the BMD model, one has by symmetry that all odd moments of the process vanish. A simple possibility, recently investigated in Ref. 25, is to correlate negatively the variable ξ_i with past values of the variables ε_j, $j<i$, through a kernel that decays as a power law. In this case, the multifractality properties of the model are preserved, but the expression for ζ_q is different for q even and for q odd.

(v) Finally, even if the BMD model can be formulated in a purely causal way, it is in fact statistically invariant under time reversal symmetry (TRS), in stark contrast with Zumbach's mug-shots of empirical data,[29] that reveals significant violations of TRS. The construction of a model displaying multifractal features compatible with the absence of TRS is still a theoretical challenge (see Refs. 29 and 30 for some recent steps in that direction).

IV. THE SUBTLE NATURE OF MARKET EFFICIENCY

We now want to come back to a simpler question, which may shed light to the volatility puzzle discussed above: why are price changes nearly uncorrelated, as postulated by Bachelier? The efficient market hypothesis (EMH) posits that all available information is included in prices, which emerge at all times from the consensus between fully rational agents, that would otherwise immediately arbitrage away any deviation from the fair price.[31,32] Price changes can then only be the result of unanticipated news and are by definition totally unpredictable. However, as pointed out by Shiller, the observed volatility of markets is far too high to be compatible with the idea of fully rational pricing.[33] More fundamentally, the assumption of rational, perfectly informed agents seems intuitively much too strong, and has been criticized by many.[34–36] There is a model at the other extreme of the spectrum where prices also follow a pure random walk, but for a totally different reason.[37–39] Assume that agents, instead of being fully rational, have "zero intelligence" and randomly buy or to sell. Suppose also that their action is interpreted by all the others agents as potentially containing some information. Then, the mere fact of buying (or selling) typically leads to a change of the ask $a(t)$ [or bid $b(t)$] price and hence of a change of the midpoint $m(t)=[a(t)+b(t)]/2$. In the absence of reliable information about the true price, the new midpoint is immediately adopted by all other market participants as the new reference price around which next orders are launched. In this case, the midpoint will also follow a random walk (at least for sufficiently large times), even if trades are not motivated by any rational decision and devoid of meaningful information. Of course, reality should lie somewhere in the middle, clearly, the price cannot wander arbitrarily far from a reasonable fundamental value, and trades cannot all be random. Here, we want to argue, based on a series of detailed empirical results obtained on trade by trade data, that the random walk nature of prices is in fact highly nontrivial and results from a fine-tuned competition between two populations of traders, liquidity providers (or market makers), and liquidity takers. Liquidity providers act such as to create antipersistence (or mean reversion) in price changes that would lead to a subdiffusive behavior of the price, whereas liquidity takers' action leads to long range persistence and superdiffusive behavior. Both effects very precisely compensate and lead to an overall diffusive behavior, at least to a first approximation, such that (statistical) arbitrage opportunities are absent, as expected. We argue that in a very precise sense, the market is operating at a critical point;[40,41] the dynamical compensation of two conflicting tendencies is similar to other complex systems such as the heart,[42] driven by two antagonist systems (sympathetic and parasympathetic), or certain human tasks, such as balancing of a long stick.[43] The latter example illustrates very clearly the idea of dynamical equilibrium, and shows how any small deviation from perfect balance may lead to strong instabilities. This near instability may well be at the origin of the fat tails and volatility clustering observed in financial data mentioned in Sec. II above. Note that these two features are indeed present in the balancing stick time series studied in Ref. 43.

A. The market response function and trade correlations

The last quote before a given trade allows one to define the sign of each trade, if the traded price is above the last midpoint $m=(a+b)/2$, this means that the trade was triggered by a market order to buy, and we will assign to that trade a variable $\varepsilon=+1$. If, one the other hand the traded price is below the last midpoint $m=(a+b)/2$, then $\varepsilon=-1$.

