# Chapter 2

# The Fractal Coastline and Dimension

ATOMS AND MOLECULES MOVE RANDOMLY. Gas molecules rush around in all directions with different speeds, zigzag, turn, reverse direction. Disordered. Random. Unpredictable.

Sometimes molecules combine to form uneven structures: branching trees, scraggly nerve cells, patchy forests, jagged coastlines. Ragged. Unpredictable.

Other times molecules combine to form structures that are ordered and even: crystals, snowflakes, seashells. Orderly. Symmetric. Predictable.

> Q2.1: Can you think of patterns in nature? Which can be described as ordered? Which can be described as disordered?

What is science good for? For description! To tell us what is happening. To describe the randomness in nature. For prediction! To tell us what is going to happen. To predict the order that often grows from randomness.

This book describes how probability and chance often shape structures around us: coastlines, nerve cells, termite tunnels, bacterial colonies, forests, root systems, river deltas, rough surfaces. These structures and processes may seem hopelessly unpredictable, but every single one has a design or behavior that can be described using the idea of a *natural fractal*. *Natural fractal* is a powerful idea used every day in scientific research. Every natural fractal is different from every other. No one has ever seen the patterns you will grow in the laboratory and by computer, because each pattern is unique. You will use fractals to investigate the dividing line between the known and the unknown.

The following exercise illustrates a strange property of a coastline: The measured length of a coastline depends on the length of the ruler you use. We will find that this and similar properties describe natural fractals.

## 2.1 Coastline

How is measuring the length of a coastline different from measuring the circumference of a circle or the distance around a square? For example: How long is the coastline along an ocean shore? Is its length equal to the mileage reading on the odometer of an automobile driving along the straight road beside the beach? Is it the distance covered by a bicyclist who pedals along the shore on a footpath? Is it the length run by a jogger whose morning exercise includes running "next to the water?" Is it the number of steps of a little bird running along the edge of the water looking for washed-up morsels? Whose "length of coastline" is the true one?

Trying to answer these innocent questions leads to deeper questions and our first look at natural fractals.

Table 2.1: Measuring the length of a coastline.

| 1. Size of step, $S$ | 2. PREDICTED Number of Steps, $N$ | 3. MEASURED Number of Steps, $N$ | 4. "Length" $= N \times S$ |
|---|---|---|---|
| | $\times \times \times \times \times$ | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |

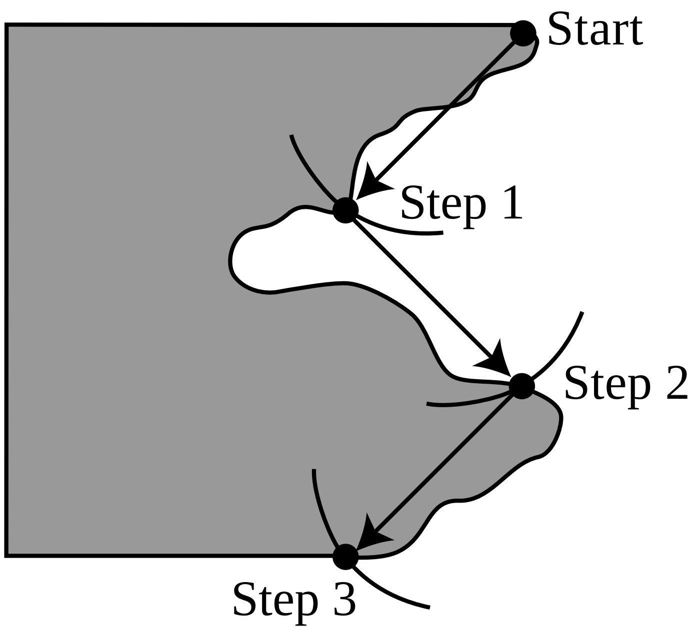



Figure 2.1: Measuring a coastline with caliper. The caliper often "steps across" a peninsula or inlet.

## HandsOn 1: "Walking" Along a Coastline

To investigate coastlines, start by "walking" along the map of a coastline as shown in Figure 2.1.

You will need:

- a map of the coastline to be measured and
- a pair of calipers, i.e., dividers (a caliper or divider is like a circle-drawing compass, except that both ends have points)

1. Make decisions about:

   A. the points on the map between which to measure the length of coastline

   B. the scaled distance on the map for the first setting between the ends of the calipers (preferably a multiple of two, such as 64 kilometers or 32 miles).

