The Relation Between Economic Growth and Economic Equality

W. Klein
Boston University

Econophysics 2014

Collaborators

- Kang Liu - Boston University
- Nick Lubbers - Boston University
- Jan Tobochnick - Kalamazoo College
- Bruce Boghosian - Tufts University and the American University of Armenia
- H. Gould - Clark University and Boston University
- Ole Peters - London Mathematical Laboratory
- Alex Adamou - London Mathematical Laboratory

Outline

- Motivation
- Geometric Random Walk
- Asset Exchange Model
- Modified Asset Exchange Model(MAEM) - Growth
- MAEM - Limiting Case
- Summary and Conclusions.

Motivation

- Despite an average annual growth rate in the GDP of about 3% over the last 40 years and an average population growth rate of under 1% the poorer segment of our economy continues to get poorer.
- Is the simple solution to this problem more growth?
- Are there other variables besides the growth rate that determine the distribution of wealth?
- Can we understand aspects of this phenomenon through simple models of the economy?

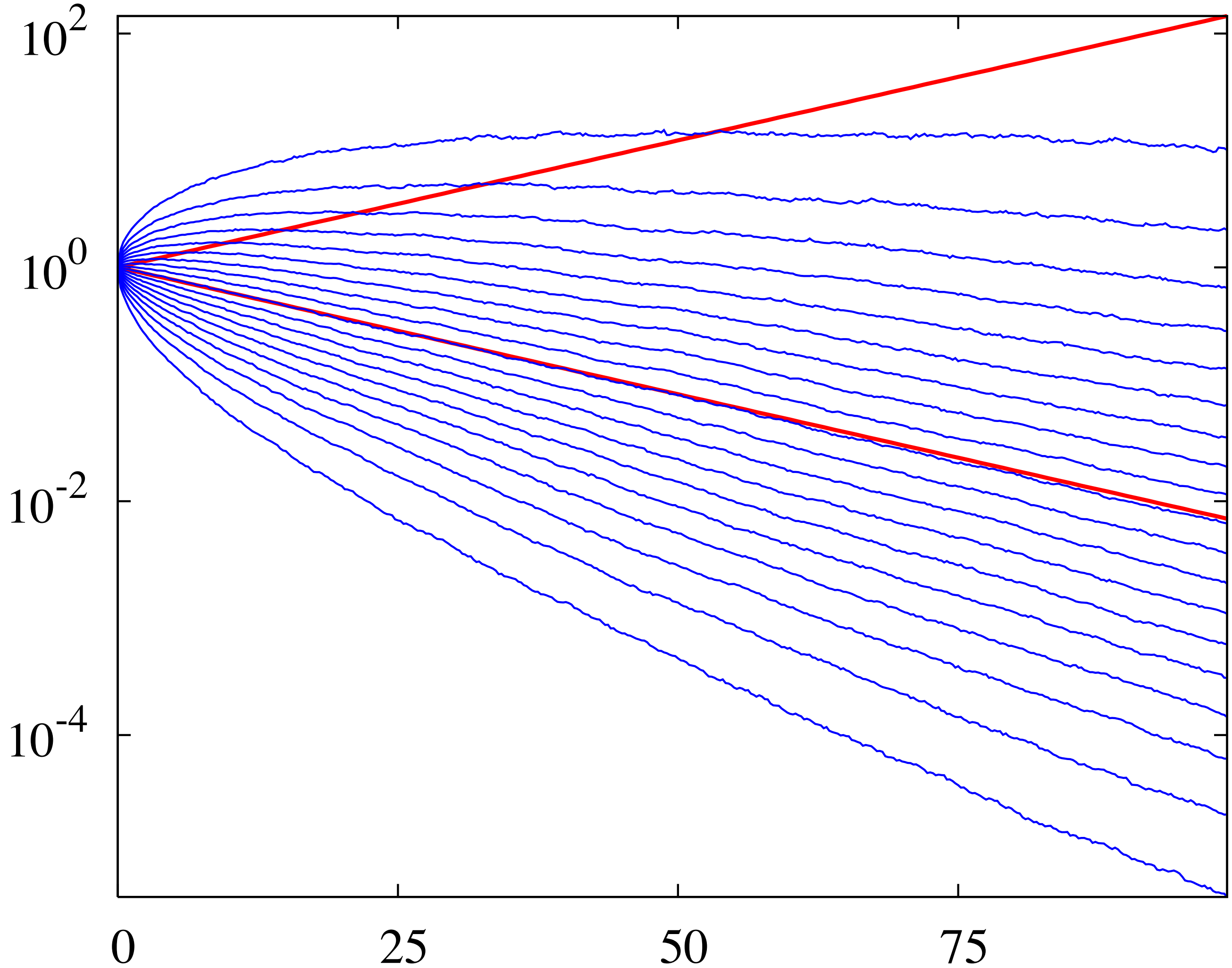
Geometric Random Walk - GRW

- The GRW has been used in economics and finance as a simple model that incorporates the effect of noise and growth.
- It is represented by the equation

$$dx(t) = \mu x(t)dt + \sigma x(t)dW \tag{1}$$

- In this context $x(t)$ is the wealth of an agent or walker.
- The quantity $\mu > 0$ is the growth rate.
- The noise dW is defined through $\int_0^t dW$ Wiener process.
- The amplitude of the noise σ is referred to as the volatility.
- The subtlety of the GRW comes from the fact that the noise is multiplicative.

- We will use three properties of dW: (1)$< dW >= 0$, (2)$< dW^2 >= dt$. (That is dW scales as $\sqrt{d}t$) (3) $< dW(t_1)dW(t_2) >= 0$ for $t_1 \neq t_2$. $< dW >$ denotes the ensemble average.
- With the ensemble average of $dW = 0$ we will see below that the ensemble average $< x(t) >= x(0)e^{\mu t}$.
- However the vast majority of paths under the time evolution do not evolve according to the ensemble average.
- Additive noise suggests that there would be an envelope around the ensemble average. Instead (O. Peters and W. Klein, PRL **110**, 100603 (2013))



- Each blue track is $\ln$ of a partial ensemble average of wealth of 5% of the agents - poorest at the bottom-richest at top. 10,000 time steps, $\mu = 0.05$, $\sigma^2 = 0.2$
- The red lines have slope μ and $\mu - \frac{\sigma^2}{2}$.

GRW Theory

- How do we understand these results?
- We rewrite eq. 1 as

$$x(t + dt) = x(t) + \mu x(t)dt + \sigma x(t)dW \tag{2}$$

- Setting $\sigma = 0$ initially

$$x(t + dt) = (1 + \mu dt)x(t) \tag{3}$$

- We can view this as a simple logistic map. Two straight lines - slope 1, slope $1 + \mu dt$.
- Map is chaotic - two arbitrarily close points separate.

$$x(t) = \lim_{N\to\infty} \left(1 + \mu \frac{t}{N}\right)^N x(0) = x(0)e^{\mu t} \tag{4}$$

- Including the noise

$$x(t + dt) = \left(1 + \tilde{\mu}\right)x(t) \tag{5}$$

$$\tilde{\mu} = \mu dt + dW \tag{6}$$

$$x(t) = (1 + \tilde{\mu}_N)(1 + \tilde{\mu}_{N-1}) \cdots (1 + \tilde{\mu}_1)x(0) \tag{7}$$

- This can be viewed as a stochastic logistic map with two straight lines - slope 1 and slope $1 + \mu dt + dW$. A new dW generated with each step.
- Since the dW at different times are uncorrelated and $< dW >= 0, < x(t) >= e^{\mu t}$.
- From logistic maps - competition between μ and σ.
- Why is the ensemble average not representative of the evolution of a typical agent?