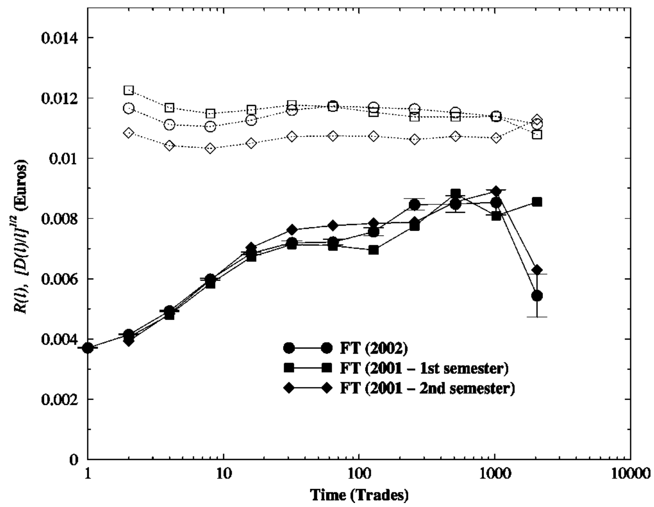

FIG. 5. Average response function $\mathcal{R}(\ell)$ for FT, during three different periods (black symbols). We have given error bars for the 2002 data. For the 2001 data, the y axis has been rescaled to best collapse onto the 2002 data. Using the same rescaling factor, we have also shown the data for $[\mathcal{D}(\ell)/\ell]^{1/2}$ (white symbols, same coding), which shows that (i) the process is indeed nearly diffusive and (ii) $\mathcal{D}\sim\mathcal{R}^2$, indicating a sort of fluctuation-response relation (Ref. 40).

The simplest quantity to study is the average mean square fluctuation of the price between (trade) time n and $n+\ell$,

$$\mathcal{D}(\ell)=\langle(p_{n+\ell}-p_n)^2\rangle. \tag{9}$$

As emphasized above, in the absence of any linear-correlations between successive price changes, $\mathcal{D}(\ell)$ has a strictly diffusive behavior, $\mathcal{D}(\ell)=D\ell$. On liquid stocks one finds a remarkably linear behavior for $\mathcal{D}(\ell)$ (see Fig. 5), even for small ℓ. The absence of linear correlations in price changes is compatible with the idea that (statistical) arbitrage opportunities are absent, even for high frequency trading.

In order to better understand the impact of trading on price changes, one can study the following *response function* $\mathcal{R}(\ell)$, defined as

$$\mathcal{R}(\ell)=\langle(p_{n+\ell}-p_n)\cdot\varepsilon_n\rangle, \tag{10}$$

where ε_n is the sign of the nth trade. The quantity $\mathcal{R}(\ell)$ measures how much, on average, the price moves up conditioned to a buy order at time 0 (or a sell order moves the price down) a time ℓ later. We show in Fig. 5 the temporal structure of $\mathcal{R}(\ell)$ for France Telecom, for different periods. Note that $\mathcal{R}(\ell)$ increases by a factor ~ 2 between $\ell=1$ and $\ell=\ell^*\approx 1000$, before decreasing back. Including overnights allow one to probe larger values of ℓ and confirm that $\mathcal{R}(\ell)$ decreases, and even becomes *negative* beyond $\ell\simeq 5000$. Similar results have been obtained for many different stocks as well. However, in some cases the maximum is not observed and rather $\mathcal{R}(\ell)$ keeps increasing mildly.[40,41] The model discussed below does in fact allow for monotonous response functions.

All the above results are compatible with a zero intelligence picture of financial markets, where each trade is random in sign and shifts the price permanently, because all other participants update their evaluation of the stock price as a function of the last trade. This model of a totally random model of stock market is however qualitatively incorrect for the following reason. Although, as mentioned above, the statistics of price changes reveals very little temporal correlations, the correlation function of the sign ε_n of the trades, surprisingly, reveals very slowly decaying correlations as a function of trade time, as discovered in Refs. 40, 44, and 41. More precisely, one can consider the following correlation function:

$$\mathcal{C}_0(\ell) = \langle \varepsilon_{n+\ell}\varepsilon_n \rangle - \langle \varepsilon_n \rangle^2. \tag{11}$$

