Generalized Preferential Attachment Model (GPMG)

The goal of the model is to explain stylized facts

(I). The size distribution of firms is highly skewed;

(II). The growth rate distribution is not Gaussian but "tent-shaped" in the vicinity of the mean growth rate;

(III). Smaller firms have a lower probability of survival, but those that survive tend to grow faster than larger firms;

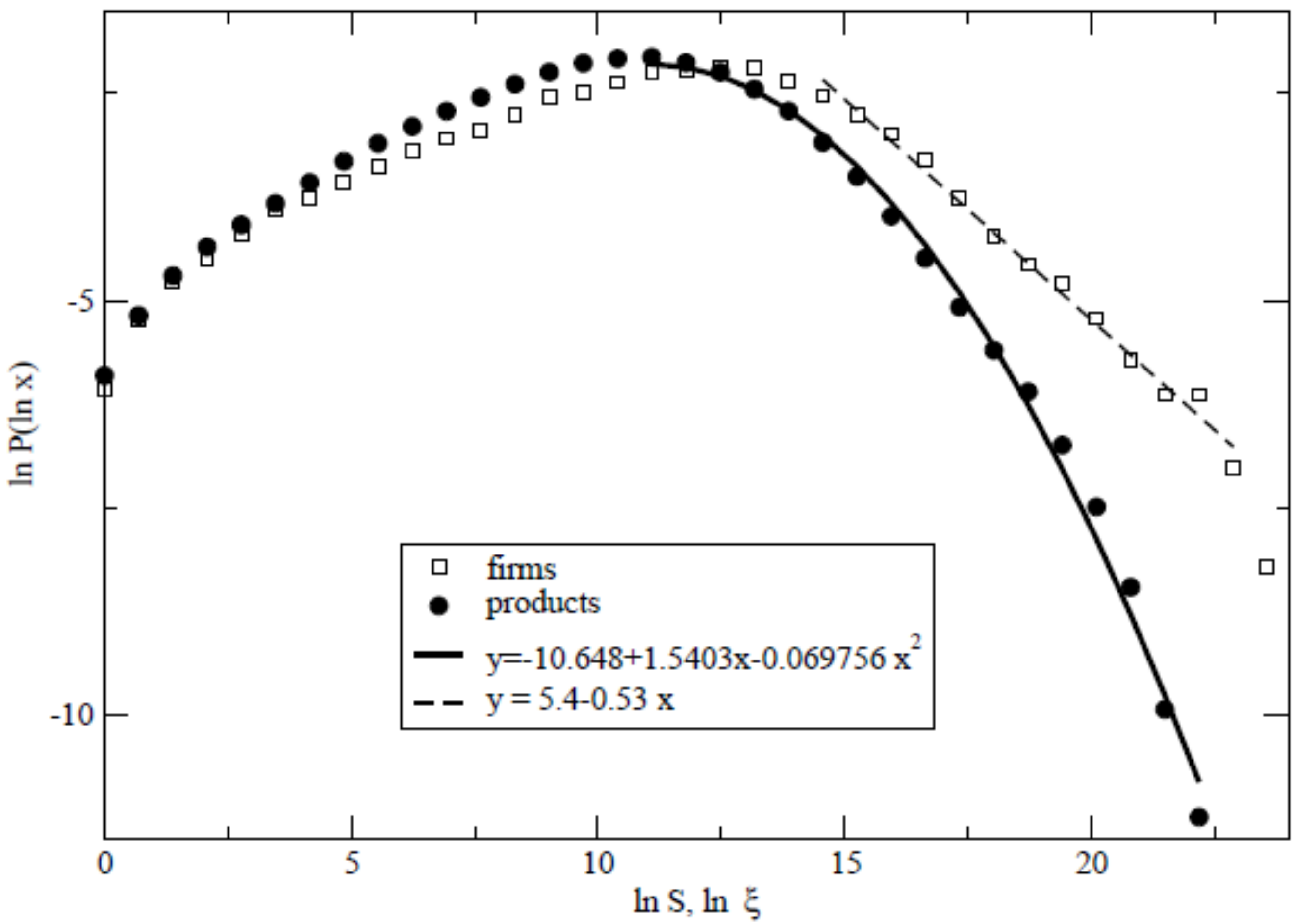
(IV). The variance of growth rates is systematically higher for smaller firms

Assumption of the model

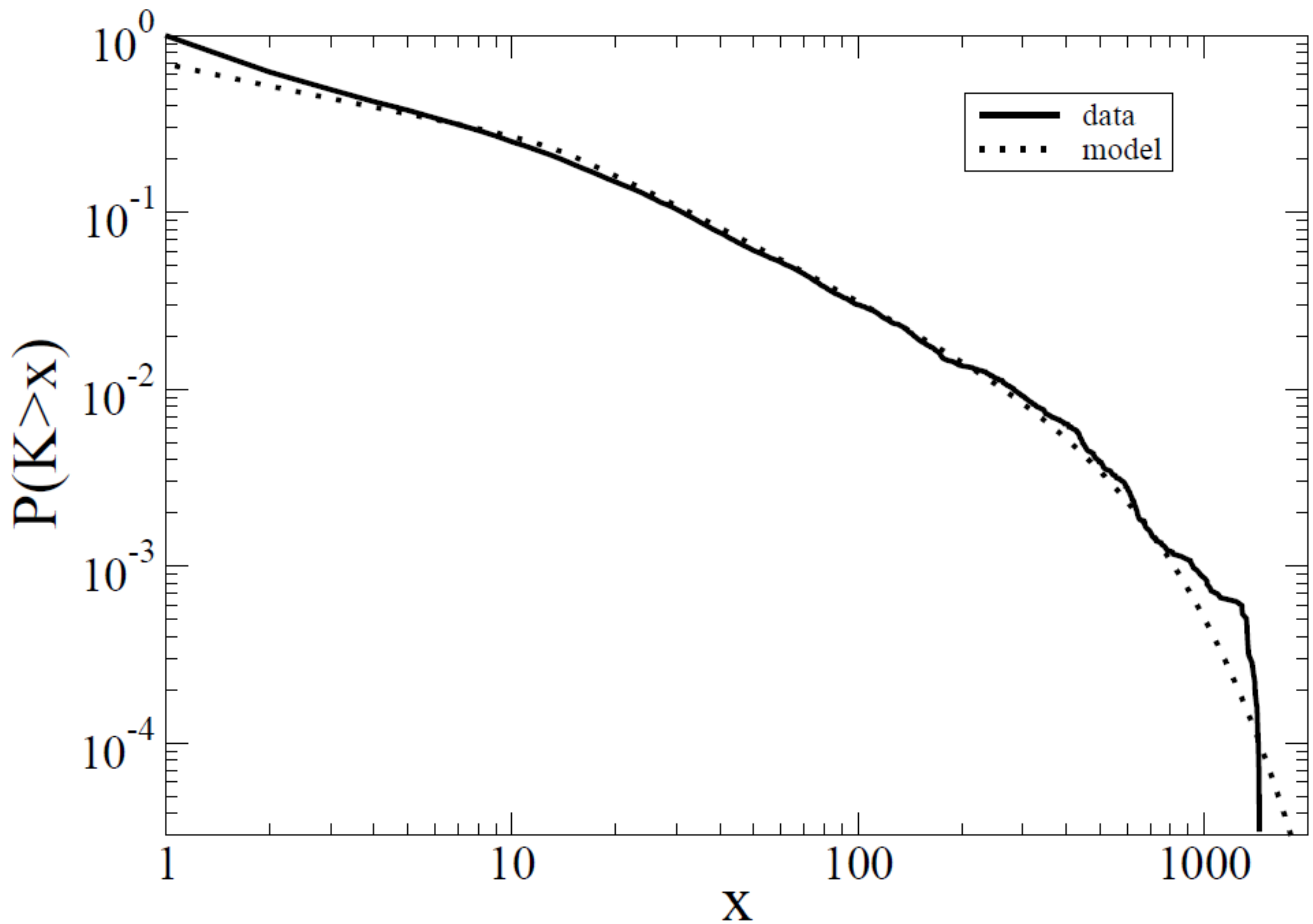
— the number of constituent elementary units in a firm grows in proportion to the number of preexisting units (proportional growth in the number of elementary units);

