Markowitz efficient frontier

Econophysics

Economics
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Assumptions of Markowitz Portfolio Theory

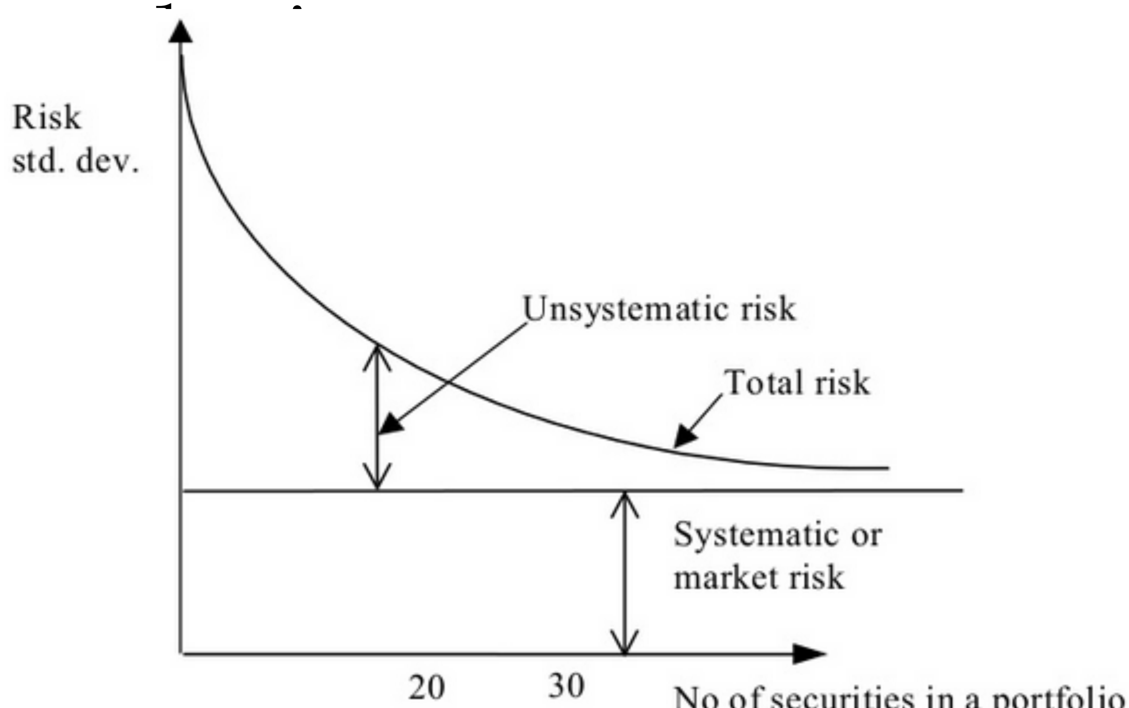
1. Investors consider each investment alternative as being presented by a probability distribution of expected returns over some holding period.
2. Investors minimize one-period expected utility, and their utility curves demonstrate diminishing marginal utility of wealth.
3. Investors estimate the risk of the portfolio on the basis of the variability of expected returns.

Assumptions of Markowitz Portfolio Theory

4. Investors base decisions solely on expected return and risk, so their utility curves are a function of expected return and the expected variance (or standard deviation) of returns only.
5. For a given risk level, investors prefer higher returns to lower returns. Similarly, for a given level of expected returns, investors prefer less risk to more risk.

Markowitz risk diversification

- Invest in assets that have low positive correlation or a negative co



Expansion of Markowitz

model	free-risk interest asset	no free-risk interest asset
short selling	Final Model	Black Model
no short selling	Tobin Model	Markowitz Model

- Assume **portfolio weights**

($\omega_t = (\omega_1, \omega_2, ..., \omega_n)^T$ $\omega_1 + \omega_2 + ... + \omega_n = 1$

- Expected Return

$$R_1 = \left\{ r = \omega_1 r_1 + \omega_2 r_2 + ... + \omega_n r_n \mid r_i \in ° ,i = 1, 2, ..., n; \sum_{i=1}^{n} \omega_i = 1 \right\}$$

- Variance (Standard Deviation) of Returns

$$\sigma_p^2 = E[(\sum_{i=1}^{n} \omega_i r_i - \sum_{i=1}^{n} \omega_i E[r_i])^2]$$

$$= \sum_{i,j=1}^{n} \omega_i \omega_j E[(r_i - E[r_i])(r_j - E[r_j])]$$

$$= \sum_{i,j=1}^{n} V_{i,j} \omega_i \omega_j$$

- Covariance： $V_{ij} = Cov[r_i, r_j]$

$$V = \begin{vmatrix} Var(r_1) & Cov(r_1, r_2) & \ldots & Cov(r_1, r_n) \\ Cov(r_2, r_1) & Var(r_2) & \ldots & Cov(r_2, r_n) \\ \ldots & \ldots & \ldots & \ldots \\ Cov(r_n, r_1) & Cov(r_n, r_2) & \ldots & Var(r_n) \end{vmatrix}$$

$$= \begin{vmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_{nn} \end{vmatrix}$$

- How to get W：

$$\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T, \quad e = (1, 1, \ldots, 1)^T,$$

$$\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T, \quad \mu_i = E(r_i), \quad i = 1, 2, \ldots, n,$$

$$V = (V_{ij})_{i,j=1,2,\ldots,n} = (Cov[r_i, r_j])_{i,j=1,2,\ldots,n}$$

$\mu_\omega = \omega^T \mu$ is the return of the portfolio, $\sigma_\omega = (\omega^T V \omega)^{1/2}$

is the risk of the portfolioi, then the question is to solve:

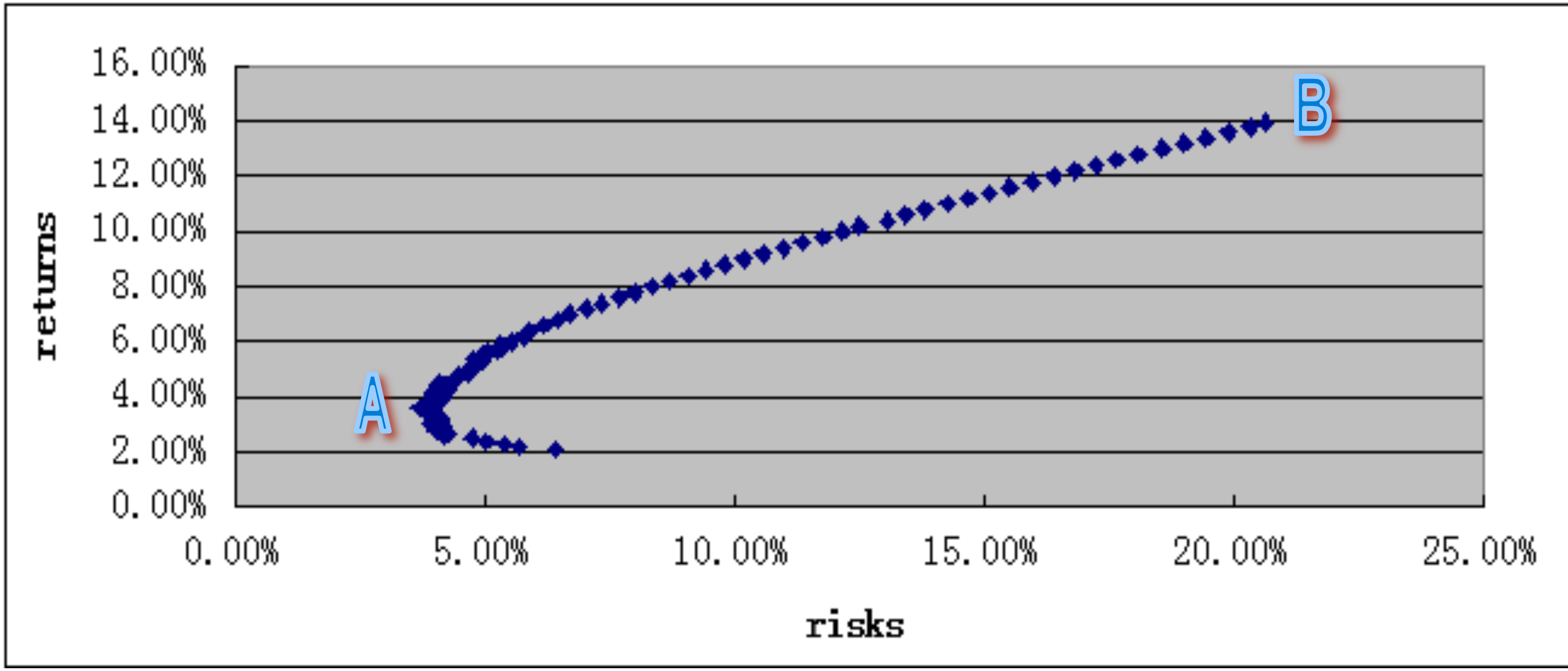
$$\begin{cases} \min \quad \sigma_\omega^2 = w^T V w = \sum_{i=1}^{n} V_{ij}\omega_i\omega_j \\ s.t \quad w^T e = \omega_1 + \omega_2 + \ldots + \omega_n = 1 \\ \quad\quad \mu_\omega = w^T \mu = \omega_1\mu_1 + \omega_2\mu_2 + \ldots + \omega_n\mu_n = \bar{\mu} \end{cases}$$

Wi>=0 (no short selling)

Portfolio	Average return%	risk
IRCP	4.96	0.087721132
APPL	4.11	0.052699075
LEVYU	4.39	0.127440516
LEVY	3.31	0.085976235
NIKE	2.33	0.051217275
PBCP	2.10	0.066363293
QURE	8.49	0.274116132
SCMP	6.11	0.158430564
MTDR	2.76	0.139361612
CHR	3.80	0.085369942
NHTC	13.93	0.213445932
TZF	3.57	0.193861817