- Extreme growth - very rare that cancels out decline of wealth in typical run.
- If you look at typical trajectories for a time t then you need of order $N \sim e^t$ samples to get outlier. (Peters and Klein, S. Redner, Am Journal of Phys. bf 58, 267 (1990))
- Re-weight by looking at logarithm

$$x(t+dt) - x(t) = (\mu dt + \sigma dW)x(t) \tag{8}$$

$$x(t+dt) = (1 + \mu dt + \sigma dW)x(t) \tag{9}$$

$$d\ln x(t) = \ln(1 + \mu dt + \sigma dW) \tag{10}$$

- Expanding $\ln$ and keeping terms up to of first order in dt

$$d \ln x(t) = \mu dt + \sigma dW - \frac{\sigma^2}{2} dW^2 \tag{11}$$

where the last term comes from dW scaling as $\sqrt{dt}$.

- Taking the ensemble average

$$< d \ln x(t) >= (\mu - \frac{\sigma^2}{2}) dt \tag{12}$$

$$d \ln x(t) = (\mu - \frac{\sigma^2}{2}) dt + \sigma dW \tag{13}$$

- This is consistent with the figure. Ito correction
- Note that the wealth of the typical agent depends on the competition between μ and σ. Fixed μ the higher the volatility the less equal the economy for the GRW.

The Asset Exchange Model - AEM

- In GRW the growth and the effect of volatility depends on a individuals wealth. No exchange of assets and no income redistribution(tax).
- Models in which there is no growth but exchange between agents are referred to as the asset exchange models.(A. Chakraborti et al Quant Finance **11**, 1013 (2011))
- Two agents are chosen randomly from N. A fraction α of the wealth of the poorer agent is transferred from the loser of a coin toss to the winner. (B. Boghosian, (arXiv 1212.6300 (2012))
- After many iterations one agent has almost all of the original wealth independent of the initial distribution.

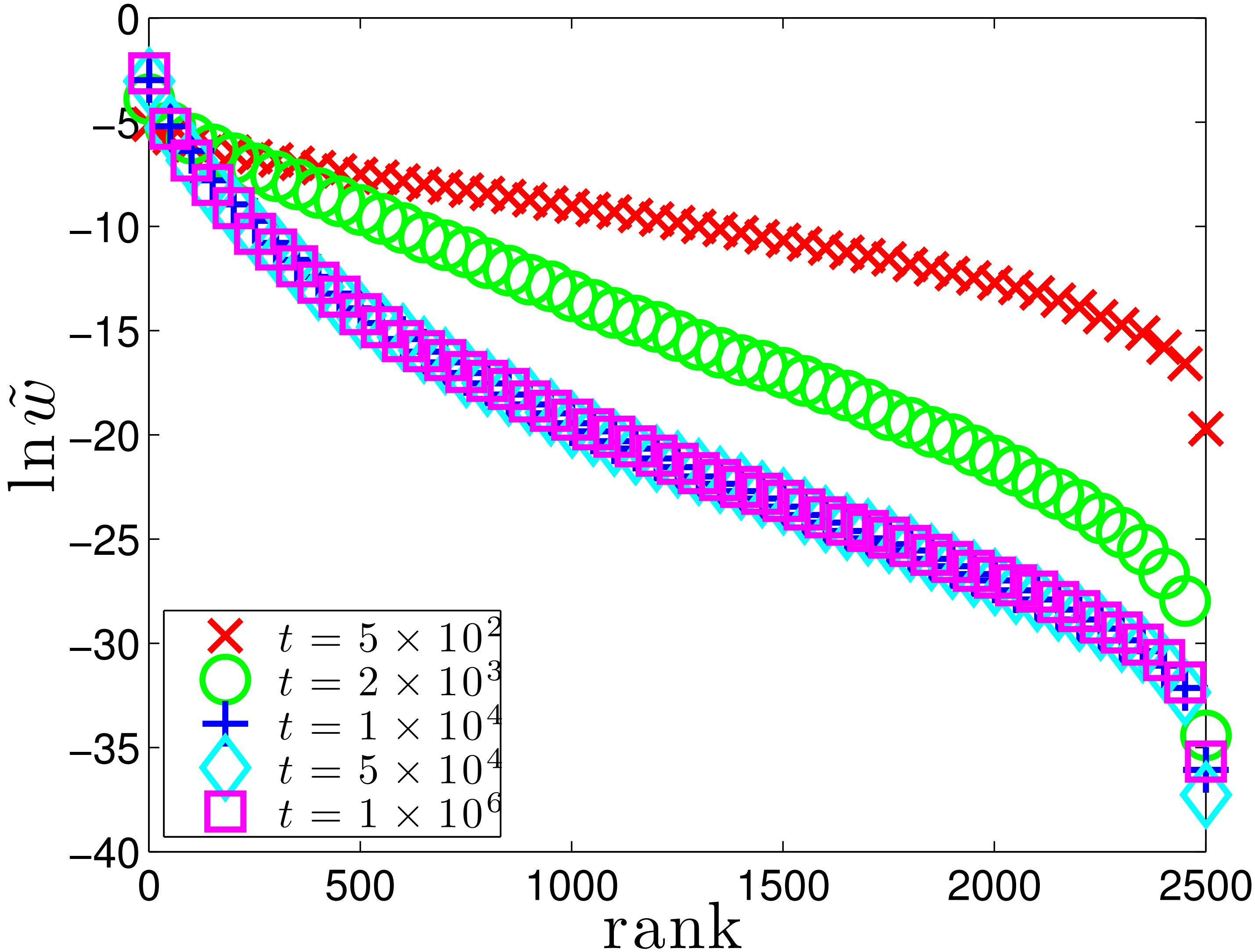
Modified AEM

- We modify the AEM by adding growth.
- After N exchanges we add to the system an amount $\Delta W(t+dt) = \mu W(t)$ where $W(t)$ is the wealth in the system at time t.
- We distribute the wealth according to