— The size of each unit grows in proportion to its size, independently of other units (proportional growth in size).

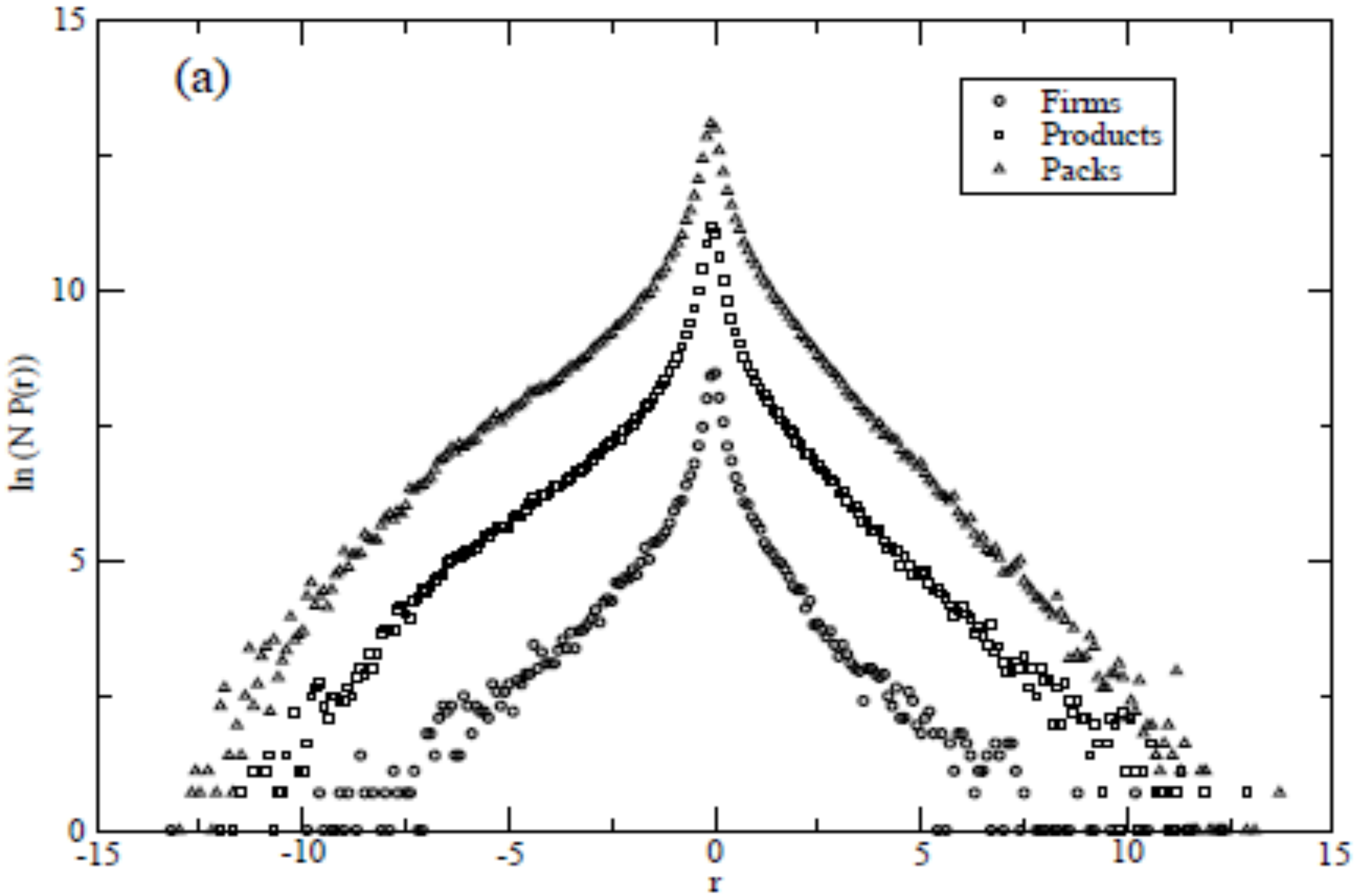
(1) Sales distribution is skewed world wide pharmaceutical database