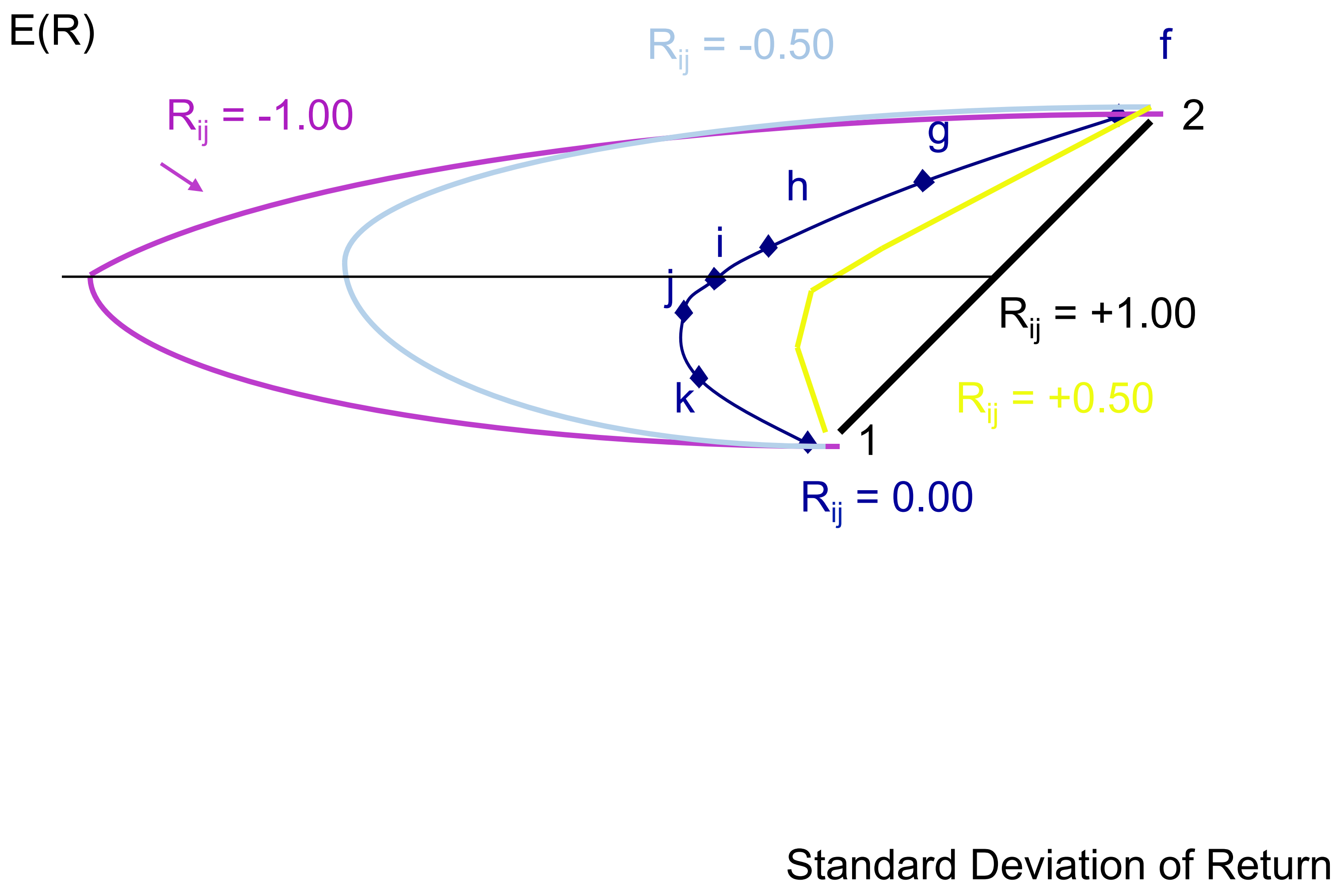
COV%	IRCP	APPL	LEVY U	LEVY	NIKE	PBCP	QUR E	SCM P	MTD R	CHR	NHT C	TZF
IRCP	0.72	-0.02	0.67	0.43	-0.01	0.30	0.27	-0.23	0.04	0.03	0.34	0.28
APPL	-0.02	0.26	-0.26	-0.18	0.04	-0.14	0.1	0.16	-0.20	0.15	0.34	-0.23
LEVY U	0.67	-0.26	1.52	1.00	-0.01	0.68	0.17	-0.17	0.15	0.07	0.72	-0.49
LEVY	0.43	-0.18	1.00	0.69	-0.02	0.43	0.28	-0.06	0.13	0.05	0.38	-0.30
NIKE	-0.01	0.04	-0.01	-0.02	0.24	-0.05	-0.16	0.18	-0.23	-0.01	0.05	0.33
PBCP	0.30	-0.14	0.68	0.43	-0.05	0.41	0.27	-0.13	0.16	0.03	0.25	-0.27
QUR E	0.27	0.12	0.17	0.28	-0.16	0.27	6.98	1.70	0.47	0.66	-0.42	-0.30
SCM P	-0.23	0.16	-0.17	-0.06	0.18	-0.13	1.70	2.34	-0.50	0.18	-0.09	-1.27
MTD R	0.04	-0.20	0.15	0.13	-0.23	0.16	0.47	-0.50	1.81	-0.06	-0.58	0.51
CHR	0.03	0.15	0.07	0.05	-0.01	0.03	0.66	0.18	-0.06	0.68	-0.11	-0.49
NHT C	0.34	0.34	0.72	0.38	0.05	0.25	-0.42	-0.09	-0.58	-0.11	4.25	-1.10
TZF	0.28	-0.23	-0.49	-0.30	0.33	-0.27	-0.30	-1.27	0.51	-0.49	-1.10	3.51

Portfolio Efficient Frontier



- A: Minimum variance porfolio
- B: Maximum return portfolio

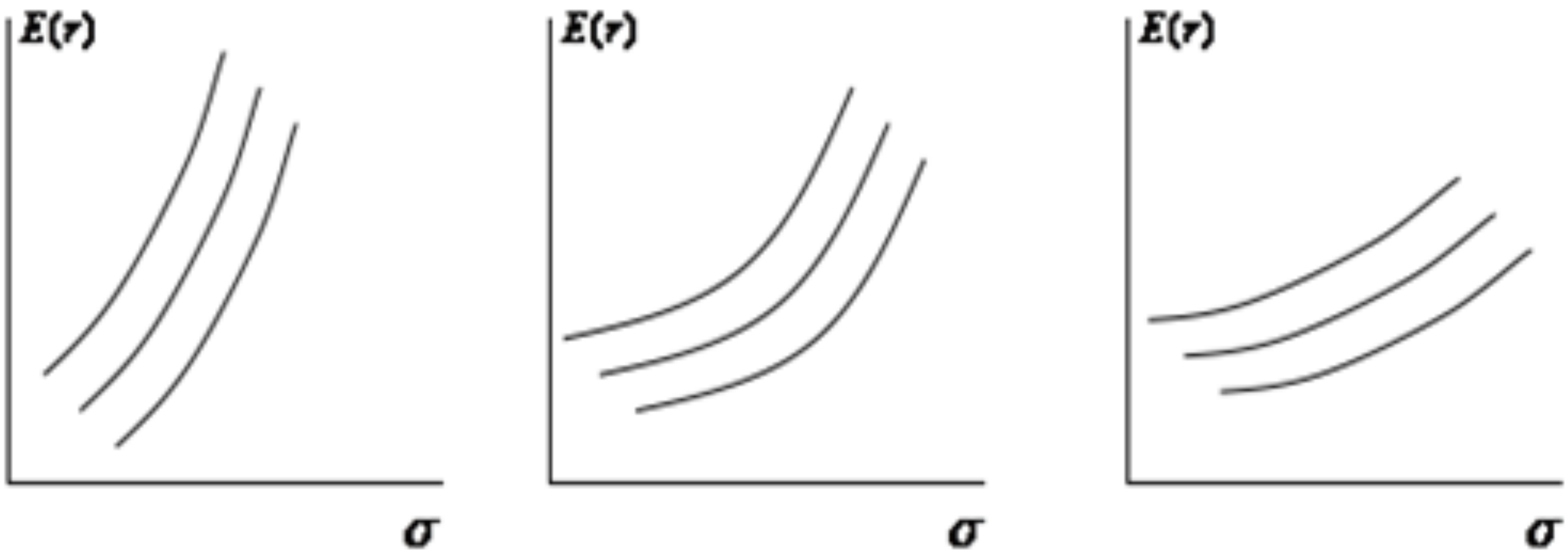
Portfolio Risk-Return Plots for Different Weights