$$\Delta w_i(t) = \mu W(t) \frac{w_i^\gamma(t)}{S(t)} \tag{14}$$

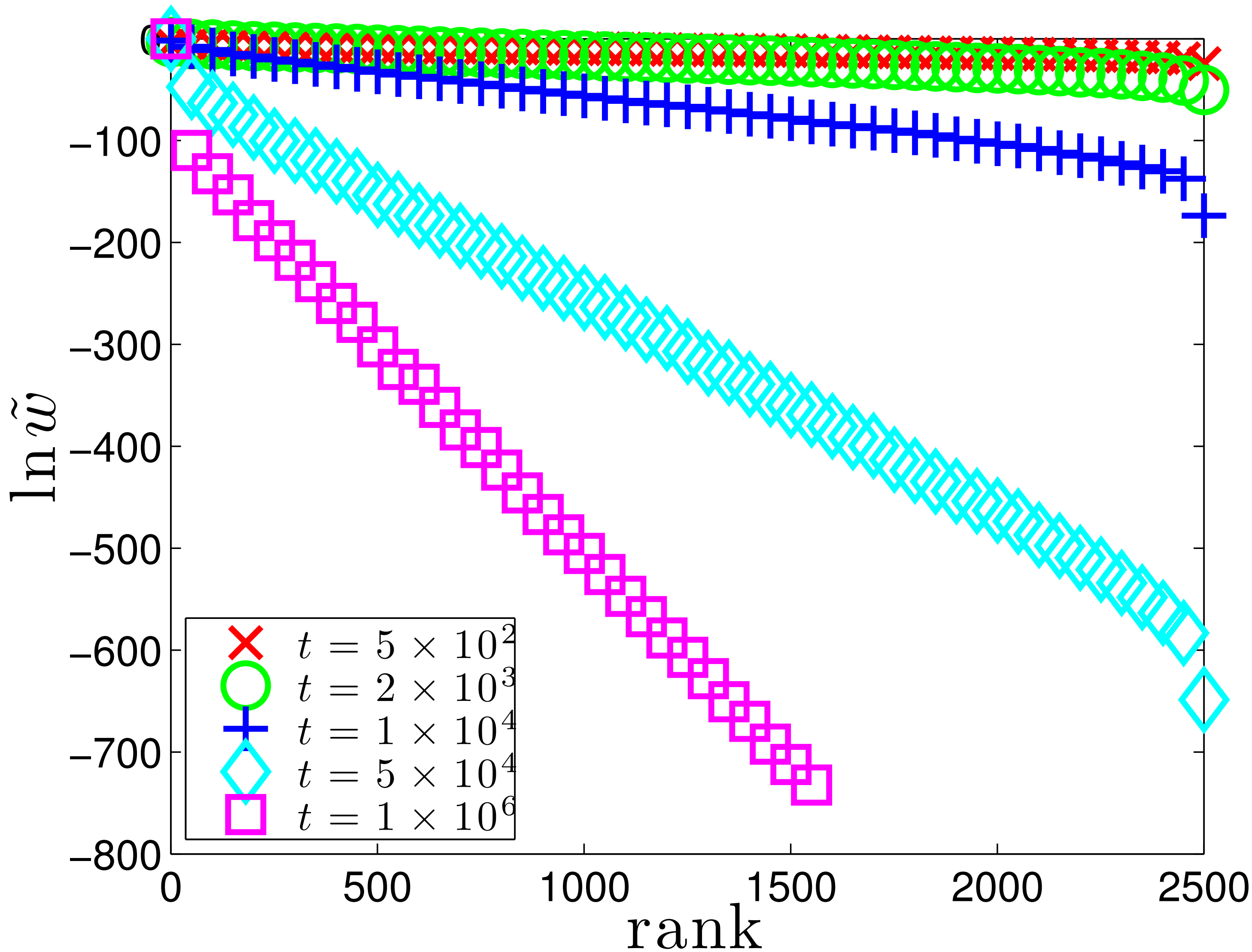
where

$$S(t) = \sum_{j=1}^{N} w_j^\gamma \tag{15}$$



$N = 2500$, $\mu = 1 \times 10^{-3}$, $\alpha = 0.1$, $\gamma = 0.9$, $\tilde{w}$ is rescaled wealth, x axis is rank.

- After a transient period the distribution of the scaled wealth reaches a steady state.

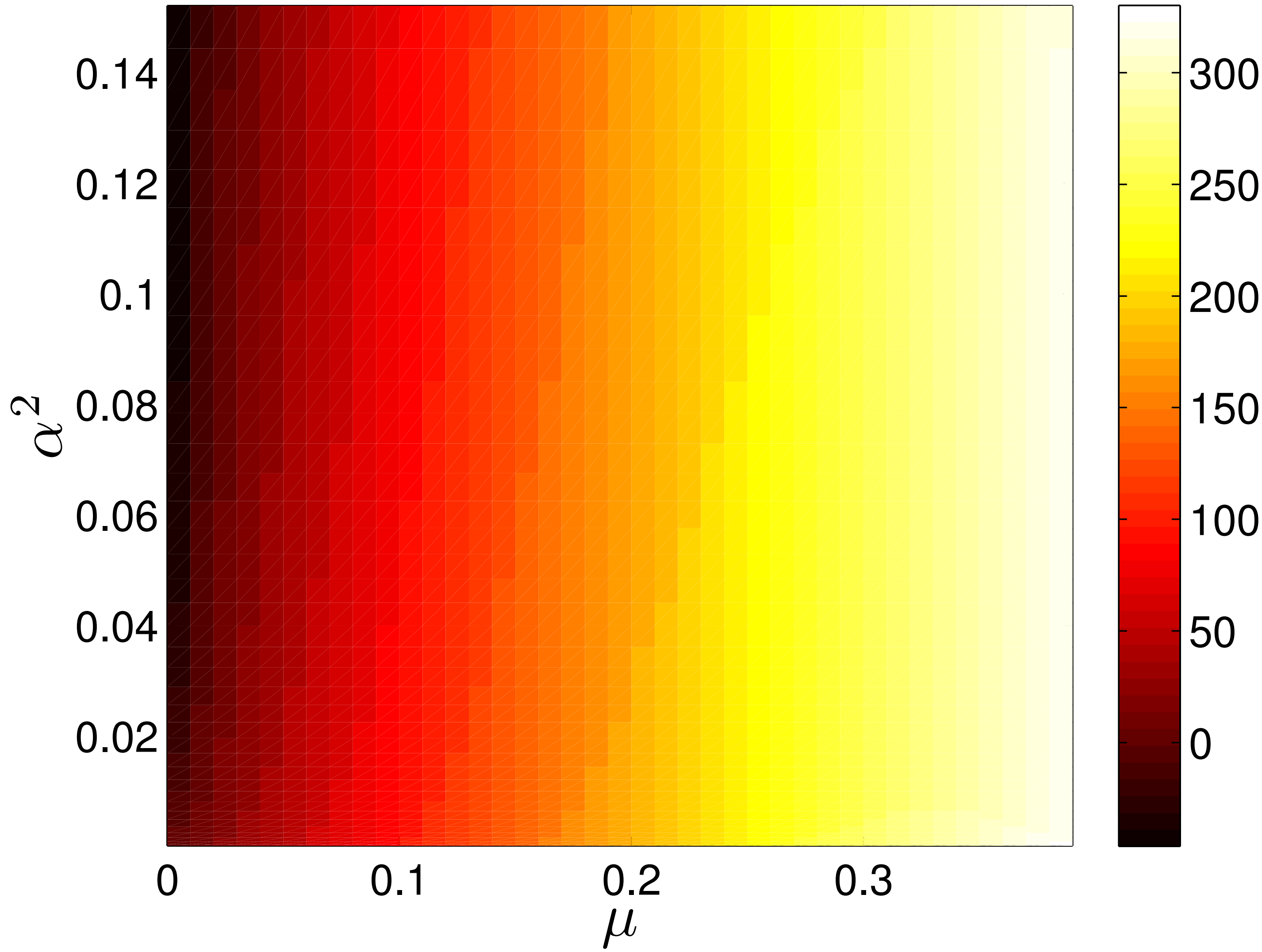


Parameters are same as previous figure except $\gamma = 1.1$

- No steady state. Richest agent gets wealth associated with the growth and the original wealth (initial condition).