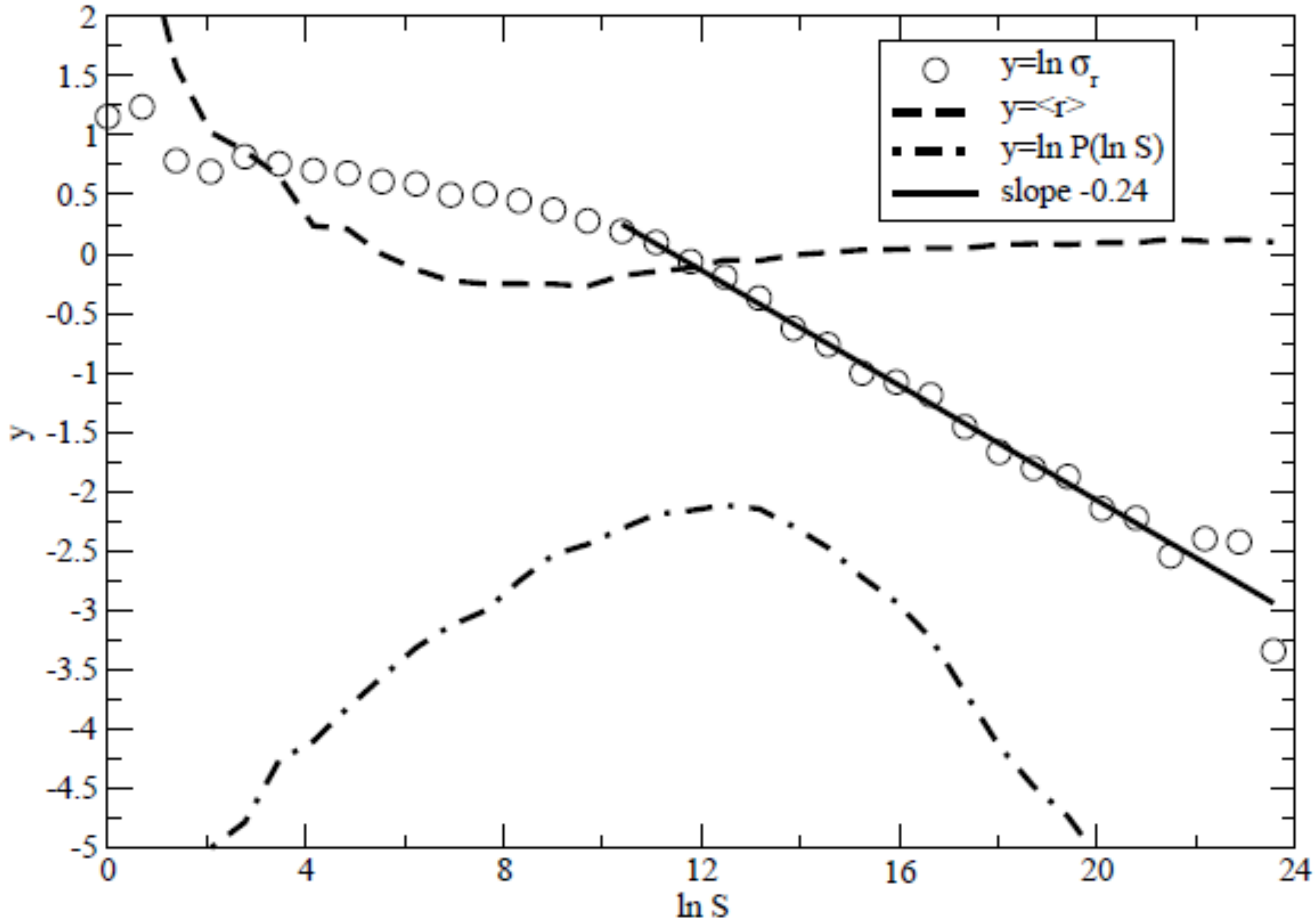
Number of products in the pharm firms



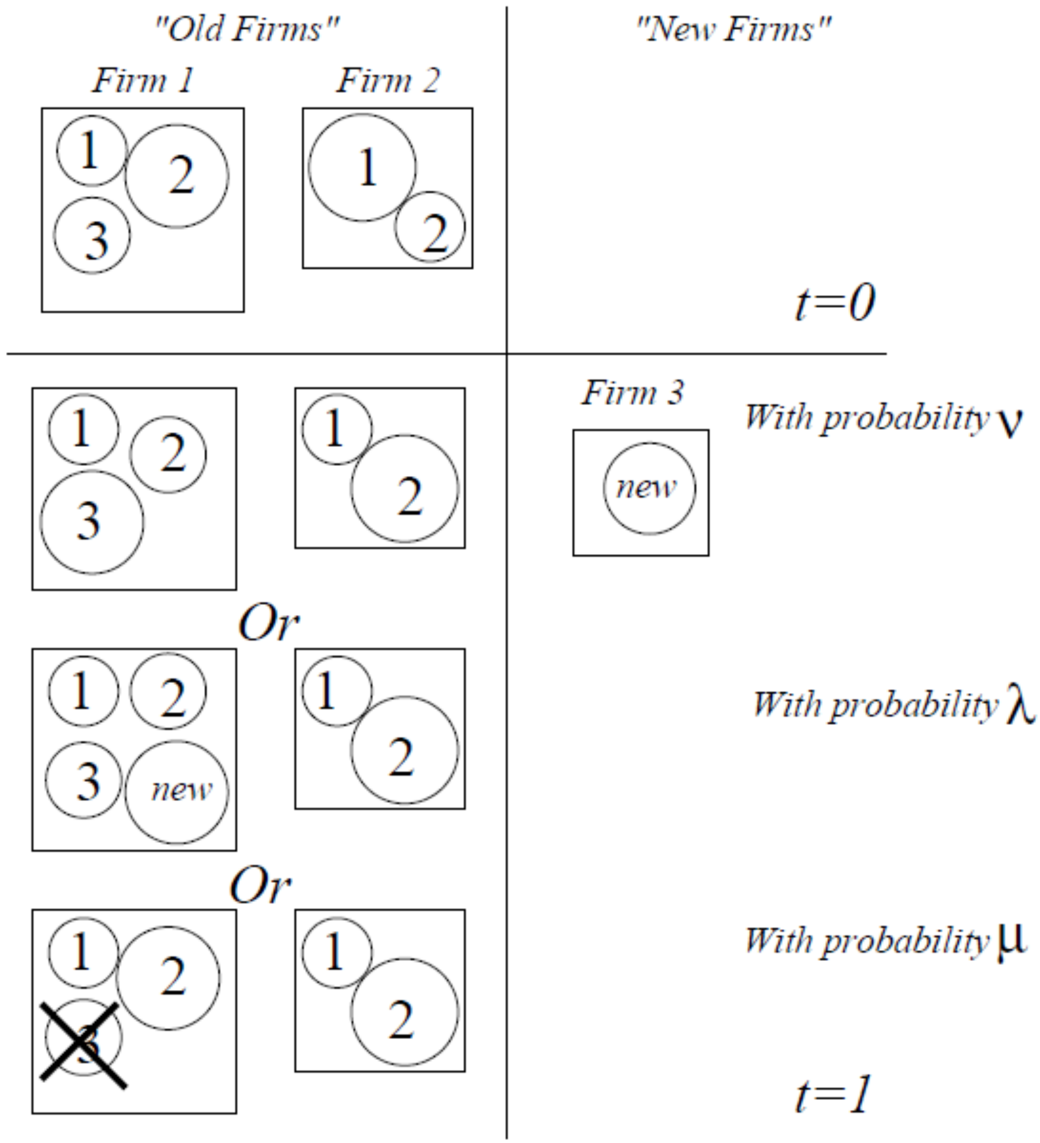
(2) Growth rate distribution



(3,4) average growth rate and its std.