2. Separate the points of the calipers by the scaled distance of step 1B. Place the first end of the calipers at the beginning of the coastline to be measured and swing your calipers so that the second end rests on the coastline as close to the starting point as possible. This is your first step along the coastline.

3. Now swing the caliper so that the first end "walks" along to the nearest next point on the coastline. This is your second step.

4. Continue this process, counting the steps as you go; stop when your next step would carry you beyond the end of the coastline. The total number of steps multiplied by the distance covered in each step gives an initial measurement of the length of your coastline. For example: 10 steps, each of a length of 64 kilometers, yields a total length of $10 \times 64$ kilometers $= 640$ kilometers.

5. Enter the size of the step (in miles, kilometers, or whatever) in the left column of your copy of Table 2.1. Enter the number of steps in the third column.

6. Now you are going to start again, this time reducing the distance between the calipers to half its original value. Before you do this, *predict* the number of steps it will take you to walk along the coastline with this smaller step size. Enter your prediction in the second column of your copy of the table.

7. "Walk" along the coastline again with the calipers set to half the original distance. Count the total number of steps and record the results in the correct columns of your copy of Table 2.1. How accurate was your predicted number of steps?

8. Repeat the process of cutting in half the distance between the caliper points as many times as you can, each time making a prediction, taking the walk, and filling in your copy of the table with your results.

9. Fill in the right-hand column of your copy of the table by multiplying each step size by the number of steps to get a total "length."

Q2.2: Did your prediction of the number of steps become more accurate with each new setting of the calipers?

Q2.3: What happens to the total length of the coastline for different step sizes?

Q2.4: What would you estimate the "final length" of the coastline would converge to as the step size gets smaller and smaller?

END ACTIVITY

## HandsOn 2: Measuring the Dimension of a Coastline

Now you are going to plot on graph paper the data you collected from your coastline measurements. A blank piece of graph paper for you to copy is given in Figure 2.2. This is a special kind of graph paper. If you have not used such graph paper before, the uneven scales may appear odd.

These scales, called logarithmic, are explained in the section *Dimensions and Logarithms* at the end of this chapter. Basically, the numbers $2, 4, 8, 16, 32, 64, \ldots$ occur at equal intervals along each axis, as do the numbers $10, 100, 1000, \ldots$ When logarithmic scales are along both axes, the graph paper is called *log-log paper*. For now, you need to know only one thing about this novel scale: You can add a zero to all the numbers along either axis and the graph paper is equally useful. For example, the vertical axis can be relabeled 10, 20, 40, ... 100, 200, 400 ... 1000, 2000, 4000, ... 10,000. The result is perfectly usable providing the numbers you are dealing with are in the corresponding range. Or you can multiply all numbers along either axis by 1/10. For example the vertical axis can be relabeled 0.1, 0.2, 0.4, ... 1, 2, 4, ... 10, 20, 40, ... 100. If the entries in your copy of the table do not match the numbers on one axis of the graph paper, just multiply all the numbers on that axis by 10 or by 1/10 until they do match.

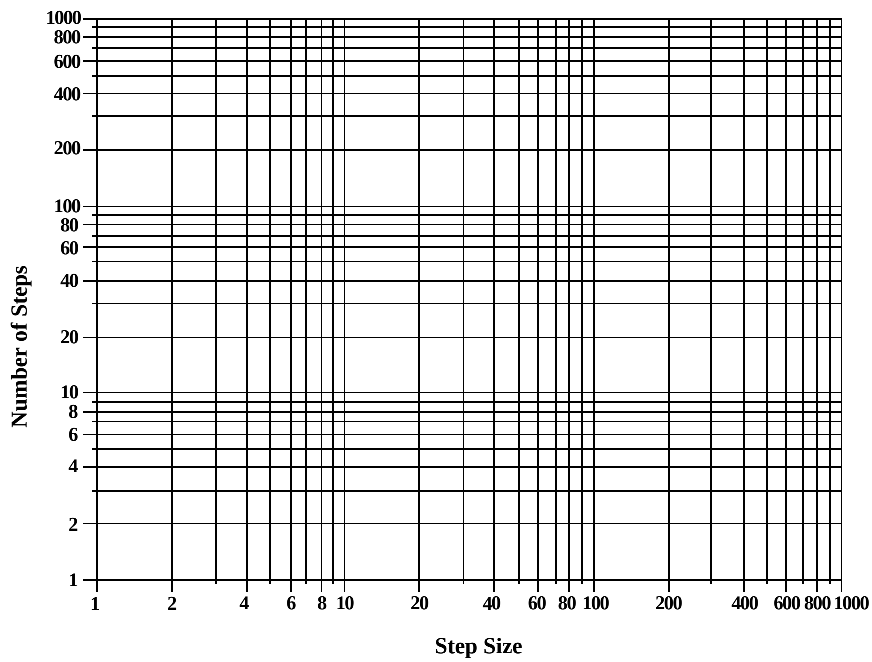



Figure 2.2: Graph paper for plotting the entries in Table 2.1. Do not draw on this grid. Photocopy it; you will need several copies. (This figure is stored in digital format in the accompanying CD-ROM.

