A Markov-Switching Stochastic Volatility Model with Jumps
Econophysics

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2 The model and its estimation
   • The model
   • Estimation Method

3 Empirical Results
   • The data
   • Estimation of the SVJMS model
   • Estimation of the SVJMS model

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Outline

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Thanks to Merton’s seminal paper of 1973, continuous-time modeling has become very popular for asset pricing.

Jump-Diffusions and more generally continuous-time Markov processes are specified in economics and finance by:

\[ dX_t = \mu(X_t; \theta)dt + \sigma(X_t; \theta)dB_t + dJ_t \]

- \( \mu(X_t; \theta) \) is the drift term
- \( \sigma(X_t; \theta) \) is the diffusion term
- \( B_t \) is a Brownian motion
- \( J_t \) is a jump process with intensity and size which are state dependent
Let introduce $Y_t = \log S_t$ ($S_t$ is the stock price) and let the volatility be stochastic: we get the famous Heston (1993) square-root stochastic volatility model:

\[
\begin{pmatrix}
  dY_t \\
  dV_t 
\end{pmatrix} = \begin{pmatrix}
  \mu + \beta V_t \\
  \kappa(\theta - V_t)
\end{pmatrix} dt + \sqrt{V_t} \begin{pmatrix}
  1 \\
  \rho \sigma_v \\
  \sqrt{(1 - \rho^2)} \sigma_v
\end{pmatrix} dW_t
\]

(1)

- Bates (1996) added jumps to the returns only, and solved the model with GMM (and option data)
- Eraker et al. (2003) added jumps to returns and volatility, and solved the model with MCMC (using prices data)

\[
\begin{pmatrix}
  dY_t \\
  dV_t 
\end{pmatrix} = \begin{pmatrix}
  \mu + \beta V_{t-} \\
  \kappa(\theta - V_{t-})
\end{pmatrix} dt + \sqrt{V_{t-}} \begin{pmatrix}
  1 \\
  \rho \sigma_v \\
  \sqrt{(1 - \rho^2)} \sigma_v
\end{pmatrix} dW_t + \begin{pmatrix}
  dZ_t^y \\
  dZ_t^v
\end{pmatrix}
\]

(2)
• Todorov (2010) emphasizes the importance of jumps in explaining the dynamics and pricing of the variance premium
• Bollerslev and Todorov (2011) offered a new nonparametric estimation of jump tails
• They found a strong temporal variation in the jump intensity and sizes
• Christoffersen et al. (2012) also provide evidence for time-varying jump intensities
Hamilton (1989) was the first to introduce this notion of regime switching. He defined the parameters of an autoregression as the outcome of a discrete-state Markov process.

This triggered a multitude of papers modeling regime-switching.

Regarding stochastic volatility, Casarin (2003 and 2013) attempted to model the volatility parameters as Markov-switching, for a log stochastic volatility model.

This motivates our choice to model the jump intensity and sizes to follow Markov process. We would expect:

- **In stress period**: a cluster of jumps, and of possibly big size
- **In calm period**: not many jumps, of medium size
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| - The data |
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| - Estimation of the SVJMS model |
Let $Y_t = \log S_t$, where $S_t$ is the stock price. Then:

$$dY_t = (\mu + \beta V_{t-}) dt + \sqrt{V_{t-}} dW_t^\gamma + \xi_{s_t}^\gamma dN_{s_t}^\gamma$$

$$dV_t = \kappa (\theta - V_{t-}) dt + \sigma \sqrt{V_{t-}} dW_t^\nu$$

where $V_{t-} = \lim_{s \to t} V_s$, $\rho = corr(W_t^\gamma, W_t^\nu)$, $W_t^i$ $(i = y, \nu)$ is a standard Brownian motion in $\mathbb{R}^2$, $N_{s_t}^\gamma$ is a Poisson process with Markov-switching intensity $\lambda_{s_t}^\gamma$, and $\xi_{s_t}^\gamma \sim N(\mu_{s_t}^\gamma, \sigma^\gamma)$ is the Markov-switching jump size in returns. The Markov chain $s_t$ have the following transition probability:

$$P(s_t = 2|s_t = 1) = \epsilon_1$$

$$P(s_t = 1|s_t = 2) = \epsilon_2$$
We use here Markov Chain Monte Carlo methods.

- In the bayesian setup, we want to draw our vector of parameters $\Theta$ from $p(\Theta|Y)$ ($Y$ is the vector of observations)
- problem: we don’t know $p(\Theta|Y)$
- We use the Gibbs algorithm
We also have one problem: we have several latent variables \((V_t, J_t, \xi_t^y, s_t)\).

Jacquier et al. (1994) were the first to implement a Gibbs cycle to estimate a stochastic volatility model. For this, they used the concept of data augmentation of Tanner and Wong (1987).