If trades were random, one should observe that $\mathcal{C}_0(\ell)$ decays to zero beyond a few trades. Surprisingly, this is not what happens, on the contrary, $\mathcal{C}_0(\ell)$ is strong and decays very slowly toward zero, as an inverse power-law of ℓ (see Refs. 40, 44, and 41),

$$\mathcal{C}_0(\ell) \simeq \frac{C_0}{\ell^\gamma} \quad (\ell \geqslant 1). \tag{12}$$

The value of γ seems to be somewhat stock dependent, but is consistently found to be smaller than unity, leading to a nonintegrable correlation function. This in general leads to superdiffusion, and is the main puzzle to elucidate: how can one reconcile the strong, slowly decaying correlations in the sign of the trades with the nearly diffusive nature of the price fluctuations, and the nearly structureless response function?

B. A micromodel of price fluctuations

In order to understand the above results, we will postulate the following trade superposition model, where the price at time n is written as a sum over all past trades, of the impact of one given trade propagated up to time n,

$$p_n = \sum_{n'<n} G_0(n-n')\varepsilon_{n'} f(V_{n'}) + \sum_{n'<n} \eta_{n'}, \tag{13}$$

where $V_{n'}$ is the volume of the n'th trade, f a certain concave function,[45–47,40] and $G_0(.)$ is the bare impact function (or propagator) of a single trade. The η_n are also random variables, assumed to be independent from the ε_n and model all sources of price changes not described by the direct impact of the trades, the bid-ask can change as the result of some news, or of some order flow, in the absence of any trades.

The bare impact function $G_0(\ell)$ represents by definition the average impact of a single trade after ℓ trades. In order to understand the temporal structure of $G_0(\ell)$, note that a single trade first impacts the midpoint by changing the bid (or the ask). But then the subsequent limit order flow due to that particular trade might either center on average around the new midpoint [in which case $G_0(\ell)$ would be constant], or, as we will argue below, tend to mean revert toward the previous midpoint [in which case $G_0(\ell)$ decays with ℓ]. If the signs ε_n were independent random variables, both the response function and the diffusion would be very easy to compute. For example, one would have

$$\mathcal{R}(\ell) = \langle f(V) \rangle G_0(\ell), \tag{14}$$

i.e., the observed impact function and the bare response function would be proportional. This case (no correlations between the ε's and a constant bare impact function) corresponds to the simplest possible zero intelligence market, where agents are memoryless, and the price is obviously a random walk. However, we have seen that in fact the ε's have long range correlations. In this case, the average response function reads

$$\mathcal{R}(\ell) = \langle \ln V \rangle G_0(\ell) + \sum_{0<n<\ell} G_0(\ell-n)\mathcal{C}_1(n) + \sum_{n>0} [G_0(\ell+n) - G_0(n)]\mathcal{C}_1(n), \tag{15}$$

where

$$\mathcal{C}_1(\ell) = \langle \varepsilon_{n+\ell}\varepsilon_n f(V_n) \rangle_c. \tag{16}$$

If the impact G_0 is constant and $\mathcal{C}_1(n)$ decays as a power law with an exponent $\gamma<1$, then the average impact $\mathcal{R}(\ell)$ should *grow* like $\ell^{1-\gamma}$, and therefore be amplified by a very large factor as ℓ increases, at variance with empirical data. Similarly, diffusion should be anomalously enhanced, $\mathcal{D}(\ell) \sim \ell^{2-\gamma}$, instead of Bachelier's first law $\mathcal{D}(\ell) \sim \ell$. The only way to resolve this paradox is to assume that $G_0(\ell)$ itself should decay with time, in such a way to offset the amplification effect due to the trade correlations. If we make the ansatz that the bare impact function $G_0(\ell)$ also decays as a power law for large ℓ, as $\ell^{-\beta}$, then one can estimate $\mathcal{D}(\ell)$ and $\mathcal{R}$ in the large ℓ limit. When $\gamma<1$, one finds $\mathcal{D} \sim \ell^{2-2\beta-\gamma}$, provided $\beta<1$. Therefore, the condition that the fluctuations are diffusive at long times imposes a relation between the decay of the sign autocorrelation γ and the decay of the bare impact function β: $\beta=\beta_c=(1-\gamma)/2$. For $\beta>\beta_c$, the price is *subdiffusive*, which means that price changes show antipersistence; while for $\beta<\beta_c$, the price is *superdiffusive*, i.e., price changes are persistent.