GPMG=Bose-Einstein + Simon + Gibrat



First set of Assumptions

(1). At time t, the system consists of $N(t)$ firms. Each firm i consists of $K_i(t)$ units, while $N_k(t)$ indicates the number of firms with exactly k units. By definition,

$$N(t) = \sum_{k=0}^{\infty} N_k(t). \tag{3.1}$$

The total number of elementary units in the system $n(t)$ is

$$n(t) = \sum_{k=0}^{\infty} kN_k(t) \equiv \langle K(t) \rangle N(t), \tag{3.2}$$

where $\langle K(t) \rangle$ is the average number of units per firm. We assume that at time $t = 0$ there are $N_k(0)$ firms consisting of k units. We denote the initial number of firms and units as $N(0) \equiv N_0$ and $n(0) \equiv n_0$, respectively. Accordingly, we introduce

$$\langle k \rangle = n_0/N_0 = \langle K(0) \rangle, \tag{3.3}$$

which indicates the initial average number of units per firm at time $t = 0$ We define the initial of firm size distribution, measured in terms of number of elementary units, as $P_k^o = N_k(0)/N_0$ [2].

(2-3) Bose-Einstein ; (4) Simon

(2). At each time interval Δt, a number of new units $\Delta_\lambda n$ is created in proportion to the total number of elementary units: $\Delta_\lambda n = \lambda n(t)\Delta t$, where λ is the growth rate. These units are distributed among existing firms with probability p_i, proportional to the number of units detained by a firm i: $p_i = K_i(t)/n(t)$.

(3). At each time step, one unit can be deleted, with probability μ. As a consequence, the number of units deleted during Δt is $\Delta_\mu n = \mu n(t)\Delta t$. The probability that a deleted unit belongs to the firm i is (proportional to the number of its units) $p_i = K_i(t)/n(t)$.

(4). At each interval Δt, a number of new firms $\Delta_\nu N = \nu' n(t)\Delta t$ is created, where ν' indicates the new firms birth rate. We assume that there is a probability P'_k that a new firm has k units. Thus, for each time interval, the total number of units added by the entry of new firms is $\Delta_\nu n = \nu n(t)\Delta t$, where

$$\nu \equiv \nu' \sum_k P'_k k = \nu' \langle k \rangle' \tag{3.4}$$

and $\langle k \rangle'$ is the average number of units in new firms.

Second set of Assumptions (Gibrat Law)

(5). At time t, each firm i in made by $K_i(t)$ units of size $\xi_j(t)$, $j = 1, 2, ... K_i(t)$ where $\xi_j > 0$ are independent random variables extracted from the distribution P_ξ. We assume that $\mathrm{E}[\ln \xi_i(t)] \equiv m_\xi$ and $\mathrm{Var}[\ln \xi_i(t)] = \mathrm{E}[(\ln \xi_i)^2] - m_\xi^2 \equiv V_\xi$, where $\mathrm{E}[x]$ and $\mathrm{Var}[x]$ are respectively the mathematical expectation and the variance of a random variable x. Accordingly, the size of a firm i is denoted by $S_i(t) \equiv \sum_{j=1}^{K_i(t)} \xi_j(t)$.

(6). For each time interval Δt, the size of each unit j is decreased or increased by a random factor $\eta_j(t) > 0$, so that

$$\xi_j(t + \Delta t) = \xi_j(t)\ \eta_j(t)$$

We assume that $\eta_j(t)$, the growth factor of unit j, is a random variable taken from a given probability distribution P_η. We assumed that $\mathrm{E}[\ln \eta_i(t)] \equiv m_\eta$, while $\mathrm{Var}[\ln \eta_i(t)] = \mathrm{E}[(\ln \eta_i)^2] - m_\eta^2 \equiv V_\eta$; η_j is independent of ξ_j, K_i and all other random variables which characterized the firm i.

(7). The size of each new unit arriving at time t is drawn randomly from the distribution of unit sizes P_ξ (cfr. Assumption 5).

Pure Gibrat Model

$$P(S) = \frac{1}{S\sqrt{2\pi V_\eta t/\Delta t}} \mathrm{e}^{-\frac{(\ln(S/S_0) - m_\eta t/\Delta t)^2}{2V_\eta t/\Delta t}},$$

Stilized Facts (2-4) are violated:
(2) The shape of the growth rate is parabolic
(3,4) The Growth rate and variance are independent of firm size.
Moreover, variance grows linearly with time

Pure Bose-Einstein: no new firms

Size distribution is Geometric

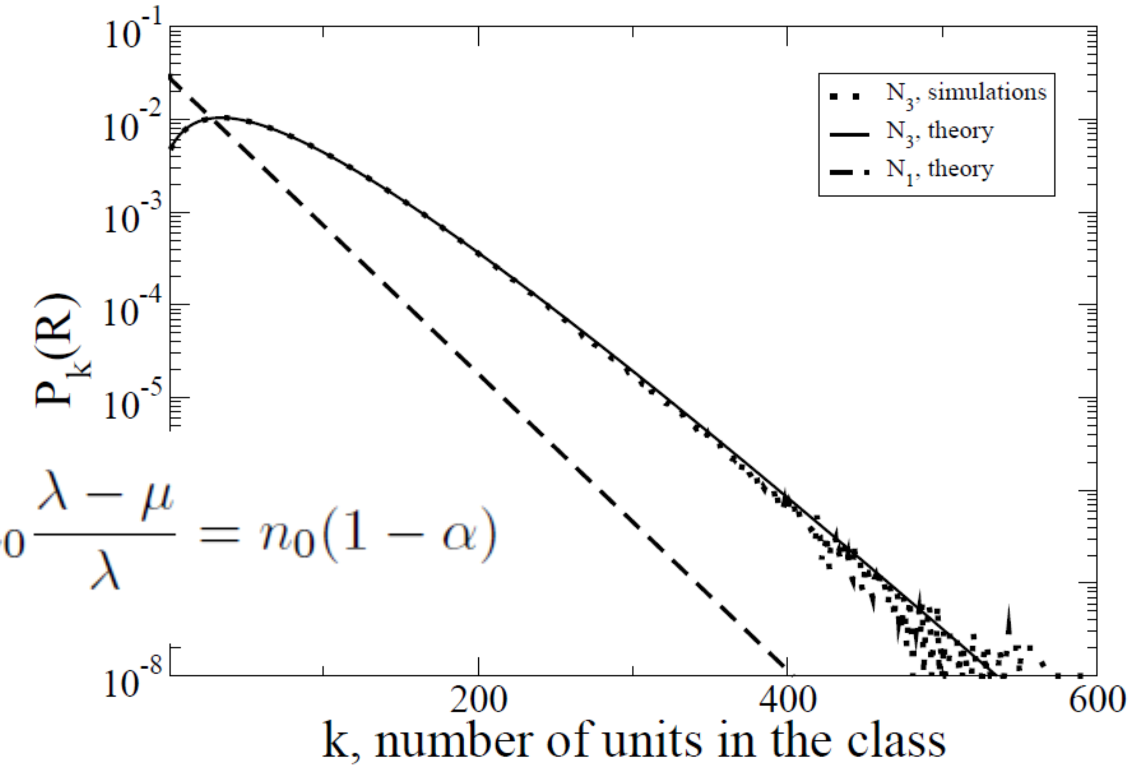
$$P_K = \frac{1}{\kappa(t)-1}\left(1-\frac{1}{\kappa(t)}\right)^K \qquad P(S) = \frac{e^{-S/\langle S\rangle}}{\langle S\rangle}$$