1. Plot the numbers from your table on a copy of the graph paper. Plot the number of steps along the vertical axis and the step size along the horizontal axis. What kind of curve do the plotted points form? Do they lie approximately along a straight line? When they lie approximately along a straight line on this kind of graph, it is possible to define the object's *dimension*. The *dimension* is equal to the magnitude of the slope of this straight line. For some objects this slope—this dimension—is not a whole number. Objects with non-integer dimension are called *fractals* (meaning they have "fractional" dimensions).

2. With a ruler, draw by eye the "best" straight line using the points on your graph. Determine the magnitude of the slope of your

graph. The slope is defined as

$$\text{slope} = \frac{\text{change in vertical rise along the line}}{\text{change in horizontal run along the line}} = \frac{\text{“rise”}}{\text{“run”}}. \quad (2.1)$$

These vertical and horizontal changes are laid out using *linear* measures, not logarithmic. Use a regular ruler to measure this slope, not the scale along the axes as shown in the example of Figure 2.3.

Q2.5: What is the magnitude of the slope (the dimension) of your coastline?

Q2.6: Why would a scientist care about the dimension of objects?

END ACTIVITY

Now, to get a better feel for the process of formation of a coastline do at least one of the following two HandsOn activities. The first (HandsOn 3) is a group or classroom activity. The second (HandsOn 4) is an individual activity.

## HandsOn 3: Creating a Rope Coastline

In order to study coastlines further, create a sample coastline using the following instructions. You will need for this activity

- a rope, such as a clothesline, about 30 meters long,
- a single die, and
- a coin.

The activity takes place with the class in a wide hall, entrance hall, or outdoors. Select one class member to roll the die and another to flip the coin.

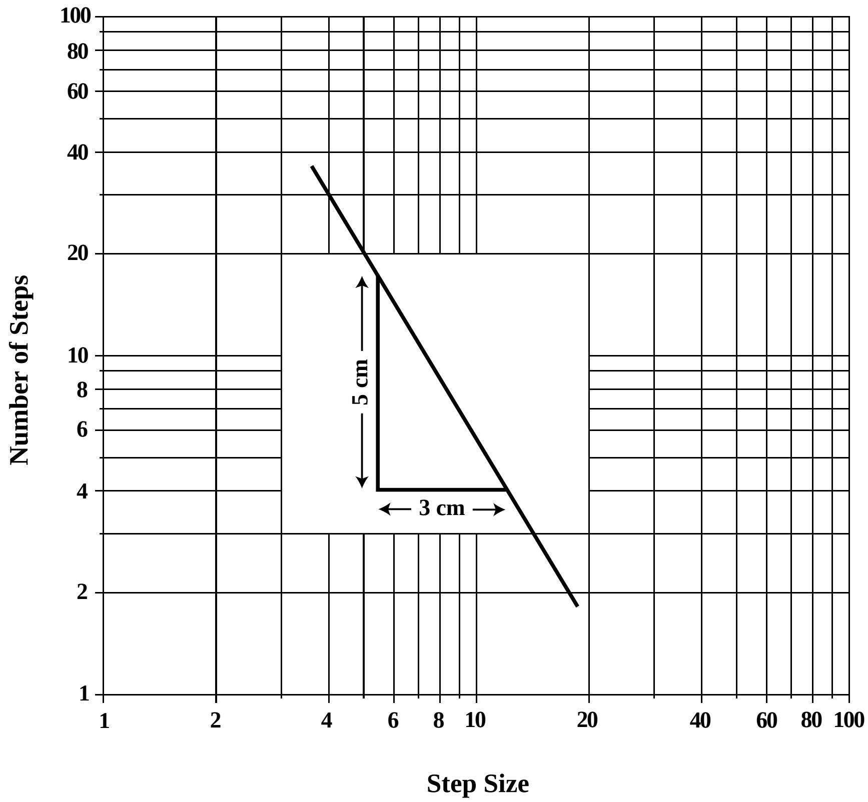



Figure 2.3: Measurement of the slope of a straight line, using the linear coordinates of a regular ruler. The slope is negative, because for the positive "run" there is a negative "rise" (see Eq. 2.1).