For the response function, one finds for large ℓ,

$$\mathcal{R}(\ell) \propto \frac{\Gamma(1-\gamma)}{\Gamma(\beta)\Gamma(2-\beta-\gamma)} \times \left(\frac{\pi}{\sin \pi\beta} - \frac{\pi}{\sin \pi(1-\beta-\gamma)} \right) \ell^{1-\beta-\gamma}. \tag{17}$$

Therefore, only when $\beta=\beta_c$, is the prefactor exactly zero, and leads to the possibility of a nearly constant impact function. Since the dominant term is zero for the critical case $\beta=\beta_c$, and since we are interested in the whole function $\mathcal{R}(\ell)$ (including the small ℓ regime), one can compute $\mathcal{R}(\ell)$ numerically, by performing the discrete sum Eq. (15) exactly, and fit to the empirical response $\mathcal{R}$. The value of β is a fitting parameters, and we show in Fig. 6 the response function computed for different values of β in the vicinity of $\beta_c=0.38$. The results are compared with the empirical data for FT, showing that one can indeed satisfactorily reproduce, when $\beta \approx \beta_c$, a weakly increasing impact function that reaches a maximum and then decays. One also sees, from Fig. 6, that the relation between β and γ must be quite accurately satisfied, otherwise the response function shows a distinct upward trend (for $\beta<\beta_c$) or a downward trend ($\beta>\beta_c$). Both shapes are actually observed on other stocks, see Ref. 41.

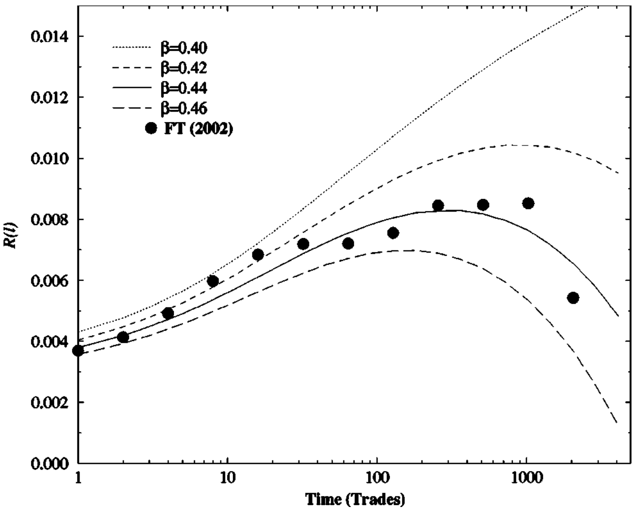

FIG. 6. Theoretical impact function $\mathcal{R}(\ell)$, from Eq. (15), and for different values of β close to $\beta_c=0.38$. The shape of the empirical response function can be quite accurately reproduced using $\beta=0.42$. Note that for $\beta>\beta_c$, the response function actually becomes *negative* at long times, as indeed observed empirically for $\ell>5000$.

C. Discussion: Critical balance of market orders vs limit orders

Although trading occurs for a large variety of reasons, it is useful to recognize that traders organize in two broad categories:

(i) One is that of liquidity takers, that trigger trades by putting in market orders. The motivation for this category of traders might be to take advantage of some information, and make a profit from correctly anticipating future price changes. Information can in fact be of very different nature, fundamental (firm based), macroeconomical, political, statistical (based on regularities of price patterns), etc. Unfortunately, information is often hard to interpret correctly—except of course for insiders—and it is probable that many of these information driven trades are misguided (on this point, see Refs. 48 and 49 and references therein). For example, systematic hedge funds which take decisions based on statistical pattern recognition have a typical success rate of only 52%. There is no compelling reason to believe that the intuition of traders in markets room fares much better than that. Since market orders are immediately executed, many impatient investors, who want to liquidate their position, or hedge, etc., might be tempted to place market orders, even at the expense of the bid-ask spread $s(t)=a(t)-b(t)$.

