Options Pricing using Monte Carlo Simulations

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1 Introduction
   • Motivation
   • Options
   • Theory

2 Calculations
   • Method
   • Results

3 Conclusions
Introduction

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Options Pricing using Monte Carlo Simulations
Why MC?

- MC has proved to be a robust way to price options
- The advantage of MC over other techniques increases as the sources of uncertainty of the problem increase
- Essential for exotic options pricing where there are no analytical solutions (e.g. Asian options)
- Compare the results of the simulation with the Black-Scholes theory
An option is a type of security which gives the holder the right (NOT the obligation) to buy or sell the underlying asset at a predefined price.

- A **call** option gives the holder the right to buy
- A **put** option gives the holder the right to sell
The two most popular types of options are

- European Options
- American Options

These are often referred to as vanilla options because of their simplicity. More non-standard options are called exotic options.
European Option Payoff

Figure: Payoff from buying a call option

Figure: Payoff from buying a put option
Options are really risky assets to have in your portfolio! Then why bother???

- Speculation
  Big money if you can predict the magnitude and the timing of the movement of the underlying security.

- Hedging
  Insurance against a risky investment
Brownian Motion

Wiener process
- $B(0) = x$
- $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots, B(t_1) - B(0)$ are independent random variables
- For all $t \geq 0$ and $\Delta t > 0$, $B(t + \Delta t) - B(t)$ are normally distributed with expectation 0 and standard deviation $\sqrt{\Delta t}$

Geometric Brownian Motion
A stochastic process $S_t$ is said to follow a GBM if it satisfies the following stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$  \hspace{1cm} (1)

where $W_t$ is a Wiener process, $\mu$ is the drift used to model deterministic trends and $\sigma$ is the volatility used to model unpredictable events.

For an arbitrary initial value $S_0$, the analytical solution of equation (1) is given by

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$  \hspace{1cm} (2)
If the variable $x$ follows the Ito process

$$dx = a(x, t) \, dt + b(x, t) \, dz$$

then a function $G$ of $x$ and $t$ follows the process

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \right) dt + \frac{\partial G}{\partial x} b \, dz$$

Hence, for $dS = \mu S \, dt + \sigma S \, dz$ we get

$$dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \, dz$$
Using Ito’s Lemma with \( G = \ln S \) produces an interesting result!

Since

\[
\frac{\partial G}{\partial S} = \frac{1}{S}, \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}, \quad \frac{\partial G}{\partial t} = 0
\]

it follows from equation (5) that

\[
dG = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma \, dz
\]

The change of \( \ln S \) between time 0 and a future time \( T \) is therefore normally distributed with mean \( (\mu - \sigma^2/2)T \) and variance \( \sigma^2 T \).

Hence,

\[
\ln S_T - \ln S_0 \sim \phi \left( \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right)
\]

(6)
If $Y$ is $LN[m, s^2]$, then $E(Y) = e^{m + \frac{s^2}{2}}$

In our case $m = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T$ and $s = \sigma \sqrt{T}$

Hence,

$$E(S_T) = S_0 e^{\mu T} \quad (7)$$

The expected return $\mu$ is driven by
- The riskiness of the stock
- Interest rates in the economy
<table>
<thead>
<tr>
<th>Introduction</th>
<th>Motivation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculations</td>
<td>Options</td>
</tr>
<tr>
<td>Conclusions</td>
<td>Theory</td>
</tr>
</tbody>
</table>

Black-Scholes Assumptions

Assumptions

- The stock price follows geometric Brownian motion
- The short selling of securities with full use of proceeds is permitted
- No transaction fees or taxes. All securities are perfectly divisible
- No dividends
- No arbitrage
- Continuous trading
- The risk-free rate $r$ is constant and the same for all maturities
The stock price process is assumed to follow

\[ dS = \mu S dt + \sigma S dz \]  \hspace{1cm} (8)

Suppose that \( f \) is the price of a derivative of \( S \), the variable \( f \) must be some function of \( S \) and \( t \). Hence from Ito’s Lemma we get,

\[ df = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz \]  \hspace{1cm} (9)
Consider the portfolio
\[ \Pi = -f + \frac{\partial f}{\partial S} S \]  
(10)