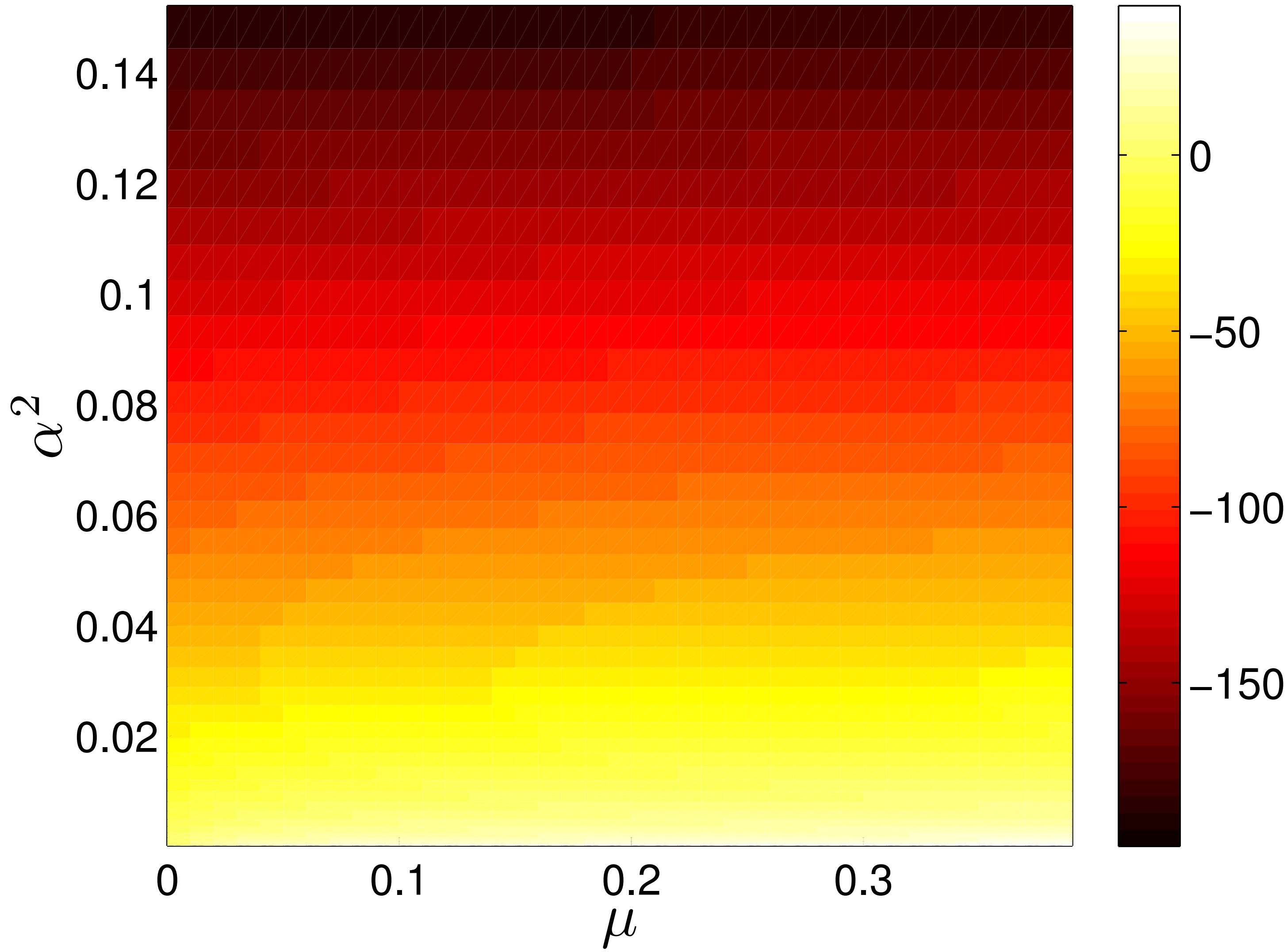
Effect of Gamma

- The parameter γ determines the way that growth is allocated.($\Delta w_i(t + dt) = \mu W(t) w_i^{\gamma}(t)/S$)
- $\gamma = 0$ distributes growth equally to all agents.
- $\gamma = 1$ distributes growth proportional to wealth or investment(GRW).
- $\gamma > 1$ wealth is distributed preferentially to the wealthy (monopoly rents - Paul Krugman- N.Y. Times, June 20, 2013).
- If $\gamma = 1$ is considered “natural” then $\gamma < 1$ could be considered income redistribution.(tax plus social programs)



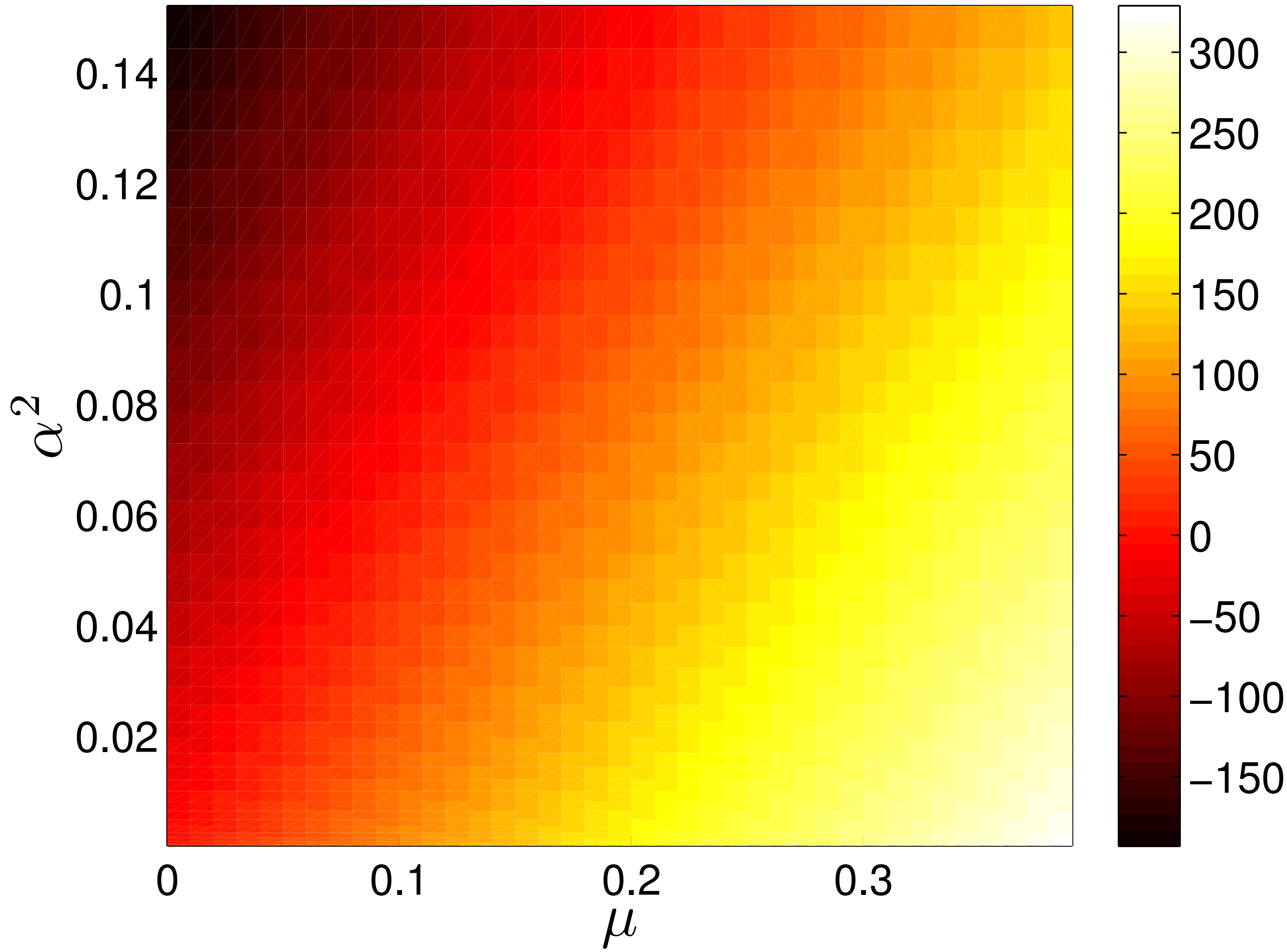
The natural log of the average wealth of the poorest 25 agents, $\gamma = 0.9$, $t \sim 10^6$ (steady state $t = 10,000$)

- Wealth depends weakly on α (volatility) and strongly on μ(growth rate)