1. Two class members, Student #1 and Student #2, stand about 20 feet apart as shown in Figure 2.4. Student #1 holds the end of the rope. Student #2 pulls the rope into a line, keeping the unused portion of the rope coiled at his or her feet.

2. Select a direction perpendicular to the rope to be the positive direction. The opposite direction will be the negative direction.

3. Student #3 grasps the rope approximately in the middle.

4. Flip the coin. Heads or tails determines the direction the person in the middle will move in Step 5 below.

5. Roll the die. The person in the middle of the rope takes a number of "baby steps" (steps that touch heel to toe) perpendicular to the rope. The *number* of steps is equal to the number on the die: from one to six. The *direction* of these steps is determined by the outcome (heads or tails) of the coin flip performed in step 4. Student #2 lets the rope slip out to provide the extra length as needed.

6. Now Students #4 and #5 take up a position in the middle of each straight segment.

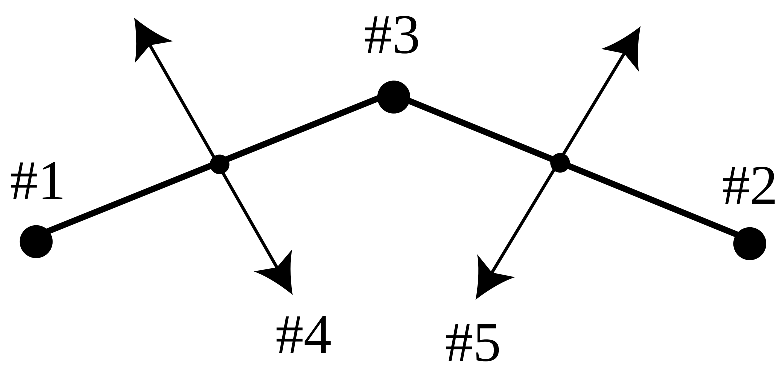



Figure 2.4: The initial steps of creating a rubber band or rope coastline as described in HandsOn 3 and 4.

7. Repeat STEPS 4 through 6 for Student #4, who takes the number of baby steps indicated on the die in a direction perpendicular to the rope indicated by the coin flip. Do the same for Student #5.

8. Continue this process, randomly deflecting the midpoint perpendicular to each straight segment, until you run out of either rope or students.

9. Take a photograph of the resulting "coastline."

END ACTIVITY

## HandsOn 4: Creating a Rubber Band Coastline

Construct a coastline using a rubber band. You will need

- a rubber band,
- a single die, and
- nine or more thumb tacks.

1. Place a blank sheet of white paper on a bulletin board and over it stretch the rubber band between two thumb tacks (as shown in Figure 2.4) a vertical distance 20 centimeters apart. Treat the doubled rubber band as a single line.

2. Locate the midpoint of the rubber band. You are going to move this midpoint right or left a random amount and hold it there with a third thumb tack. Here is how:

3. Roll the die.

   – If you get 1, push the midpoint *left* by 4 centimeters, hold it with the thumb tack.
   – If you get 2, push the midpoint *left* by 2 centimeters, hold it with the thumb tack.
   – If you get 3, leave the midpoint where it is, but hold it with a thumb tack.
   – If you get 4, push the midpoint *right* by 2 centimeters, hold it with the thumb tack.
   – If you get 5, push the midpoint *right* by 4 centimeters, it with the thumb tack.
   – If you get 6, roll again.

4. Now look at the two segments that make up your coastline. Repeat the above process on each of the segments, pushing the midpoint to one side or the other perpendicular to the direction of the segment. Pin each relocated midpoint to obtain a four-segment coastline.

5. Repeat one more time on the four segments to get an eight-segment coastline. Trace the final coastline on the white paper backing and leave it on the bulletin board.

Q2.7: Did the previous two activities teach you anything about the process of forming natural fractals? Can you draw parallels between your own creation of an artificial coastline using a rope or rubber band and the natural process that creates real coastlines?

**END ACTIVITY**

Before we go on to the computer activities, we delve a bit deeper into the meaning of *natural fractals* and how they are formed. There are two types of fractals: non-random (or deterministic) and random (or natural). You will learn more about non-random or mathematical fractals in Section 2.3 on page 26.