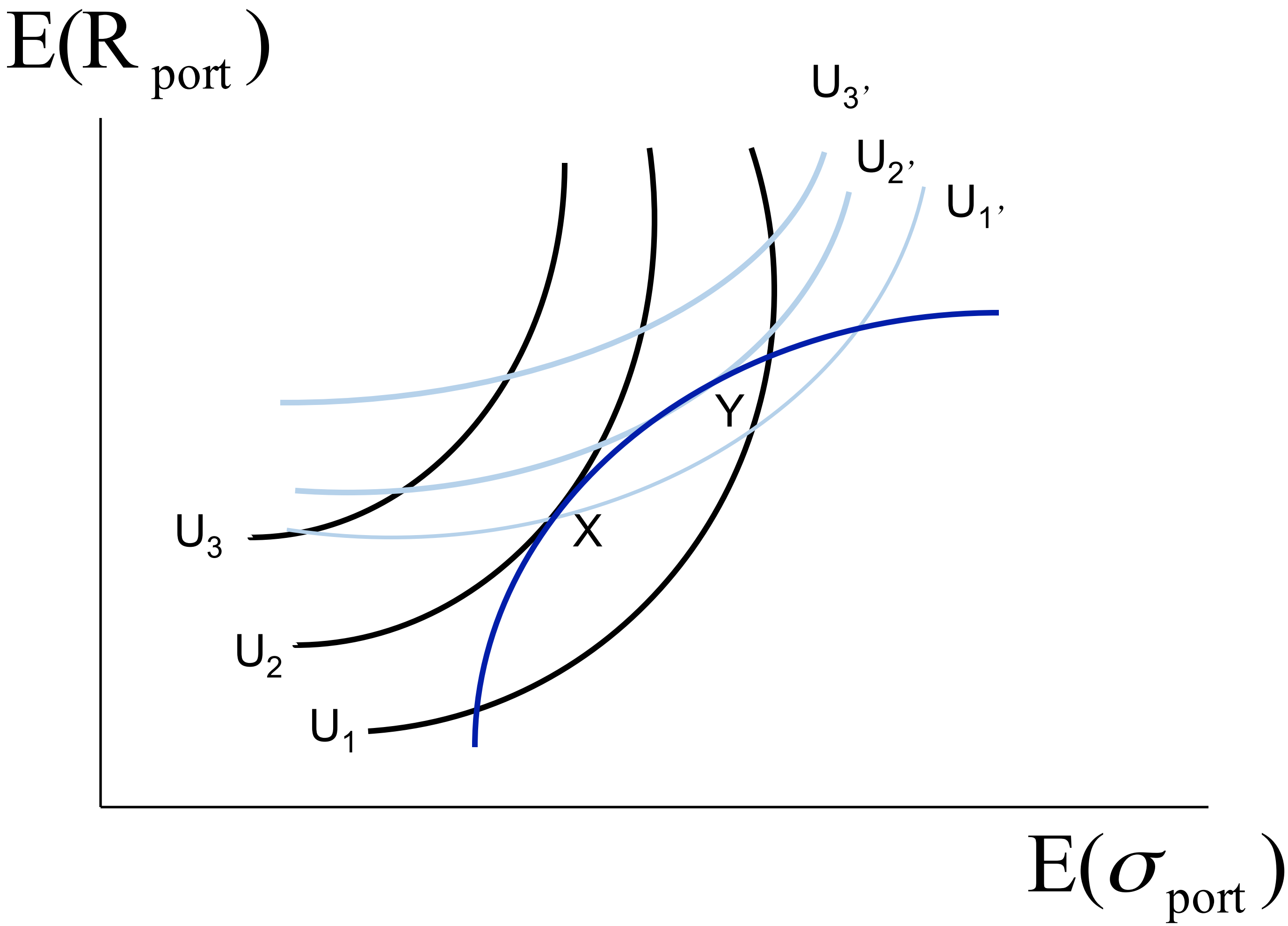
The Efficient Frontier and Investor Utility

Different degree of risk aversion:

High Medium

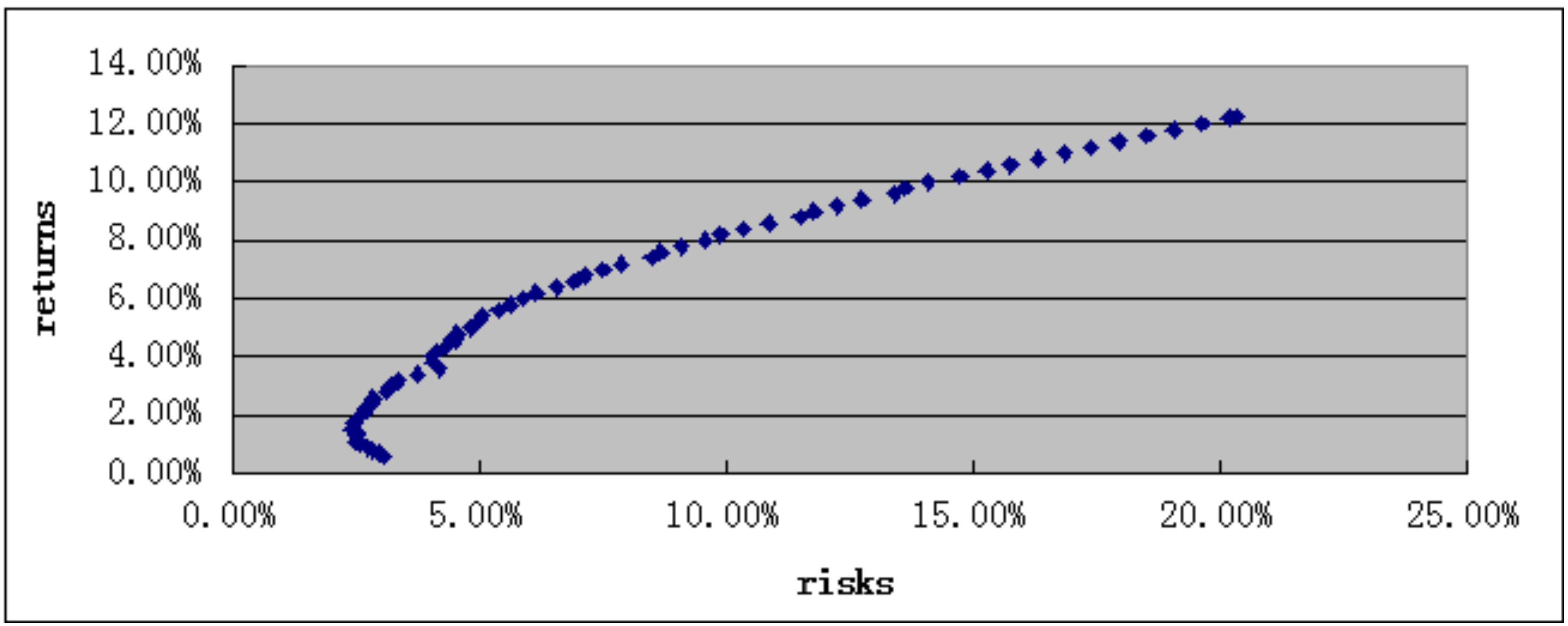


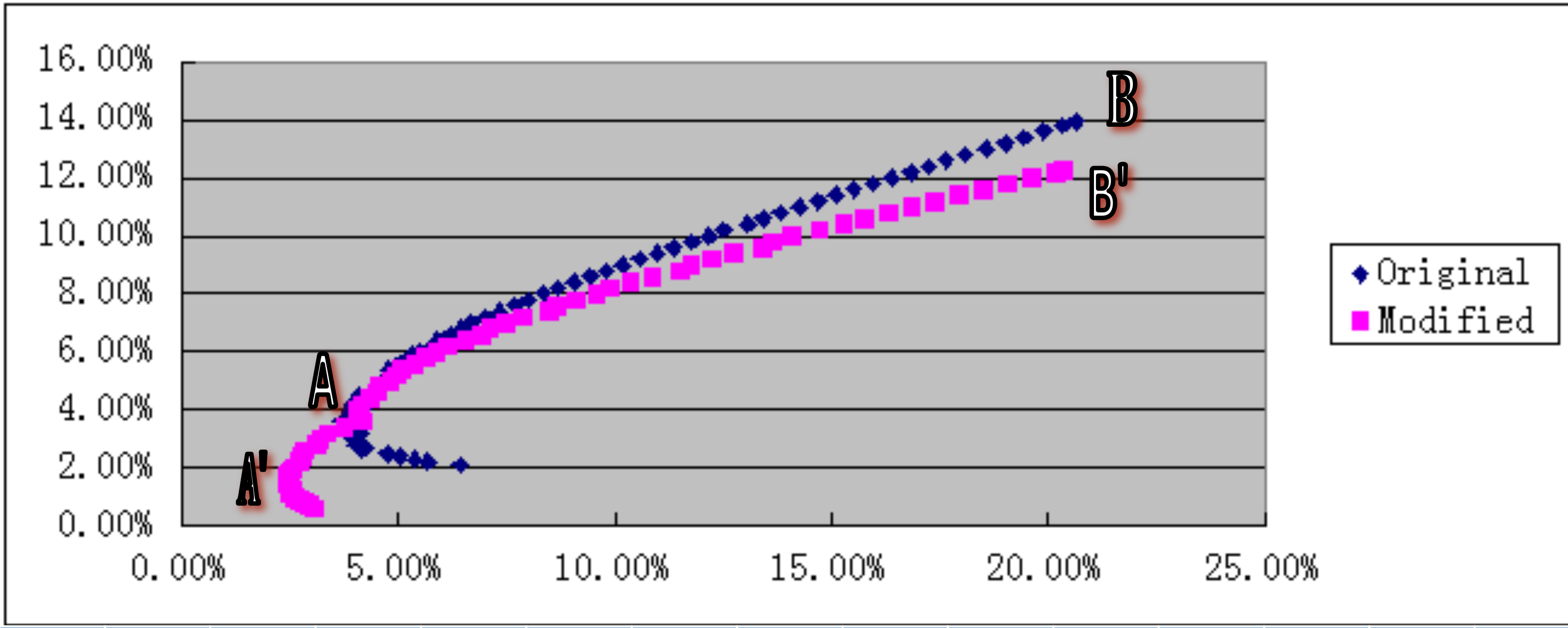
Selecting an Optimal Risky Portfolio



Modified Efficient Frontier

return%	IRCP	APPL	LEVYU	LEVY	NIKE	PBCP	QURE	SCMP	MTDR	CHR	NHTC	TZF
201503	21.43	-3.14	48.32	31.50	3.31	23.38	4.65	1.17	1.20	5.84	37.38	-18.83





			IRCP	APPL	LEVYU	LEVY	NIKE	PBCP	QURE	SCMP	MTDR	CHR	NHTC	TZF
original	Max E[r]	13.93%	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00
	Min stdev	2.76%	0.00	0.41	0.00	0.00	0.38	0.22	0.00	0.00	0.00	0.00	0.00	0.00
modified	Max E[r]	12.26%	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00
	Min [illegible]	1.92[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Short selling allowed