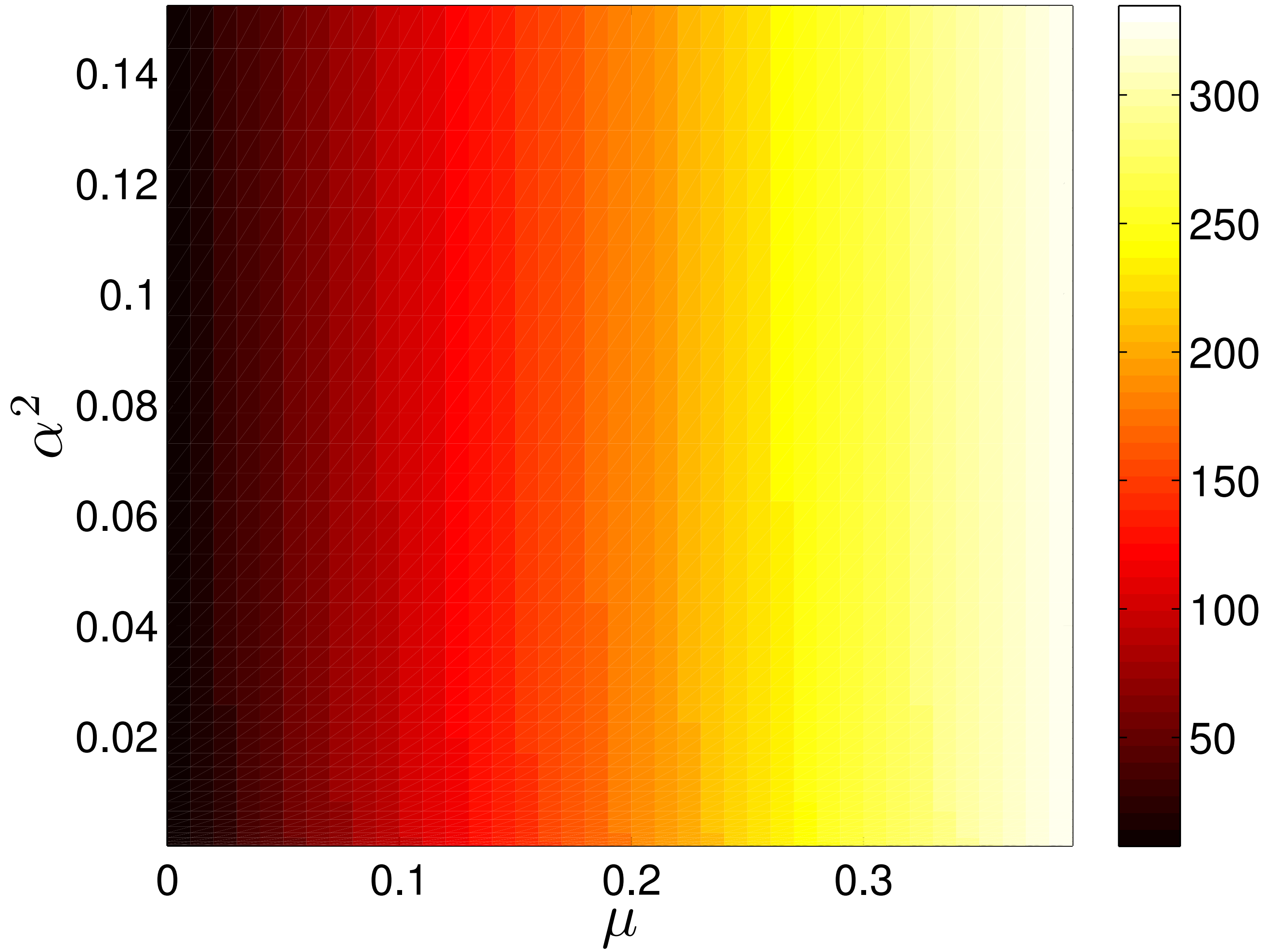


Same as above except $\gamma = 1.1$ $t \sim 10^6$(no steady state)

- Wealth depends “strongly” on α and weakly on μ.