Coastlines are examples of *natural fractals*, also called *random fractals*. Natural fractals are formed through random processes, such as erosion of a beach to create coastlines. This erosion was modeled by our coin flipping and die throwing. No two coastlines are ever exactly the same; neither are two snowflakes or two lightning bolts. Though two such patterns may have some overall features in common, if you look closely you will see that they differ in the details of their structure. The same is true for other natural fractals as well. Each example is unique, because the chances are almost zero that exactly the same sequence of random events will occur in the growth of two different patterns, such as two snowflakes. In the previous exercise, you created a model of a fractal coastline using a rubber band, thumb tacks, and a die (or a rope, a die, and a coin). It is unlikely that you and your classmate would get exactly the same sequence of numbers when you throw your die, and therefore it is unlikely that your coastlines will be identical to one another.

The erosion process on a beach is random: the resistance of rocks along the shore as well as the force of incoming waves at a particular spot vary more or less randomly. Thus, when the outcome of the toss of a coin or die tells you to move your "coastline" back one step, you are mimicking this process, and randomly identifying a "weak spot" on the shore that gives in to the force of incoming waves.

Q2.8: Could you have *estimated* the dimension of your coastline to be within certain values just by looking at it? How? Can you think of other natural fractals with:
(a) dimension between 0 and 1
(b) dimension between 1 and 2
(c) dimension between 2 and 3

Q2.9: Can you say anything about the processes that form these fractals?

## SimuLab 1: "Walking" Along a Coastline

*Coastline is also available as a Java applet.*

Now we use a computer program to draw the model of a coastline and to measure the dimension of that model using the "ruler method" (see Figure 2.5).

1. Open the program **Coastline**. Click on the title panel to start the program. An initial straight vertical coastline separates the green land on the left from the blue water on the right.

2. Click on the **Step-by-step** button, so that you can watch the construction of the coastline.

3. Click on the **Draw** button once. What rule do you think the computer uses to start the coastline?

4. Click on the **Draw** button a second time. What does the computer do?

5. Go on clicking the **Draw** button (you can do this up to 10 times).

6. Make additional coastlines by selecting **New** under the **File** menu. You can speed up the drawing by clicking on the **Automatic** button.

Figure 2.5: Screen display in the **Coastline** program.

7. Make several different coastlines. Each time change the **Roughness** setting by clicking on the control boxes containing the symbols $\ll$, $<$, $>$, or $\gg$. Does every roughness setting give you a realistic coastline?

8. Find a roughness setting that gives you a coastline about as jagged as the actual coastline you measured before (even though the overall shape may not be the same). Give this coastline a name using the **Save as** ... command in the **File** menu.

9. Now measure the dimension of your constructed coastline. Go to the **Measure** menu. There are two options, one using a ruler (calipers) and one using the grid. Choose **Using Ruler**, set the ruler length with the size bar to a rather large value, then select **Manual Measure** from the panel at the right side of the screen.

10. One end of the ruler is anchored at one end of the coastline and the other end points toward the cursor. Move the cursor to set the free end of the ruler at a point on the coastline and click the mouse to anchor it, which creates a new ruler.

11. Repeat the process until you have stepped along the whole coastline. When you reach the bottom, press the **Done** button. If you make a mistake, press the **Cancel** button to start over.

12. Change the ruler length, and this time click on the **Auto Measure** button. Then the computer will do the "stepping" for you.

13. Repeat the process with several different ruler lengths. Each result is automatically entered into the data table.

14. Now let the computer analyze the data. Press the **Done Measuring** button. Explore for yourself the **Table** and **Graph** commands under the **Data** menu. This shows the data you have collected and different ways to graph it, similar to what you did by hand in previous activities.

15. The kind of graph we drew in HandsOn 1 to measure dimension is called a log-log graph. (See Section 2.4 on page 29 for an explanation of the log-log graph.) Under **Graph Type** choose **y vs. x (log-log scale)**. The size $L$ of the ruler is plotted along the horizontal axis, and the number $N$ of ruler lengths to follow the coastline along the vertical axis. Click on the **Curve Fit** button to draw the best-fit straight line through the points. The slope is the *dimension* of the coastline, which appears as the exponent in an equation next to the graph. Write down the value of this dimension; you will need it later. Does this dimension have a value approximately equal to the dimension of the map coastline you measured in HandsOn 1?