(ii) The other category is that of liquidity providers (or market makers, although on electronic markets all participants can act as liquidity providers by putting in limit orders), who offer to buy or to sell but avoid taking any bare position on the market. Their profit comes from the bid-ask spread s: the sell price is always slightly larger than the buy price, so that each round turn operation leads to a profit equal to the spread s, at least if the midpoint has not changed in the mean time (see below).

This is where the game becomes interesting. Assume that a liquidity taker wants to buy, so that an increased number of buy orders arrive on the market. The liquidity providers are tempted to increase the offer (or ask) price a because the buyer might be informed and really know that the current price is too low and that it will most probably increase in the near future. Should this happen, the liquidity provider, who has to close his position later, might have to buy back at a much higher price and experience a loss. In order not to trigger a sudden increase of a that would make their trade costly, liquidity takers obviously need to put on not too large orders. This is the rationale for dividing one's order in small chunks and disperse these as much as possible over time so as not to reveal their intentions. Doing so liquidity takers *necessarily create some temporal correlations* in the sign of the trades. Since these traders probably have a somewhat broad spectrum of volumes to trade,[50] and therefore of trading horizons (from a few minutes to several weeks), this can easily explain the slow, power-law decay of the sign correlation function $\mathcal{C}_0(\ell)$ reported above.

Now, if the market orders in fact do *not* contain useful information but are the result of hedging, noise trading, misguided interpretations, errors, etc., then the price should not move up on the long run, and should eventually mean revert to its previous value. Liquidity providers are obviously the active force behind this mean reversion, again because closing their position will be costly if the price has moved up too far from the initial price. However, this mean reversion cannot take place too quickly, again because a really informed trader would then be able to buy a large volume at a modest price. Hence, this mean reversion must be slow enough.

These are the basic ingredients ruling the game between liquidity providers and liquidity takers. The subtle balance between the positive correlation in the trades (measured by γ) and the liquidity molasses induced by liquidity providers (measured by β) is a self-organized dynamical equilibrium, if the liquidity providers are too slow to revert the price $[\beta<(1-\gamma)/2]$, then the price is superdiffusive and liquidity providers lose money. If they are too fast $[\beta<(1-\gamma)/2]$, the residual anticorrelations can be used by liquidity takers to buy larger quantities of stocks at a low price in a given time interval, which is an incentive to speed up the trading and increase γ. A dynamical equilibrium where $\beta\approx(1-\gamma)/2$ therefore establishes itself spontaneously, with clear economic forces driving the system back towards this equilibrium see Fig. 7). Interestingly, fluctuations around this equilibrium leads to fluctuations of the local volatility, since persistent patches correspond to high local volatility and anti-persistent patches to low local volatility. The extreme crash situations are well known to be liquidity crisis, where the liquidity molasses effect disappears temporarily, destabilizing the market (on that point, see the detailed recent study of Refs. 51 and 52).

To summarize, liquidity takers must dilute their orders and create long range correlations in the trade signs, whereas liquidity providers must correctly handle the fact that liquidity takers might either possess useful information (a rare situation, but that can be very costly since the price can jump as a result of some significant news), or might not be informed

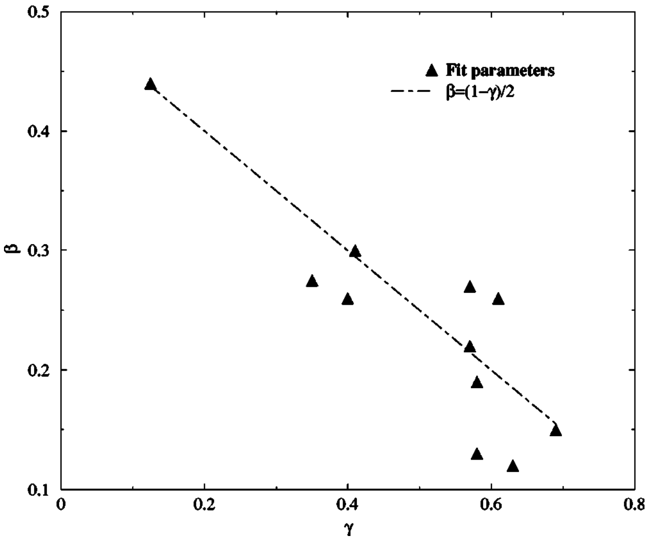

FIG. 7. Scatter plot of the exponents β, γ extracted from the fit of G_0 and $\mathcal{C}_\infty$, see Ref. 41 for details. These exponents are seen to lie in the vicinity of the critical line $\beta = (1-\gamma)/2$ (dotted line), as expected from the nearly diffusive behavior of prices.