- We cannot draw from \(p(\Theta|Y)\)
- We can then add some parameters, for instance \(\lambda\)
- Consider then \(p(\Theta, \lambda|Y)\) via \(p(\Theta|\lambda, Y)\) and \(p(\lambda|\Theta, Y)\).
To apply the Gibbs algorithm here, we also need to discretize our continuous-time jump-diffusion:

\[
Y_{(t+1)\Delta} - Y_{t\Delta} = \mu \Delta + \sqrt{V_{t\Delta} \Delta} \epsilon^y_{(t+1)\Delta} + \xi^y_{s_{(t+1)\Delta}} J^y_{s_{(t+1)\Delta}}
\]

\[
V_{(t+1)\Delta} - V_{t\Delta} = \kappa (\theta - V_{t\Delta}) \Delta + \epsilon_v \sqrt{V_{t\Delta}} \epsilon^v_{(t+1)\Delta}
\]

where \( J^y_{s_{(t+1)\Delta}} \) equals 1 if a jump occurs, \( \epsilon^i_{(t+1)\Delta} (i = y, v) \) are standard normal variables with correlation \( \rho \), and \( \Delta \) is the time-discretization interval (one-day here). Once discretized, the jump times are Bernoulli random variables with intensity \( \lambda^y_{s_t} \Delta \).
We need to be able to compute the posterior distribution, which summarizes all the information regarding the latent variables (volatility $V$, the jump times $J$ and size $\xi^y$, the Markov chain $S$) and the parameters $\Theta = (\mu, \kappa, \theta, \sigma_v, \rho, \mu_1^\gamma, \mu_2^\gamma, \sigma^\gamma, \lambda_1, \lambda_2, \epsilon_1, \epsilon_2)$. The posterior distribution combines the likelihood and the prior:

$$p(\Theta, J, \xi^y, V, S|Y) \propto p(Y|\Theta, J, \xi^y, V, S)p(\Theta, J, \xi^y, V, S)$$

(4)
The model and its estimation

Empirical Results

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\[ p(V_t | V_{(t-1)\Delta}, V_{(t+1)\Delta}, \Theta, J, \xi^y, S, Y), t = 1,\ldots, T \]

\[ p(J_{t\Delta} = 1 | \Theta, \xi^y, V, S, Y), t = 1,\ldots, T \]

\[ p(\xi^y_s | \Theta, J_{t\Delta} = 1, V, S, Y), t = 1,\ldots, T \]

\[ p(s_{t\Delta} | s_{(t-1)\Delta}, s_{(t+1)\Delta}, \Theta, J, \xi^y, V, Y), t = 1,\ldots, T \]

\[ p(\Theta_i | \Theta_{-i}, J, \xi^y, V, Y), i = 1,\ldots, k \]

where \( \Theta_{-i} \) denotes the parameter vector except the \( i^{th} \) one, and \( k \) is the number of parameters. Drawing from these distributions is not always straightforward. We need to add a Metropolis Step to draw the volatility.
The algorithm will provide a set of draws
\[ \{ V^{(m)}, J^{(m)}, \xi^{y(m)}, \xi^{v(m)}, S^{(m)}, \Theta^{(m)} \}_{m=1}^{M}, \]
M being the number of Monte Carlo simulations. These draws are samples from the posterior distribution. Because the spot volatilities, the jump times and sizes and the states are drawn from the posterior distribution, the Monte Carlo estimates of these processes is given by:

\[
\mathbb{E}[X_t | Y] \approx \frac{1}{M} \sum_{m=1}^{M} X_{t \Delta}^{(m)}
\]  

(5)

where \( X \) can be \( V, \xi^y, J, S \). Eraker et al. (2003) also notes a desirable feature of the MCMC methods: \( \mathbb{E}[X_t | Y] \) is estimated and not \( \mathbb{E}[X_t | Y, \hat{\Theta}] \), which means that we integrate out all the parameter uncertainty.
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A Markov-Switching Stochastic Volatility Model with Jumps
Estimate the models using the S&P500 (SPX) and the Nasdaq (NSX) returns.

We use the January 2, 1990 to December 31, 2013 period.

We have 6,049 daily observations - all from the CRSP.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Volatility</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>SP500</td>
<td>6.82</td>
<td>18.35</td>
<td>-0.24</td>
<td>11.62</td>
<td>-9.47</td>
<td>10.96</td>
</tr>
<tr>
<td>Nasdaq</td>
<td>9.20</td>
<td>24.05</td>
<td>-0.08</td>
<td>9.04</td>
<td>-10.17</td>
<td>13.25</td>
</tr>
</tbody>
</table>
The intensity of the jumps is Markov-switching ($\lambda_{st}$).