Innovation parameter

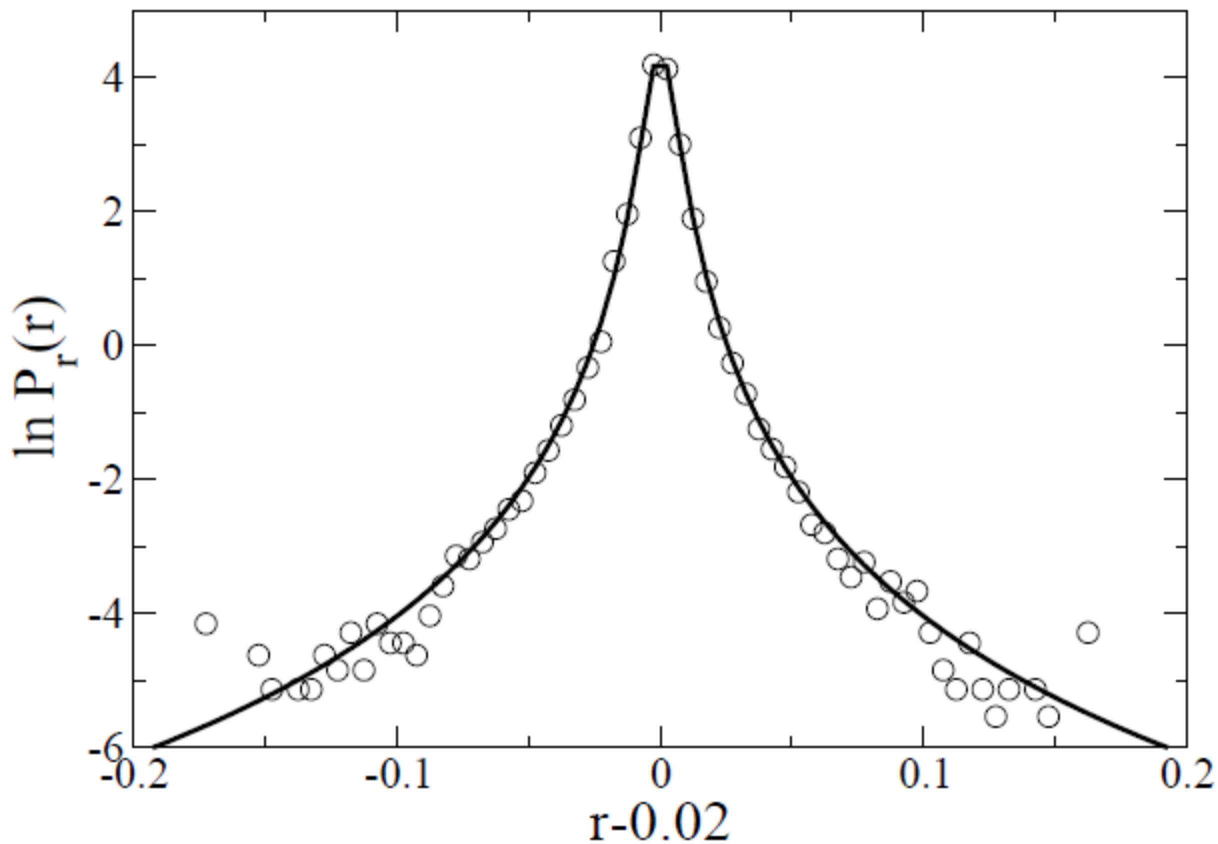
$$\kappa(t) = \frac{n_\lambda(t)+n_0}{n_0}.$$

Number of active firms

$$N_a(t) = n_0\frac{n(t)}{n(t)+n_\mu(t)} \to n_0\frac{\lambda-\mu}{\lambda} = n_0(1-\alpha)$$

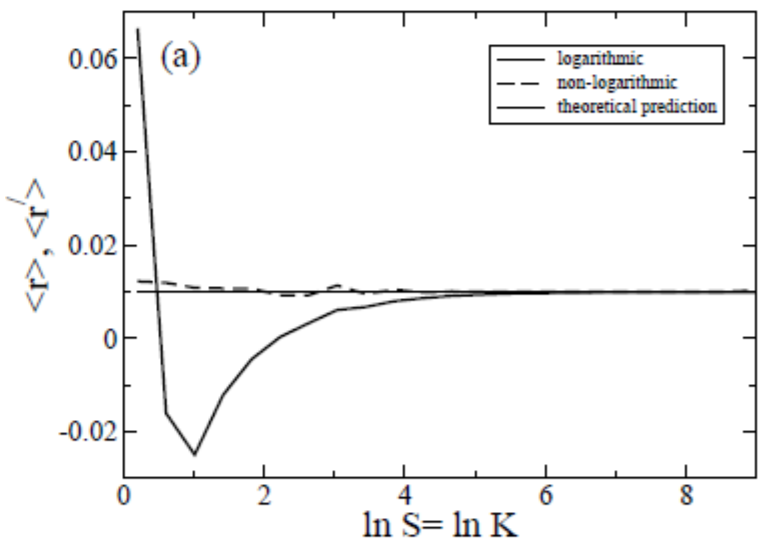


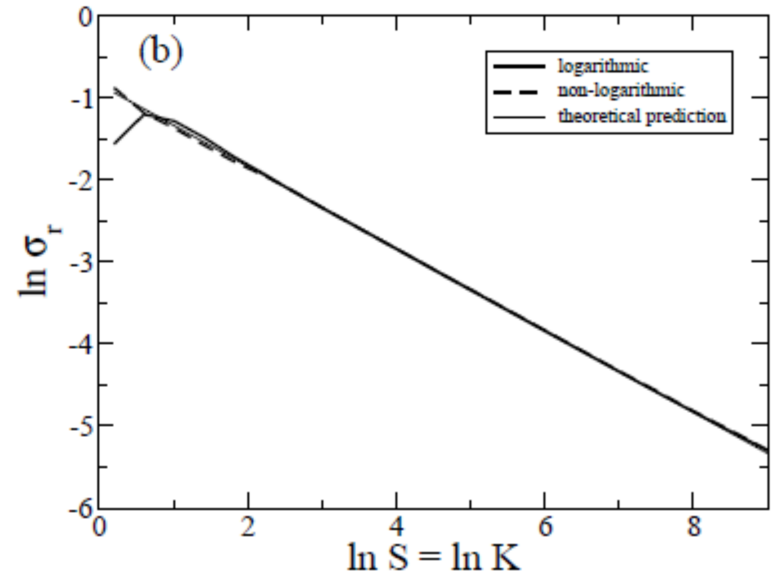
Growth rate is tent shape



$$P_r(r) = \frac{\sqrt{\kappa(t)}}{2\sqrt{2\,V_r}} \left(1 + \frac{\kappa(t)}{2V_r}(r - m_r)^2\right)^{-\frac{3}{2}},$$