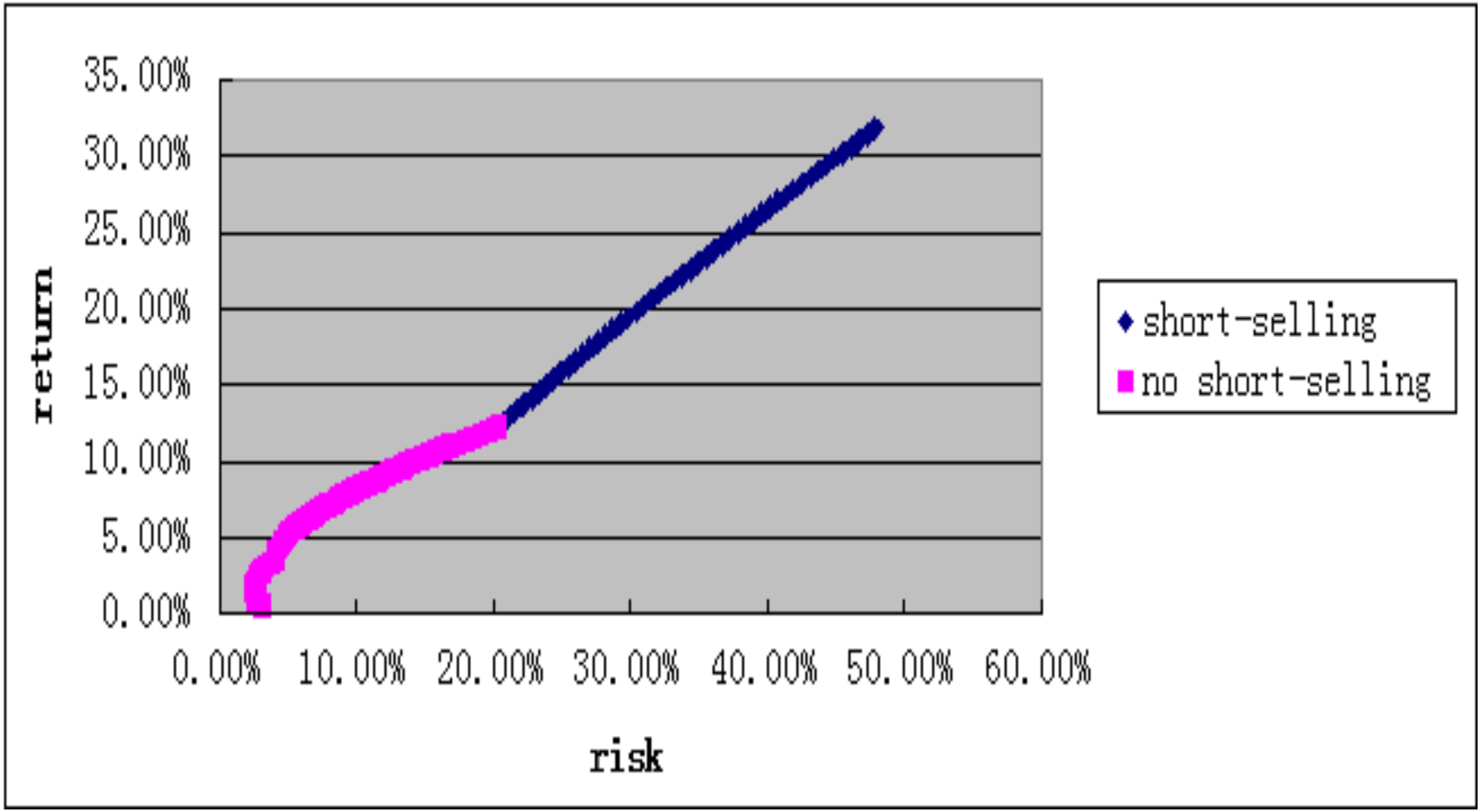
- Short sell

In order to profit from a decrease in the price of a security, a short seller can borrow the security and sell it expecting that it will be cheaper to repurchase in the future.

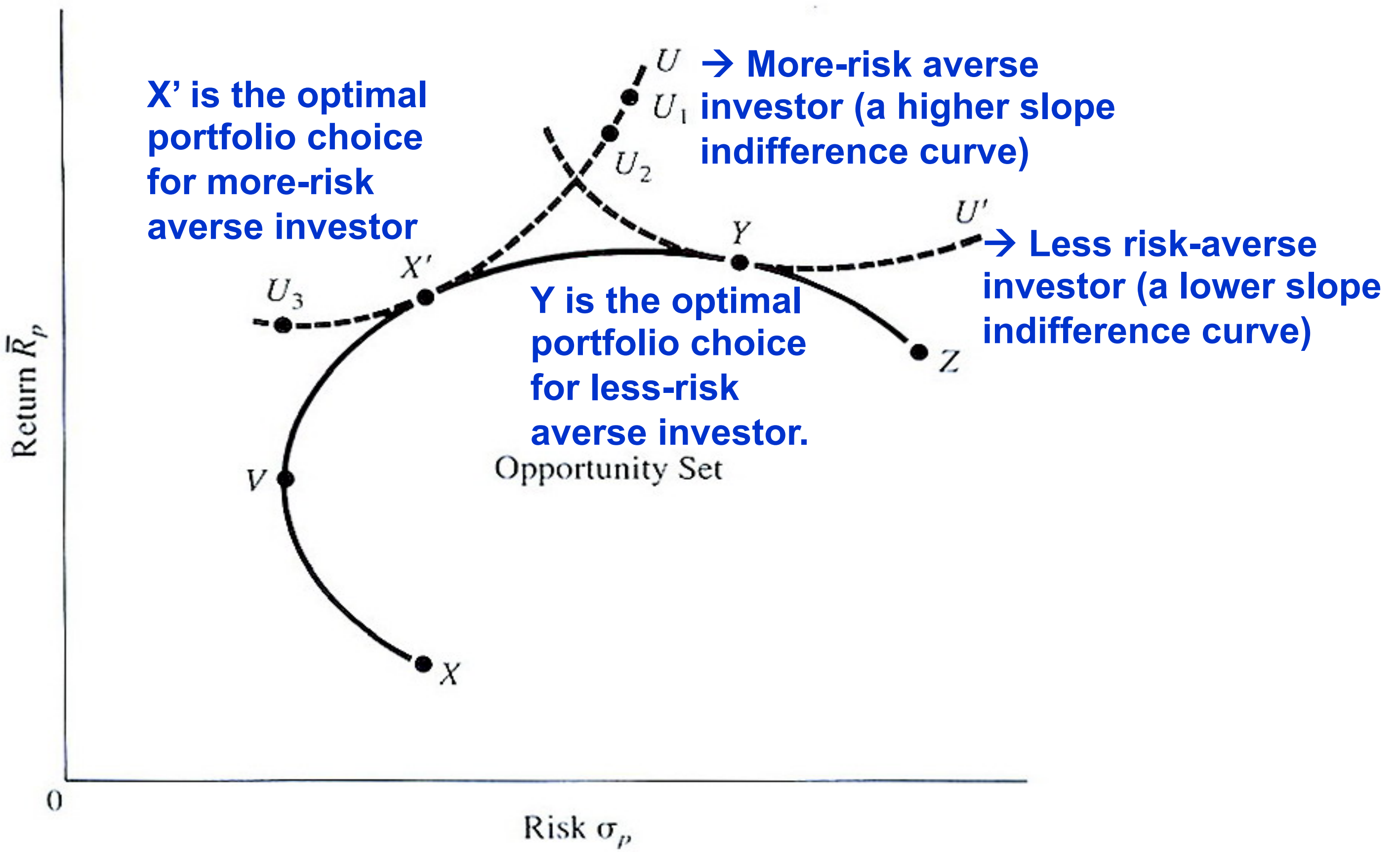
- solve (without individual weight constraint)

$$
\begin{cases}
\min \quad \sigma_\omega^2 = w^T V w = \sum_{i=1}^{n} V_{ij}\omega_i\omega_j \\
s.t \quad w^T e = \omega_1 + \omega_2 + \ldots + \omega_n = 1 \\
\quad\quad \mu_\omega = w^T \mu = \omega_1\mu_1 + \omega_2\mu_2 + \ldots + \omega_n\mu_n = \bar{\mu}
\end{cases}
$$