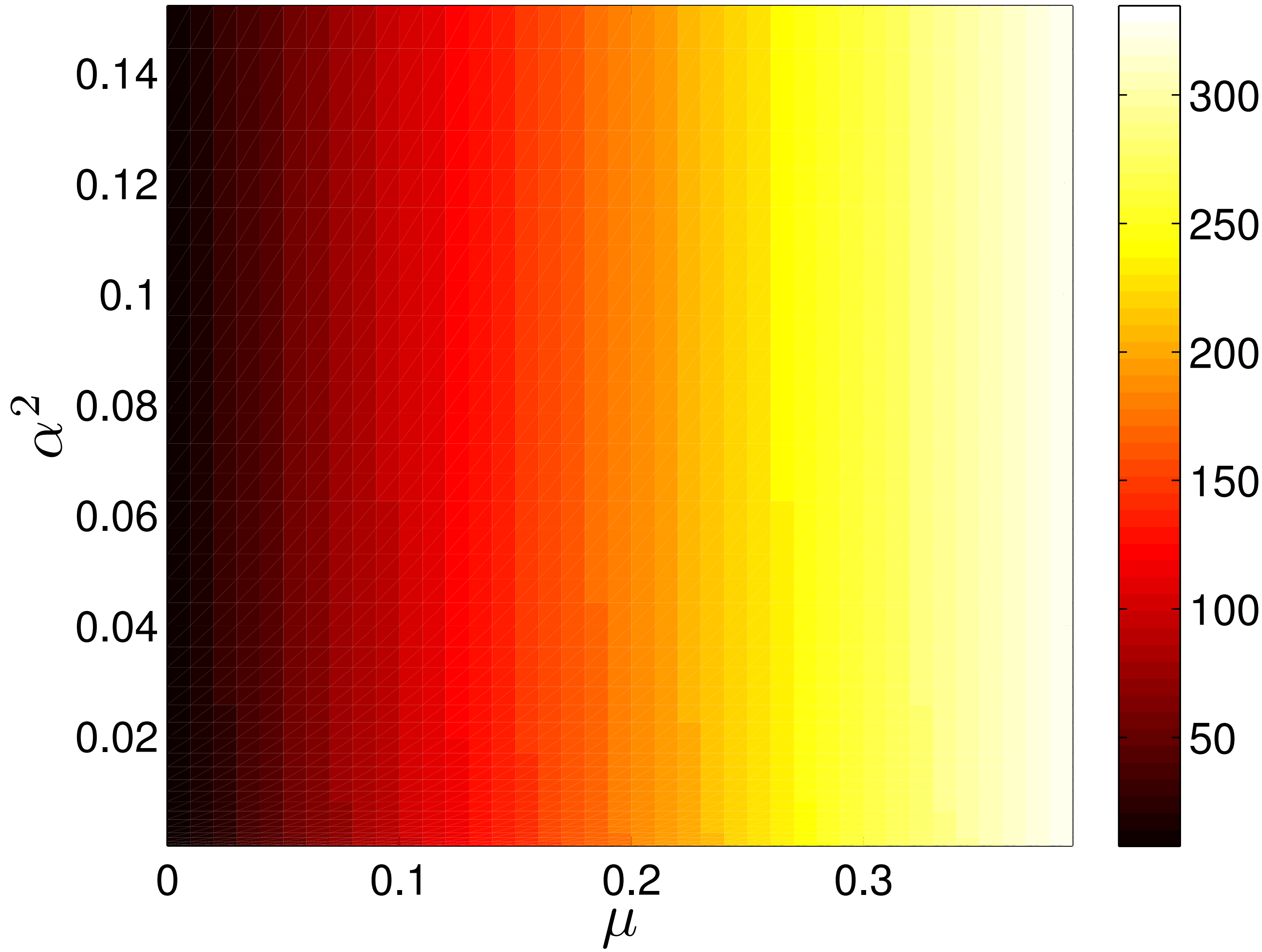


Same as above except $\gamma = 1.0$. Similar to GRW

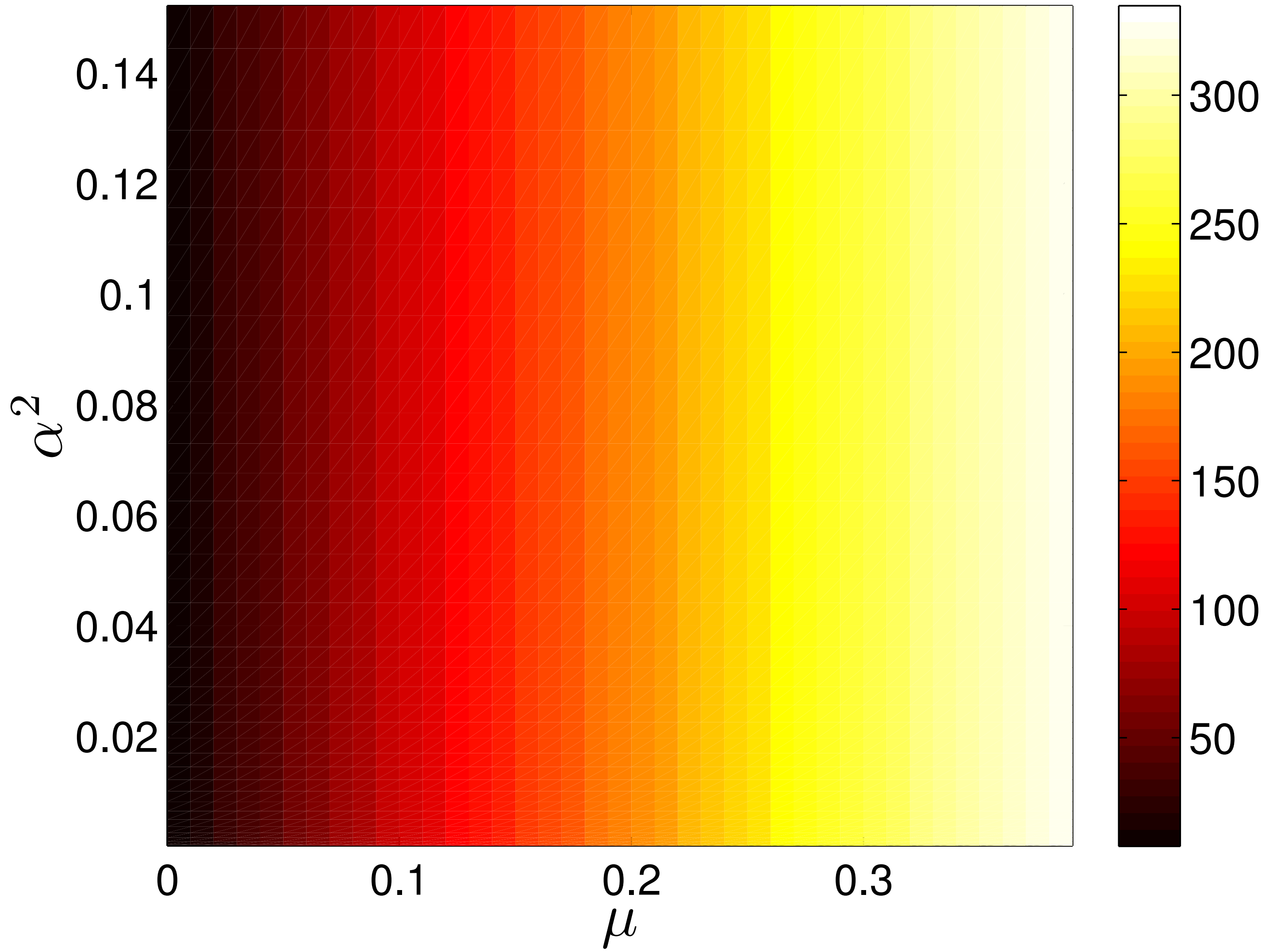


Average wealth of the richest 25, $\gamma = 0.9$

- Virtually no dependence on α.

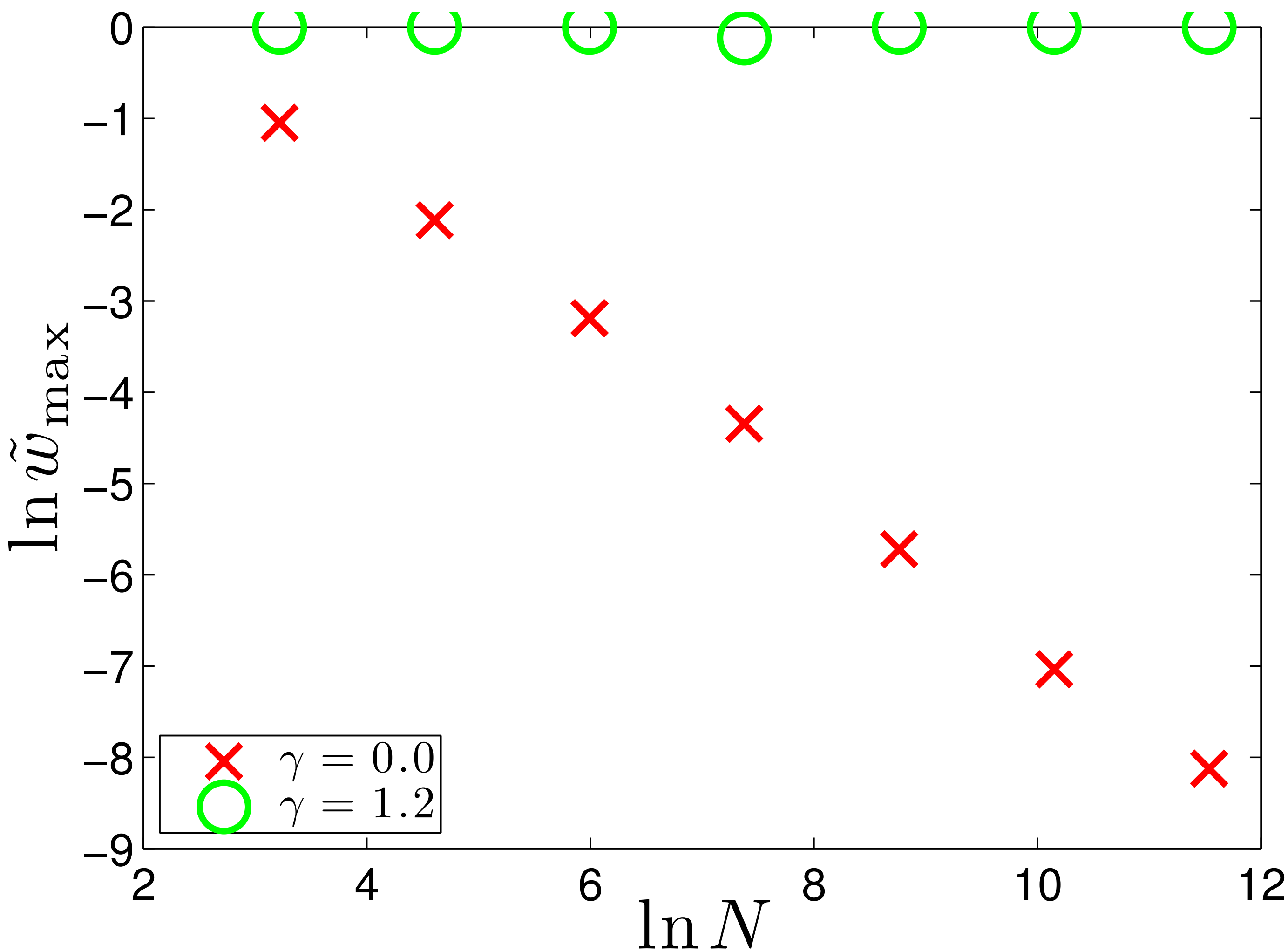


Same as above except $\gamma = 1$.