Stylized Facts 3-4 are violated





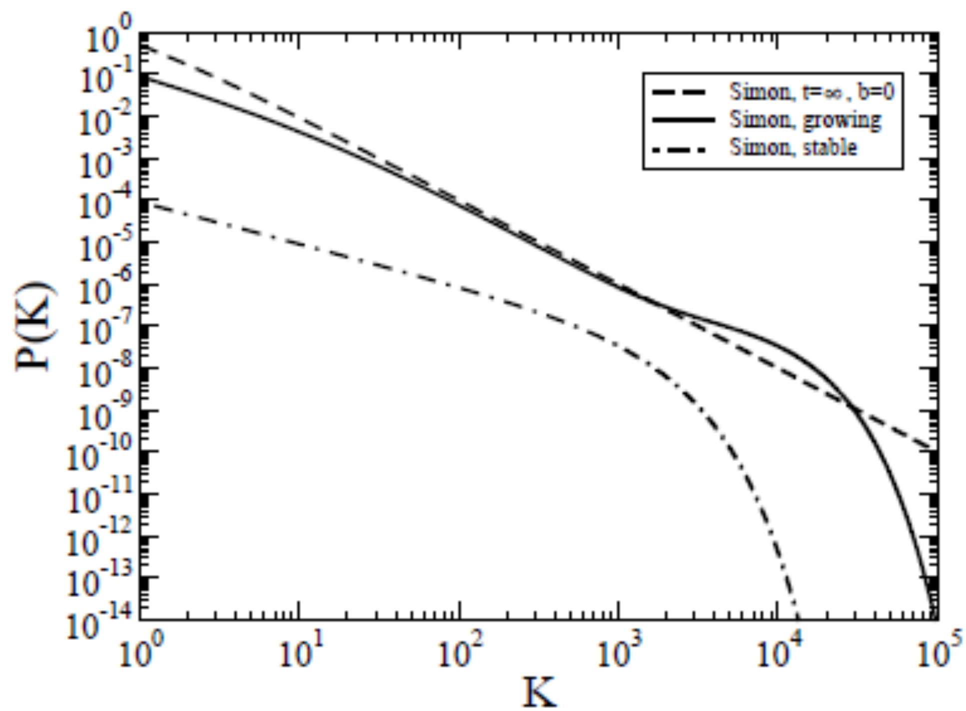
$$\sigma_r = \sqrt{(\lambda + \mu)/S} \quad \beta = 1/2$$

In reality [illegible]=0.24

Simon Model: new firms can be created

$$P_K = \frac{1}{K^{2+b}}[C + o(1)] \qquad b = \frac{\nu}{\lambda - \mu} \qquad t \to \infty$$

$$P(S) = \frac{1}{S^{2+b}}[C + o(1)]$$ Finite time



Finite time Behavior

$$P_k(t) = P_k^o(t)\frac{N_0}{N(t)} + \frac{1}{N(t)}\int_0^t dN(t_0)P_k'(t,t_0)$$

Growth Factor of the Economy

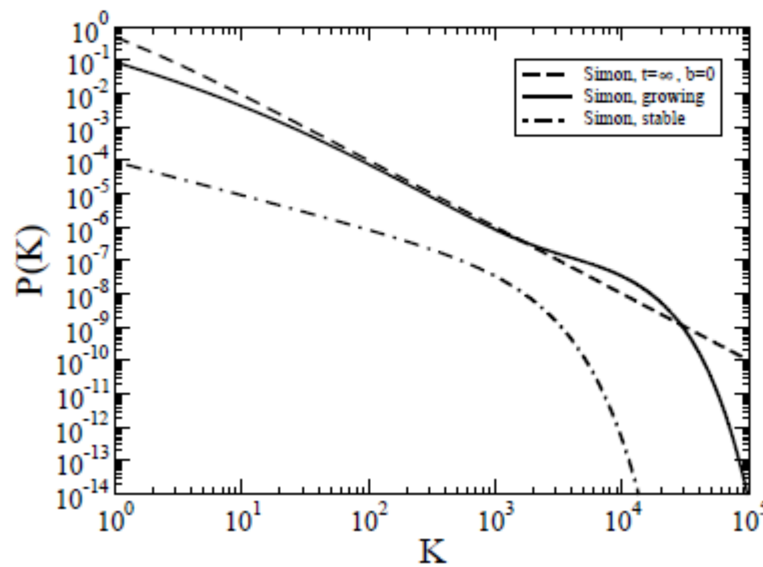
$$R(t) = \left(\frac{n(t)}{n_0}\right)^{1/(1+b)} = \mathrm{e}^{t(\lambda-\mu)}$$