at all and trade randomly. By controlling the order flow such as to slowly mean reverting the price, liquidity providers minimize the probability that they either sell too low, or have to buy back too high. The delicate balance between these conflicting tendencies conspire to put the market at the border between persistence (if mean reversion is too weak, i.e., $\beta < \beta_c$) or antipersistence (if mean reversion is too strong, i.e., $\beta > \beta_c$), and eliminate arbitrage opportunities. Therefore, the mere fact of trading such as to minimize impact for liquidity takers, and to optimize gains for liquidity providers, does lead to a random walk dynamics of the price, even in the absence of any real information.

V. CONCLUSION

The delicate competition between liquidity takers and liquidity providers is at the heart of Bachelier's first law, i.e., that price changes are nearly uncorrelated. The resulting absence of linear correlations in price changes, and therefore of arbitrage opportunities is often postulated *a priori* in the economics literature, but the details of the mechanism that removes these arbitrage opportunities are left rather obscure. The main message of our work is that the random walk nature of price changes on short time scales may not be due to the unpredictable nature of incoming news, but appears as a dynamical consequence of the competition between antagonist market forces. In fact, the role of *real* (and correctly interpreted) information appears to be rather thin, the fact that the intraday volatility of a stock is nearly equal to its long time value suggests that the volatility is mostly due to the trading activity itself, which is dominated by noise trades. This result is most probably one of the mechanism needed to explain the excess volatility puzzle first raised by Shiller,[33] and the anomalous, long ranged dynamics of the volatility leading to the multifractal properties reviewed in Secs. II and III.

The conclusion that price changes are to a large extent induced by the trading activity itself seems to imply that the price random walk will, on the long run, wander arbitrarily far from the fundamental price, which would be absurd. But even if one assumes that the fundamental price is independent of time, a typical daily 3% noise trading volatility would lead to a significant (say a factor 2) difference between the traded price and the fundamental price only after a few years.[53] Since the fundamental price of a company is probably difficult to determine better than within a factor 2, say (see, e.g., Refs. 35 and 54), one only expects fundamental effects to limit the volatility on very long time scales as indeed suggested by the empirical results of de Bondt and Thaler,[5] but that these are probably negligible on the short (intraday) time scales of interest in most statistical analysis of financial markets.

From a more general standpoint, the finding that the absence of arbitrage opportunities results from a critical balance between antagonist effects is quite interesting. It might justify several claims made in the (econo-)physics literature that the anomalies in price statistics (fat tails in returns described by power laws, long range self-similar volatility correlations, the long ranged correlations in signs reported here, and many others) are due to the presence of a critical point in the vicinity of which the market operates (see, e.g., Ref. 55, and in the context of financial markets.[56,57]). If a fine-tuned balance between two competing effects is needed to ensure absence of arbitrage opportunities, one should expect that fluctuations are crucial, since a local unbalance between the competing forces can lead to an instability. In this respect, the analogy with the balancing of a long stick is quite enticing.[43] In more financial terms, the breakdown of the conditions for this dynamical equilibrium is, for example, a *liquidity crisis*, a sudden cooperativity of market orders, that lead to an increase of the trade sign correlation function, can out-weight the liquidity providers stabilizing (mean-reverting) role, and lead to crashes. This suggests that one should be able to write a mathematical model, inspired by our results, to describe this on–off intermittency scenario, advocated (although in a different context) in Refs. 43, 58, and 59.

ACKNOWLEDGMENTS

The author thanks L. Borland, J. D. Farmer, Y. Gefen, J. Kolkelkoren, Y. Lemperiere, T. Lux, J. F. Muzy, B. Pochart, M. Potters, M. Wyart, and G. Zumbach for many discussions on the content of this paper.

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