Same as above except $\gamma = 1.1$.

- Clearly the richest are not affected by the volatility.



The $\ln$ of the rescaled wealth of the wealthiest agent vs $\ln N$ for $\gamma = 0$ and $\gamma = 1.2$.

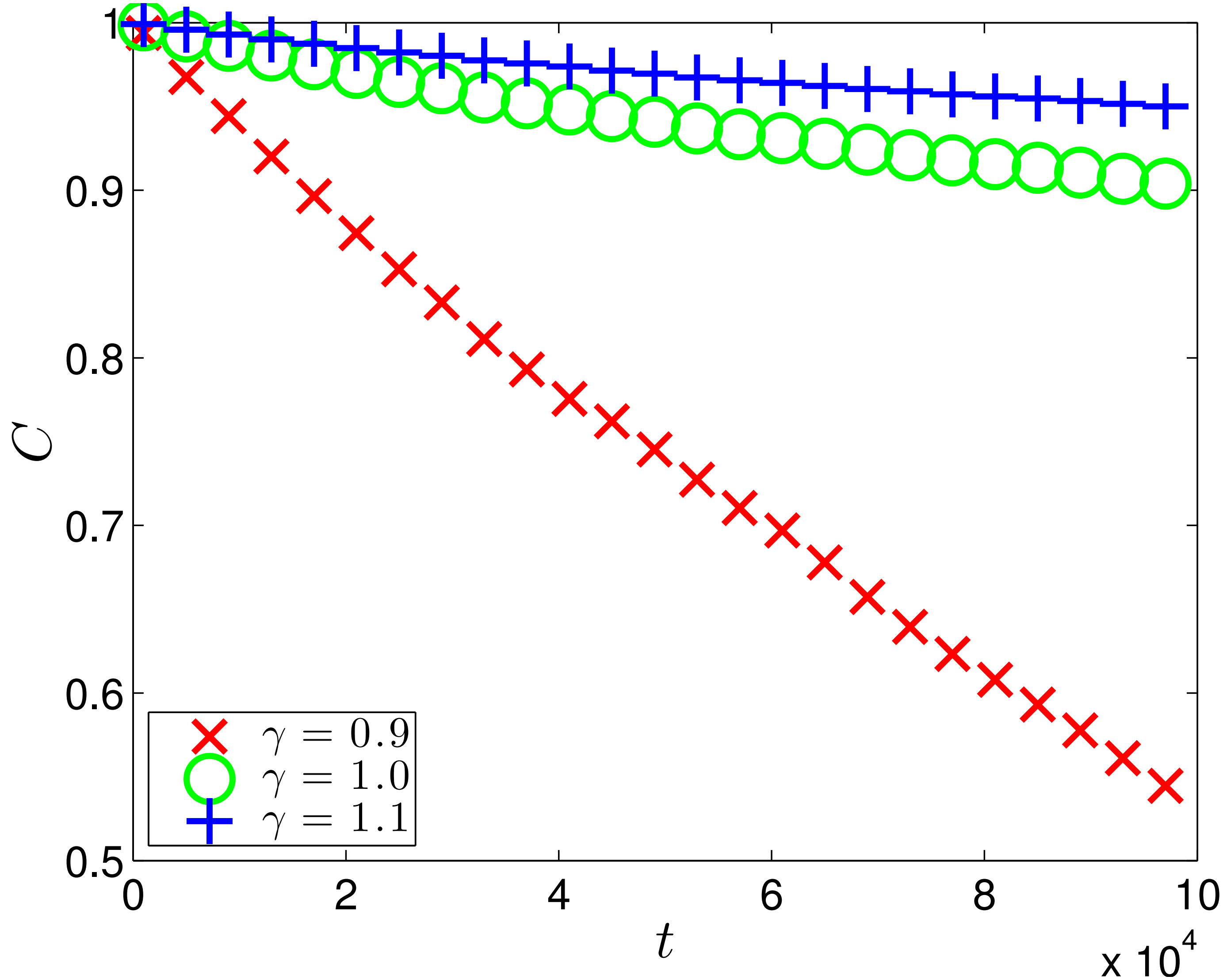
Economic Mobility

- How does the parameter γ affect mobility. i.e. Is the system ergodic?
- We use two measures: One is the Pearson correlation function, J. L. Rogers and W. A. Nicewander, Amer. Stat. **42**, 59 (1988)

$$C(t) = \frac{\sum_i \left[R_i(t) - \overline{R}(t)\right]\left[R_i(0) - \overline{R}(0)\right]}{\sqrt{\left[\sum_j \left(R_j(t) - \overline{R}_j(t)\right)^2\right]\left[\sum_k \left(R_k(0) - \overline{R}_k(0)\right)^2\right]}}; \tag{16}$$

where $R_j(t)$ is the rank of the jth agent and $\overline{R}(t) = N/2$ is the ensemble average of the rank.

- We plot $C(t)$ for three different values of γ.



- $C(t) \to 0$ as $t \to \infty$ for $\gamma<1$. While $C(t)$ appears to approach a non-zero constant for $\gamma \geq 1$

- The second measure is the Thirumalai-Mountain(TM) metric D. Thirumalai and R. Mountain, Phys. Rev. A 42, 4574 (1990) and Phys. Rev. E 47, 479 (1996).
- Take a quantity associated with one of the N agents such as the rescaled wealth.
- Form the time average for each agent $\bar{w}_j(t)$ and the ensemble average of the time average $<\bar{w}(t)>$.