Old Firms

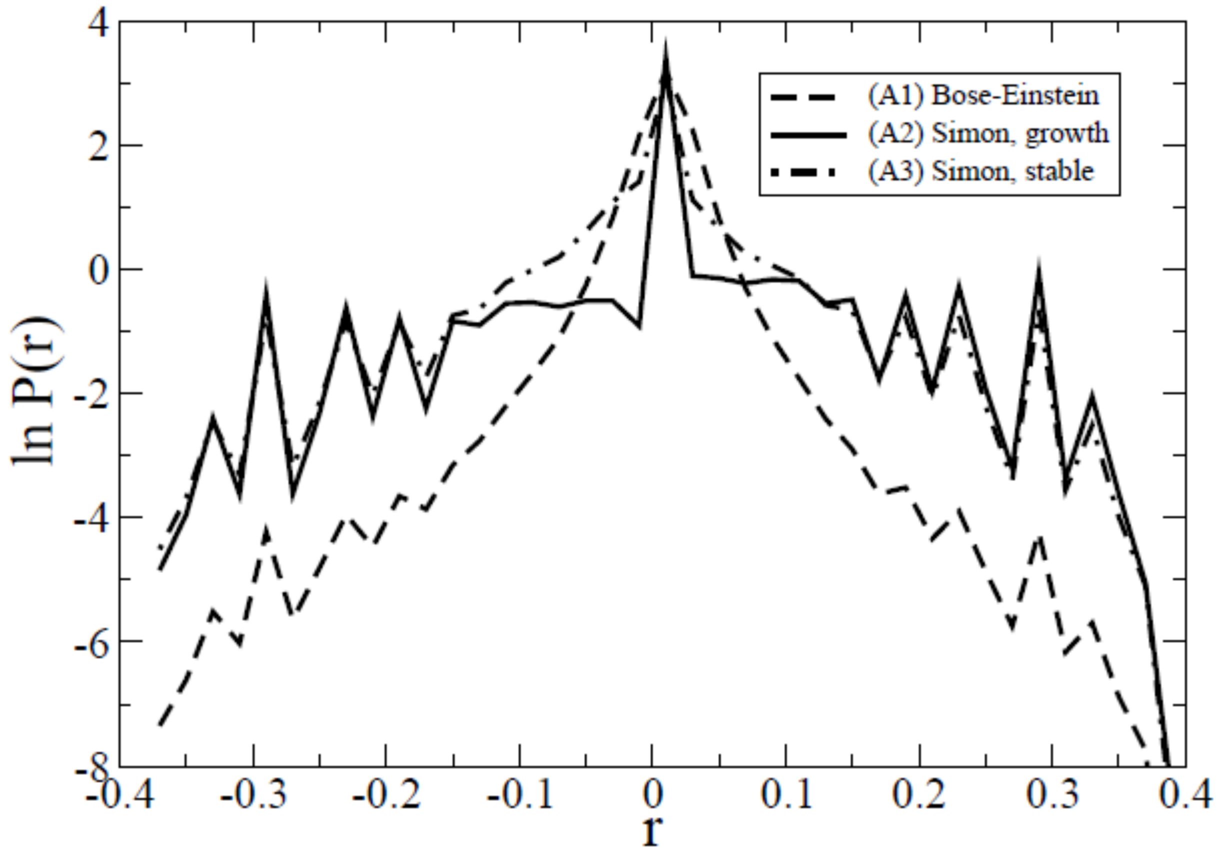
$$P_K^o(t) \sim \exp[-(1-\alpha)K/R]$$

New Firms : Crossover from power law for K<<R to exponential for K>>R

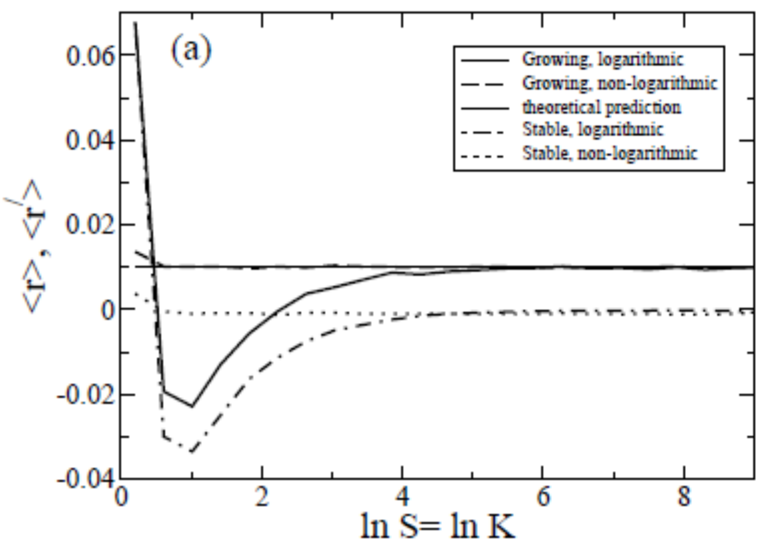
$$P_K'(t) \sim \exp[-(1-\alpha)K/R]/K,$$

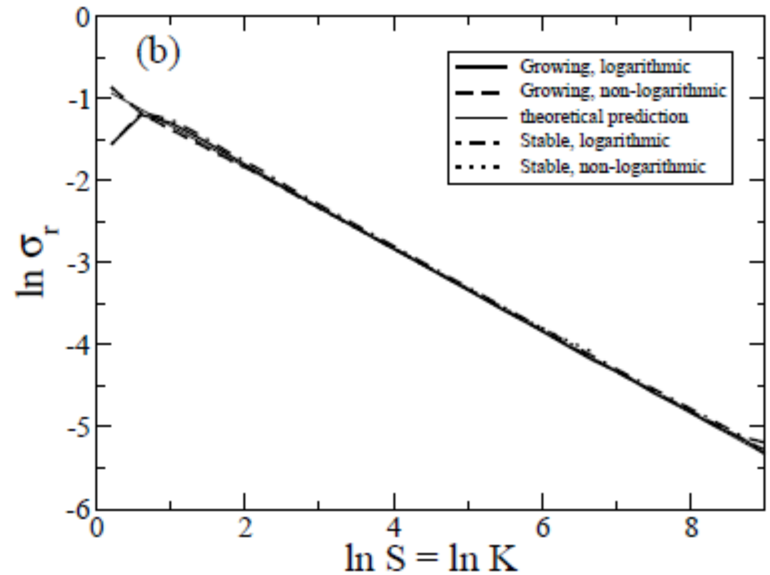


Growth rate : disaster!

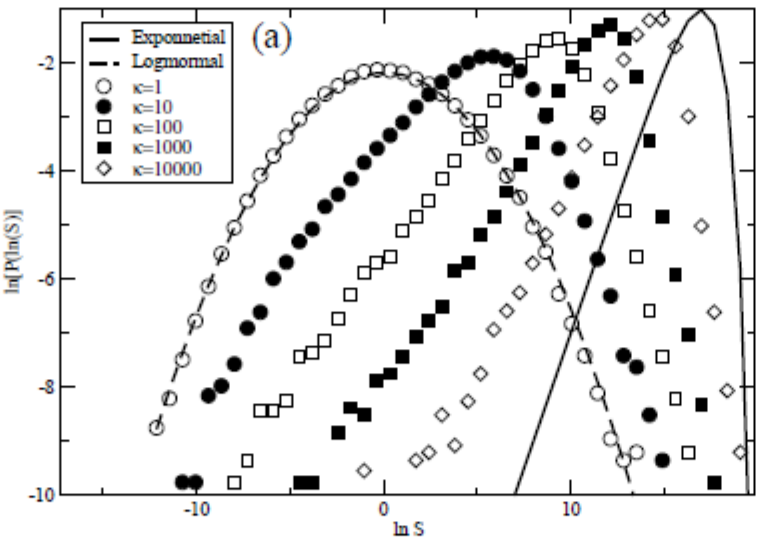


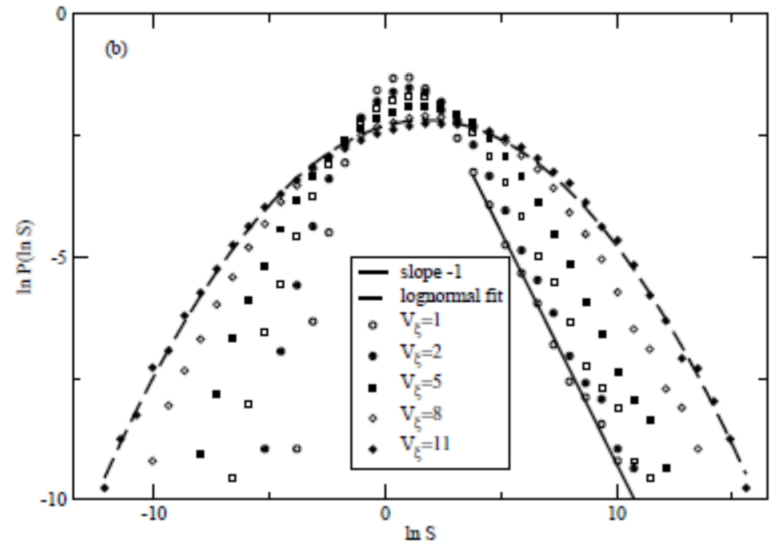
Stylized Facts (3,4): Same as Bose.





