Methods of Pricing American Options

Arshan Tarapore
Department of Economics, Boston University
Agenda

- Introduction/Recap of Options
- Binomial Options Pricing Model (BOPM) (in-depth)
- Quadratic Approximations
- Longstaff-Schwartz Method
- Future Work/Conclusion
Introduction/Recap of Options

• American v. European
  • Main difference is in the right to exercise

• Call
  • Right but not obligation to buy an asset

• Put
  • Right but not obligation to sell an asset

• Main uses of Options
  • To hedge risk
  • Speculation

• What is Arbitrage, why do we assume it doesn’t exist?
Binomial Options Pricing Model (BOPM)

- Simple discrete-time model to value options.
- “the value of the call can be interpreted as the expectation of its discounted future value in a risk-neutral world” (Cox, Ross, Rubinstein 1979)
- $u > r > d$

\[
\begin{align*}
S & \quad \text{with probability } q, \\
S & \quad \text{with probability } 1-q.
\end{align*}
\]
\[
\begin{align*}
C_u &= \max[0, uS - K] \quad \text{with probability } q, \\
C_d &= \max[0, dS - K] \quad \text{with probability } 1-q.
\end{align*}
\]
\[
\begin{align*}
\Delta uS + rB & \quad \text{with probability } q, \\
\Delta S + B & \quad \text{with probability } 1-q.
\end{align*}
\]
\[
\begin{align*}
\Delta dS + rB & \quad \text{with probability } 1-q.
\end{align*}
\]
Binomial Options Pricing Model (BOPM)

- We select $\Delta$ and $B$, so:
  - $\Delta uS + rB = Cu$
  - $\Delta dS + rB = Cd$

- Solving the above leads to:
  - $\Delta = C \downarrow u - C \downarrow d / (u - d) S$
  - $B = uC \downarrow d - dC \downarrow u / (u - d) r$

- This combination is known as the “hedging portfolio”
  - No riskless arbitrage
Binomial Options Pricing Model (BOPM)

• With the assumption of no riskless arbitrage:
  \[ C = \Delta S + B \]

  \[
  C = C\downarrow_u - C\downarrow_d / (u-d) + uC\downarrow_d - d \\
  C\downarrow_u / (u-d) r
  \]

\[
C = [(r-d/u-d)C\downarrow_u + (u-r/u-d)C\downarrow_d]1/r
\]

• To simplify, define:
  \[ p \equiv (r-d/u-d) \]
  \[ (1-p) \equiv (u-r/u-d) \]

• Thus:
  \[
  C = [pC\downarrow_u + (1-p)C\downarrow_d]1/r
  \]
  if \( C > (S-K) \)
  otherwise; \( C = S-K \)

Call Option:
Binomial Options Pricing Model (BOPM)

• The BOPM can be taken further to a more complicated scenario

• In this “2-phase” case:
  \[ C_{\downarrow u} = \frac{pC_{\downarrow uu} + (1-p)C_{\downarrow ud}}{r} \quad C_{\downarrow d} = \frac{pC_{\downarrow du} + (1-p)C_{\downarrow dd}}{r} \]

• \( \Delta, B \) chosen; and \( C_{\downarrow du} = C_{\downarrow ud} \):
  \[ C = \left[ p \uparrow 2 \ C_{\downarrow uu} + 2p(1-p)C_{\downarrow ud} + (1-p) \uparrow 2 \ C_{\downarrow dd} \right] \frac{1}{r \uparrow 2} \]

• These get very complicated very quickly

\[ C = \left[ p \uparrow 2 \ max[0, u^2 S - K] + 2p(1-p)\ max[0, du(S-K)] + (1-p) \uparrow 2 \ max[0, d^2 S - K] \right] \frac{1}{r \uparrow 2} \]
Quadratic Approximations

- First attempt to tackle options pricing from a heavily analytical sense
- Attempt to link American Options to Black-Scholes-Merton
- Directed economists to simulation methods (Longstaff-Schwartz)

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0
\]
Longstaff-Schwartz Method

- Use least-squares regression to estimate expected payoff of an option
- Regress discounted future option cash flows on the current price of the underlier associated with *in-the-money* sample paths
- The Risk-Neutral market model used in the simulation is the Stochastic Differential Equation:
  \[ dS = rSdt + \sigma SdZ \]
  - \( r \) is the riskless rate; constant
  - \( \sigma \) is the exposure matrix; constant
  - \( Z \) follows a standard Brownian motion
- Simulation methods like the LSM are accurate, under assumptions known to be wrong (Risk-Neutral)
Future Work/Conclusion

- Multiple-Factor Models/Simulations
- Optimal Exercise
- Behavioral Economics
- Why bother??
References