The change \( \Delta \Pi \) in the value of the portfolio in the time interval \( \Delta t \) is given by
\[ \Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S \]  
(11)

Using equations (8) and (9) into equation (11) yields
\[ \Delta \Pi = \left( -\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t \]  
(12)

Notice that this is a riskless portfolio!
For a risk-free portfolio we have \( \Delta \Pi = r\Pi \Delta t \). Hence, substituting in equation (12) we get the Black-Scholes-Merton differential equation
\[ \frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \]  
(13)
For European options we have

\[ f_{\text{call}} = \max(S - K, 0) \quad \text{and} \quad f_{\text{put}} = \max(K - S, 0) \]

For a non-dividend-paying stock, the prices at time 0 are

\[
\begin{align*}
c &= S_0 N(d_1) - Ke^{-rT} N(d_2) \\
p &= -S_0 N(-d_1) + Ke^{-rT} N(-d_2)
\end{align*}
\] (14)

and

\[
\begin{align*}
d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \\
d_2 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}
\end{align*}
\]

\(N(x)\) is cumulative probability distribution function for a standardized normal distribution \(\phi(0, 1)\).
Notice that the variables that appear in the equation (13) are all independent of the risk preferences of the investors.

No $\mu \rightarrow$ No risk dependence $\rightarrow$ Any rate can be used when evaluating $f$

**We assume that all investors are risk neutral!**

Risk-neutral valuation of a derivative

- Assume $\mu = r$
- Calculate the expected payoff from the derivative
- Discount the expected payoff using $r$
Calculations
The Monte Carlo method can be divided in three main steps:

- Calculation of the potential future price using GBM
- Calculation of the pay-off for this price
- Discount the pay-off back to today’s price

Repeating the above procedure for a reasonable number of times, gives a good estimate of the average pay-off and the price of the option.
Input parameters of the Monte-Carlo simulation

- The initial price of the underlying stock: 100
- The strike price at maturity: 102
- The expected annual return: 1%
- The risk-free annual rate: 1%
- The expected annual volatility: 20%
- Number of steps: 252
- Years to maturity: 1
- Number of trials: 2500
Figure: Typical output of the simulation for 1% drift

Figure: Typical output of the simulation for 10% drift
Figure: Lognormal distribution of the stock price at maturity
Figure: Options Price versus Initial Price
Figure: Options Price versus Strike Price
Figure: Options Price versus Return Rate
Figure: Options Price versus Volatility
Convergence Tests versus the Expectation Value $\overline{S_T}/E(S_T)$

Figure: Time steps versus number of trials compared with $E(S_T) = S_0 e^{rT}$
Convergence Tests

Figure: Price versus number of trials
Figure: Time steps versus number of trials compared with the normalized call price
Figure: Options price versus time steps
Effects of the Strike Price on Convergence

Figure: Normalized call price versus time steps for different K-values

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Effects of the Strike Price for different Volatilities

Figure: Normalized call price versus strike price

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Effects of the Strike Price

Figure: Normalized put price versus strike price
Trials and Effects of the Strike Price

![Graph showing normalized call price versus number of trials for different strike prices and trial numbers.](image)

Figure: Normalized call price versus number of trials for $K = 150$ for different number of trials
Steps and Number of Trials

N=2500

Figure: Normalized call price versus steps for different K values
Figure: Normalized call price versus steps for different K values
Figure: Normalized expected return rate versus risk-free rate \( \frac{\mu}{r} \)
Girsanov’s Theorem

A powerful theorem that allows us to translate any result at any expected rate to the risk-free world.
In other words, the rate we use to price the derivative is irrelevant.
Conclusions
Very few time steps may not give converged results. They become irrelevant after some point.

More trials produce results with less uncertainty but the computational cost increases.

Important to use the same rate both for Monte Carlo and Black-Scholes.

The results of the Monte Carlo approach are very accurate compared to the Black-Scholes for reasonable parameters.
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