Full Blown GPMG





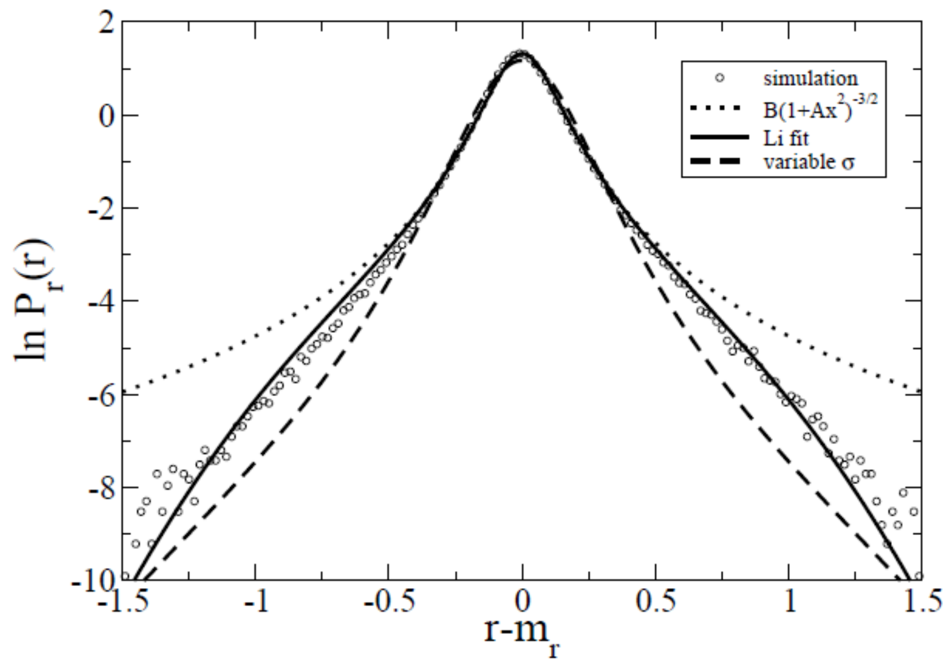
$$P(S) = \sum_{K=1}^{\infty} P(S|K)P_K$$

No analytical formula for the sum of Lognormals!!!

$$P(S|K) = P_\xi^{(K)}(S)$$

$$P_\xi(\xi) = \frac{1}{\xi\sqrt{2\pi V_\xi}} \mathrm{e}^{-\frac{(\ln \xi - m_\xi)^2}{2MV_\xi}}$$

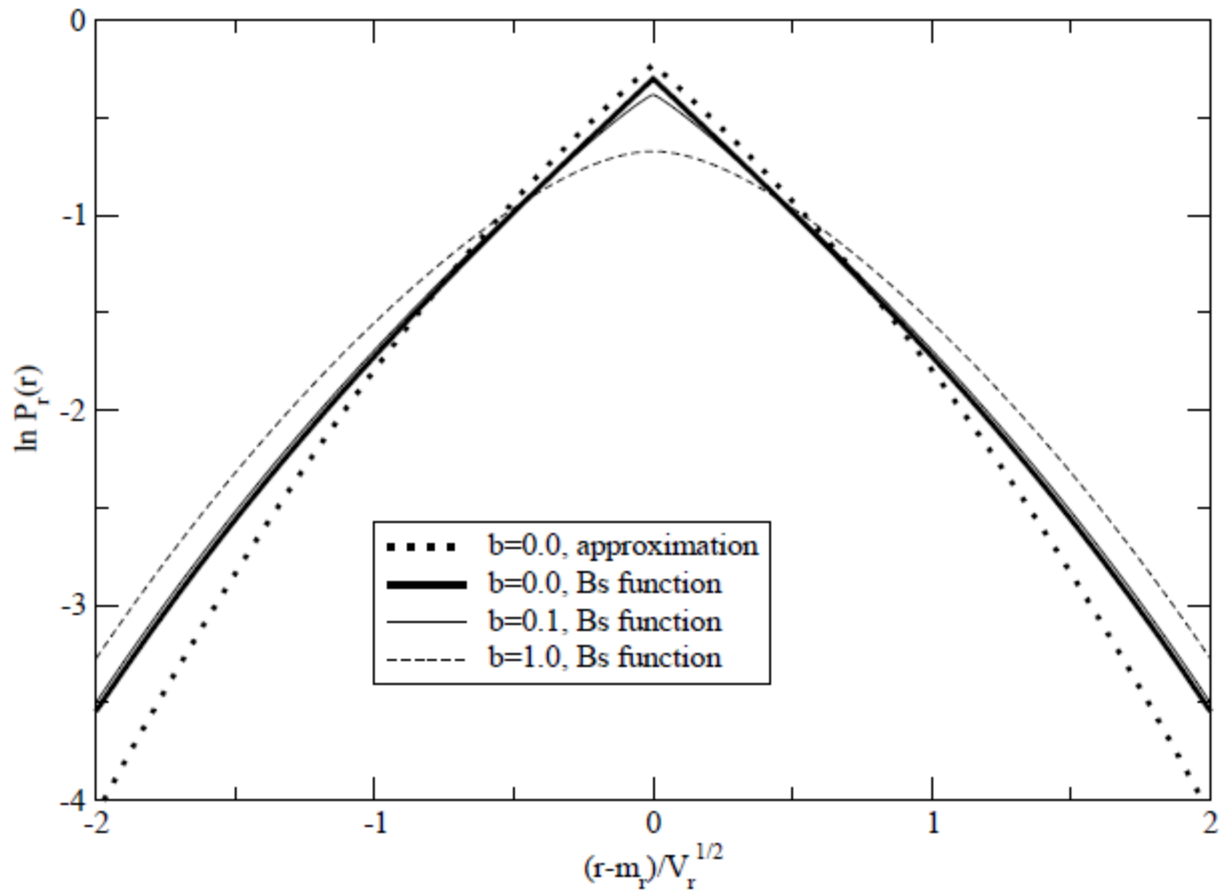
Growth rate for Bose-Einstein



$$P_r(r) \approx \frac{1}{\sqrt{2\pi V_r}} \int_0^\infty \frac{1}{\kappa(t)} \exp\left(\frac{-K}{\kappa(t)}\right) \exp\left(-\frac{(r-m_r)^2\, K}{2\, V_r}\right) \sqrt{K}\; dK,$$

$$= \frac{\sqrt{\kappa(t)}}{2\sqrt{2\, V_r}} \left(1 + \frac{\kappa(t)}{2V_r}\,(r-m_r)^2\right)^{-\frac{3}{2}}, \qquad (3.69)$$

Growth rate for Simon



Laplassian Cusp

Size variance