<table>
<thead>
<tr>
<th></th>
<th>SVJ</th>
<th>SVJMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.00289(0.0098)</td>
<td>0.0243(0.0096)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>1.1921(0.3290)</td>
<td>1.2058(0.3137)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.0166(0.0026)</td>
<td>0.0180(0.0020)</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>0.1496(0.0097)</td>
<td>0.1559(0.0089)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.7168(0.0281)</td>
<td>-0.7124(0.0303)</td>
</tr>
<tr>
<td>$\mu_y$</td>
<td>-1.3900(0.6998)</td>
<td>-0.4717(0.5037)</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>1.7214(0.3475)</td>
<td>3.3840(0.4940)</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>0.0094(0.0050)</td>
<td>0.8278(0.1011)</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0</td>
<td>0.0025(0.0014)</td>
</tr>
<tr>
<td>$\epsilon_1$</td>
<td>1</td>
<td>0.0426(0.0165)</td>
</tr>
<tr>
<td>$\epsilon_2$</td>
<td>0</td>
<td>0.0005(0.0003)</td>
</tr>
</tbody>
</table>
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We consider jumps in returns only, and the jump size and the jump intensity to be Markov-switching \((\lambda_{st}, \mu_{st}^y)\).

**Table**: Parameter Comparison for SVJMS model with the SVJ model - NDX

<table>
<thead>
<tr>
<th></th>
<th>SVJ</th>
<th>SVJMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu)</td>
<td>0.10 (0.0160)</td>
<td>0.0721(0.0125)</td>
</tr>
<tr>
<td>(\theta)</td>
<td>1.3874(0.4459)</td>
<td>1.7000(0.4981)</td>
</tr>
<tr>
<td>(\kappa)</td>
<td>0.013(0.0020)</td>
<td>0.011(0.0025)</td>
</tr>
<tr>
<td>(\sigma_v)</td>
<td>0.1459(0.0124)</td>
<td>0.1405(0.0104)</td>
</tr>
<tr>
<td>(\rho)</td>
<td>-0.5596(0.0407)</td>
<td>-0.5224(0.0432)</td>
</tr>
<tr>
<td>(\mu_1^y)</td>
<td>-0.979(0.2500)</td>
<td>-0.4432(0.3032)</td>
</tr>
<tr>
<td>(\mu_2^y)</td>
<td>-0.979(0.2500)</td>
<td>-2.1321 (0.9219)</td>
</tr>
<tr>
<td>(\sigma^y)</td>
<td>1.2226(0.1610)</td>
<td>3.0431(0.3005)</td>
</tr>
<tr>
<td>(\lambda_1^y)</td>
<td>0.0495(0.0203)</td>
<td>0.8696(0.0616)</td>
</tr>
<tr>
<td>(\lambda_2^y)</td>
<td>0.0495(0.0203)</td>
<td>0.0053(0.0019)</td>
</tr>
<tr>
<td>(\varepsilon_1)</td>
<td>0.0268(0.0104)</td>
<td></td>
</tr>
<tr>
<td>(\varepsilon_2)</td>
<td>0.0011(0.0004)</td>
<td></td>
</tr>
</tbody>
</table>
Nasdaq, 1990-2013

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A Markov-Switching Stochastic Volatility Model with Jumps

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Nasdaq, 1998-2003, Tech Bubble

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A Markov-Switching Stochastic Volatility Model with Jumps
Nasdaq, 2008-2009, Financial Crisis

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Total variance decomposition

\[
\frac{\mathbb{E}[\xi^2_t] \lambda_y}{\hat{V} + \mathbb{E}[\xi^2_t] \lambda_y}
\]

Table: Variance Decompositions

<table>
<thead>
<tr>
<th></th>
<th>Spot</th>
<th>Volatility</th>
<th>Total</th>
<th>Volatility</th>
<th>Jump Variance</th>
<th>(% total)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SPX</td>
<td>NDX</td>
<td>SPX</td>
<td>NDX</td>
<td>SPX</td>
<td>NDX</td>
</tr>
<tr>
<td>SVJ</td>
<td>17.33</td>
<td>18.7</td>
<td>17.66</td>
<td>21.15</td>
<td>3.72</td>
<td>8.05</td>
</tr>
<tr>
<td>SVJMS</td>
<td>17.43</td>
<td>21.15</td>
<td>18.42</td>
<td>23.78</td>
<td>10.47</td>
<td>20.87</td>
</tr>
</tbody>
</table>
SVJMS model seems promising, we find evidence of two regimes and the volatility dynamics are changed in crisis times

• The convergence of the model can be sensitive to the parameters

• Explore option pricing implications of having different volatility dynamics, especially during stressed periods

• Compute the odds ratios to evaluate which model is more likely
ANY QUESTIONS?