Efficient Frontier with Short Selling

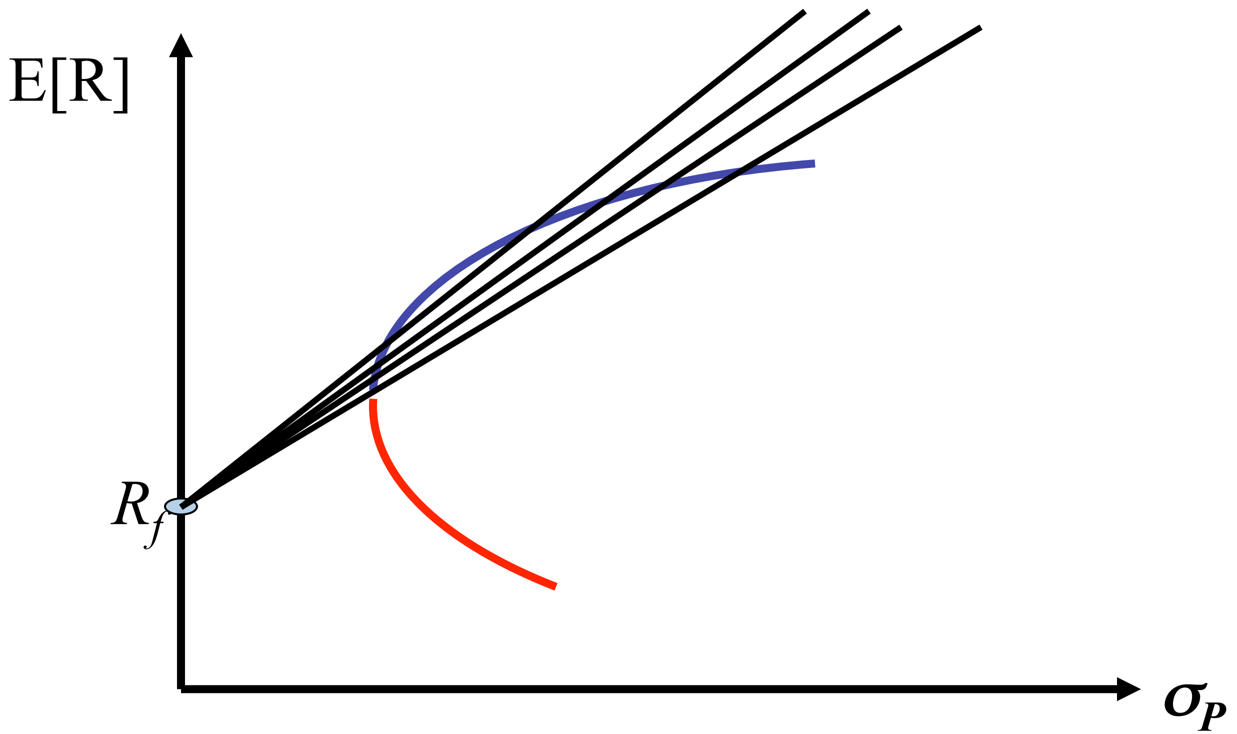


Efficient Frontier with Short Selling



The riskfree asset: riskless lending and borrowing

- ***risk-free rate***
  - risk-free lending
  - risk-free borrowing



The riskfree asset: riskless lending and borrowing

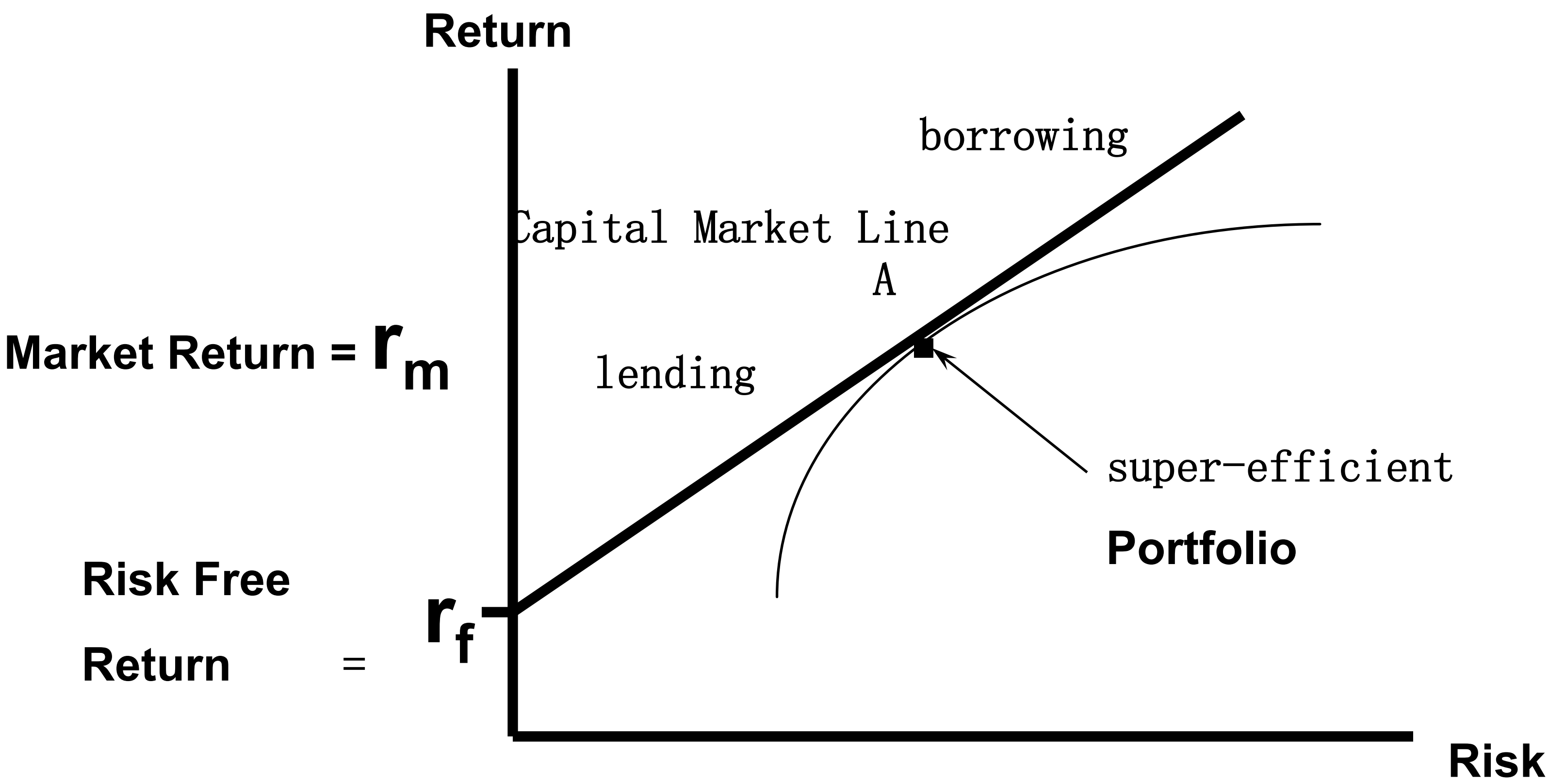
- Assume portfolio P consists of a risk-free asset and (n-1) risky assets. Then:

$$\mu_P = x_F r_F + (1 - x_F)\mu_R$$

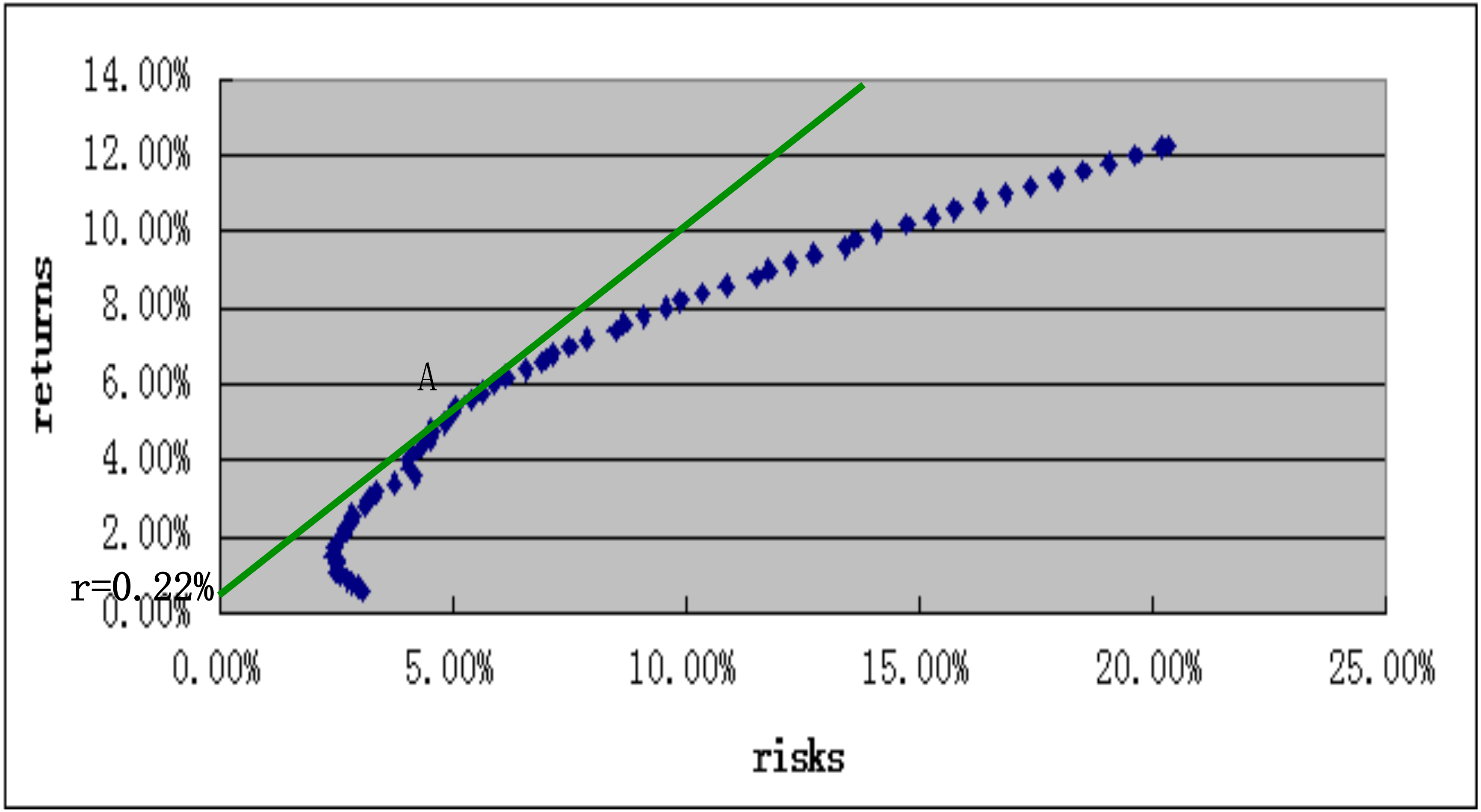
$$\sigma_P = (1 - x_F)\sigma_R$$

- so $\mu_P = r_F + \frac{\mu_R - r_F}{\sigma_R}\sigma_P$

The Efficient Frontier when Lending and Borrowing Possibilities Are Allowed



The Efficient Frontier when Lending and Borrowing Possibilities Are Allowed



Thank you

- Thanks for Chester and Antonio's help.