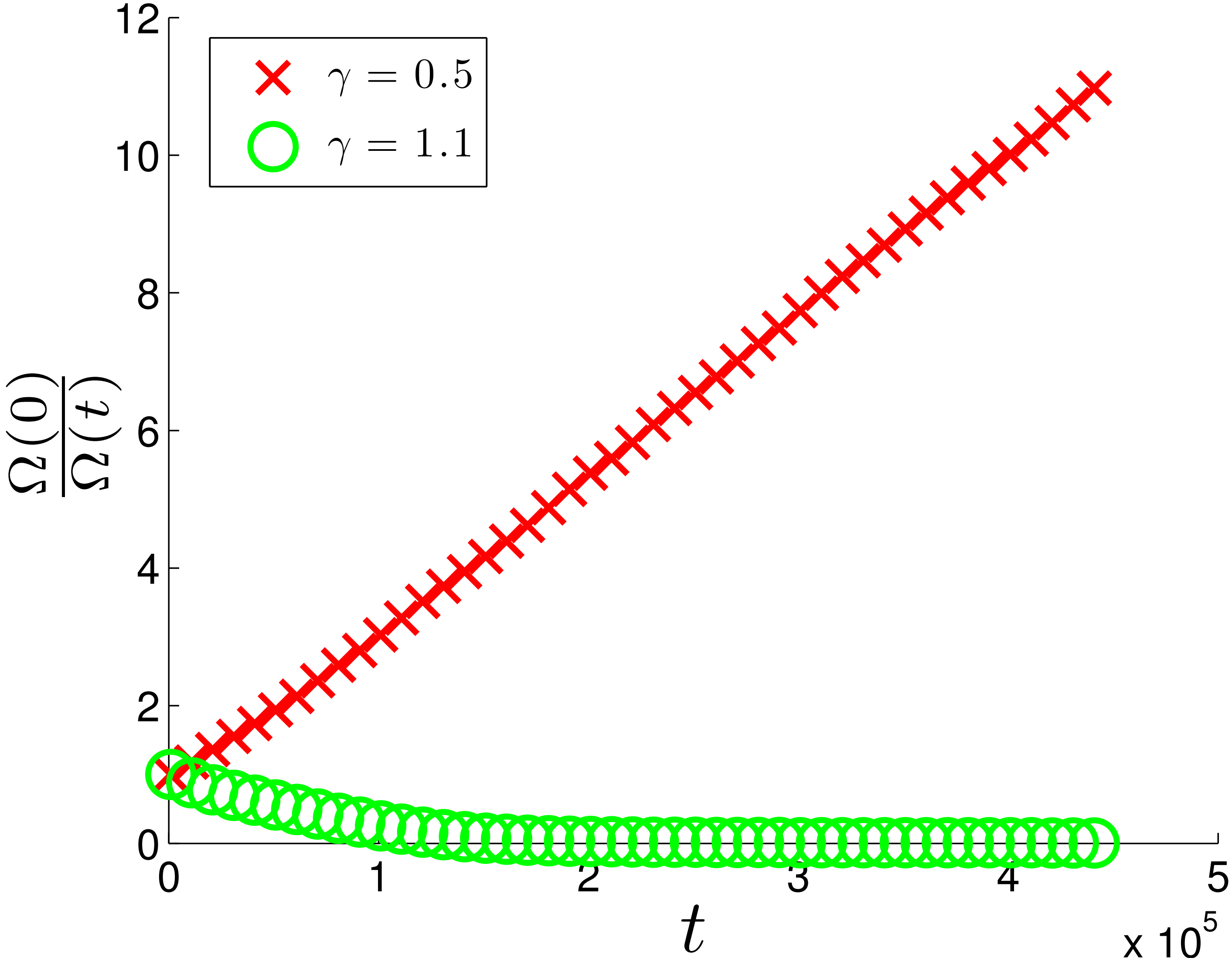
$$\bar{w}_j(t) = \frac{1}{t}\int_0^t w_j(t')dt' \tag{17}$$

$$<\bar{w}(t)> = \frac{1}{N}\sum_{j=1}^{N} \bar{w}_j(t) \tag{18}$$

The TM metric is defined as

$$\Omega_w(t) = \frac{1}{N}\sum_{j=1}^{N}\left[\bar{w}_j(t) - <\bar{w}(t)>\right]^2 \tag{19}$$

- Compares time average to ensemble average of time average.
- Does not measure ergodicity but effective ergodicity.
- If the system is effectively ergodic, then $\Omega_f(t) \propto 1/t$.(TM)
- Time averages are the same.



$\gamma = 0.5$
$\gamma = 1.1$
$\frac{\Omega(0)}{\Omega(t)}$
t
x 10^5
0
2
4
6
8
10
12
0
1
2
3
4
5

Phase Transition

- Data indicates that $\gamma = 1$ is a phase transition.
- For $\gamma < 1$ a distribution of wealth is established during a transient period.
- Once the steady state is reached each individuals wealth grows as $e^{\mu t}$.
- As γ approaches 1 from below steady state becomes less equal. Bigger spread between richest and poorest.
- The time to establish the steady state diverges as

$$\tau = \frac{1}{(1 - \gamma)} \tag{20}$$

MAEM Limiting Case

- To make "physics" of the MAEM clearer and to make contact with the GRW look at a limit of the MAEM.
- Differential equation for the MAEM (coin flip 1/2)

$$dx_j(t) = \frac{\alpha}{2}\sum_k \Theta(x_j(t) - x_k(t))\eta_{jk}x_k(t)dt + \tag{21}$$

$$\frac{\alpha}{2}\sum_k \big(1 + \Theta(x_j(t) - x_k(t))\big)\eta_{jk}x_j(t)dt + \mu e^{\mu t}\frac{x_j^\gamma(t)}{S}dt$$

$S = \sum_k x_k^\gamma(t)$ as above, $\Theta(x_j(t) - x_k(t))$ is the step function (= 1 if $x_j(t) > x_k(t)$, zero otherwise) η_{jk} is a random anti symmetric matrix with all zeros except for one $\pm$ 1 pair.$(x(0) = 1)$

- Restrict the number of agents to two with one considerably poorer than the other. Equation 21 reduces to

$$dx(t) = \frac{\alpha}{2}\eta x(t)dt + \mu e^{\mu t}\frac{x^{\gamma}(t)}{x^{\gamma}(t) + (e^{\mu t} - x(t))^{\gamma}}dt \quad (22)$$

  $x(t)$ is the wealth of the poorer of the two agents. $e^{\mu t}$ is the total wealth at time t. η is a random variable with values ± 1. $\eta dt = dW$ is a Wiener process.
- For $\gamma = 1$ eq.22 is the GRW. Factoring $e^{\mu t}$ for arbitrary γ

$$dy(t) = \frac{\alpha}{2}y(t)dW + \mu\frac{y^{\gamma}(t)}{y^{\gamma}(t) + (1 - y(t))^{\gamma}}dt - \mu y(t)dt \quad (23)$$

  $y(t) = x(t)/e^{\mu t}$ is the rescaled wealth.

Ignoring the noise and discretizing (logistic map)

$$y(t+dt) = y(t) + \mu \frac{y^{\gamma}(t)}{y^{\gamma}(t) + (1-y(t))^{\gamma}} dt - \mu y(t) dt \tag{24}$$

- Three fixed points: $y(t) = 0, 1/2, 1$: Phase transition.
- For $\gamma<1$ fixed point at 1/2 stable, other two unstable.
- For $\gamma>1$ fixed point at 1/2 unstable other two stable.
- The slope of the r. h. s. of eq.24 approaches 1 as $\gamma \to 1$.
- At $y(t) = 1/2$ slope equals $1-(1-\gamma)\mu$ consistent with MAEM critical slowing down.
- Noise should be rescaled by N (number of agents) and will have minimal effect.

Conclusions-Models

- Models are not necessarily ergodic. - Growth in GDP is not indicative of the growth of wealth of individuals.
- With "natural" growth of individual wealth ($\gamma = 1$) the growth of most agents depends on the relation between μ(growth parameter) and α or σ(volitility).
- $\gamma<1$ (income redistribution-tax) after transient all agents' wealth grows. System appears to be ergodic. Economic mobility.
- $\gamma>1$ (monopoly rents) only richest agents' wealth grows. Lacks economic mobility.
- GRW is a special case of the MAEM. Lacks economic mobility.

Future Work-Models

- Model with finite range wealth transfer - globalization.
- Model on a network.
- Different forms of wealth transfer.
- Pareto index - proportion of population with wealth x greater than x_m is $(x_m/x)^\beta$. May only apply to upper end of income scale.
- Effect of time dependent growth rate (μ) and volatility (α)
- Study how inequality might lead to growth.
- Model is driven dissipative - Nature of phase transition.
- Does ergodicity imply equilibrium?

Future Work-Data

- In GRW the volatility is related to fluctuations in the growth parameter μ. Not so in MAEM or its limiting version. Can we relate fluctuations in stock indices, unemployment, consumer confidence etc. to inequality?
- Are there periods of time when the real economy is ergodic as described by the models?
- If so, what are relaxation times to return to equilibrium after a perturbation?
- Is economy in punctuated equilibrium?
- Does inequality spur growth

Models and the Real World

- Clearly we can never have a totally realistic model of something as complicated as the economy.

  Use of Simple models:

- Essence of the "physics"
- Force us to think quantitatively - expose bias.
- New paradigm-suggest new questions and approaches.