16. Give this coastline a name you can remember using the **Save as...** command in the **File** menu.

END ACTIVITY

## HandsOn 5: Covering a Coastline with Boxes

Thus far we have used the ruler method for measuring the dimension of a coastline. An alternative method is called the *grid method* or *box method* or *covering method.* The grid method is a bit more versatile than the ruler method, and can be used for different kinds of fractals. Here is how it works:

1. Get out the map whose coastline you measured originally.

2. Take a second piece of blank paper approximately the same size and cut out a *square* that just covers the entire coastline to be measured. Call the edge-length of this square $L = 16$.

3. Now fold the covering square into fourths and cut along the fold-lines. Each of the four squares has an edge length $L = 8$.

4. *Predict:* How many of these $L = 8$ squares will it take to *cover* the coastline? The word "cover" means "cover up": the squares lie along the coastline without overlapping, but so that no piece of the coastline is visible. Enter your prediction in a copy of Table 2.2.

5. Now cover the coastline with $L = 8$ squares and enter the result in your copy of the table. How good was your prediction?

6. Fold each of the smaller covering squares into equal fourths and cut along the fold lines. *Predict* how many of these squares of length $L = 4$ it will take to cover the coastline. Enter your prediction in your copy of Table 2.2.

7. Now cover the coastline with squares of $L = 4$ and enter the result in your copy of Table 2.2.

8. Repeat this process with squares of edge-length 2, and 1. In each case *predict* the number of squares that will be needed to cover the coastline, and enter the result in your copy of the table before measuring and entering the actual number of squares needed to cover the coastline. Were your predictions closer to your measured

values this time than when you predicted results for the ruler method?

10. Get out your original log-log graph of the coastline results. Plot the data from your copy of Table 2.2 on the same graph. This time the vertical scale will be "number of squares to cover coastline" and the horizontal scale will be "edge length of square." Is the result approximately a straight line? If so, is the slope of the line the same as you found using the ruler method?

Table 2.2: Covering the Coastline with Boxes (used in HandsOn 5 and SimuLab 2)

| Edge length of Square | PREDICT: Number of Squares to cover Coastline | MEASURE: Number of Squares to cover Coastline |
|---|---|---|
| 16 | | |
| 8 | | |
| 4 | | |
| 2 | | |
| 1 | | |

Q2.10: Based on your use of the ruler and grid methods, which do you find more efficient? Does it make sense to ask which method is more "accurate"? Explain.

END ACTIVITY

## SimuLab 2: Covering a Coastline

*Coastline is also available as a Java applet.*

Now let the computer help you carry out the grid method quickly and easily.

1. Call up the **Coastline** program again.
2. From the **File** menu, select **Open** and call up the coastline you saved earlier.

3. Under the **Measure** menu, choose **Using Grid**. The size control allows you to adjust the size of the grid.
4. Click on the **Manual Measure** button, as you did previously.
5. Click on every grid square in which any part (even a very small piece) of the coastline shows. When you are done, the resulting set of grid squares should completely cover the coastline. Click on **Done**.
6. Try another grid size. This time click on **Auto Measure** and let the computer cover the coastline. Each result will be entered into the data table automatically.
7. Let the computer cover the coastline with grids of a variety of sizes. When you have used as many grids as you want, click on **Done Measuring**.
8. Now have the computer analyze the data using a log-log plot to find the dimension of your coastline according to the grid method.
9. Compare the value of the dimension measured using the grid method with the value of the dimension you obtained earlier using the ruler method.
10. Using **Save** in the **File** menu, save your coastline under a new name. You will need this file later.
11. Summarize this part of the project by entering values of dimension for both coastlines in a copy of Table 2.3.

Table 2.3: Measured Dimensions of Coastline

| Measurement method | Dimension of actual coastline | Dimension of computer coastline |
|---|---|---|
| Ruler/Caliper | | |
| Grid/Box | | |

END ACTIVITY

## 2.2 Dimension of "Non-Fractals"

We have been measuring the non-integer dimension of coastlines. The shape of a coastline is called a *fractal*. Non-fractal objects have integer dimensions. For example, a line is 1-dimensional. A non-crumpled piece of paper is 2-dimensional. A cube is 3-dimensional, as shown in Figure 2.6.

### HandsOn 6: 1-, 2-, and 3-Dimensions

Fill in a copy of Table 2.4. Now plot the entries in this table on a fresh copy of the log-log graph paper. Along the horizontal axis plot the number of segments and, on the vertical axis, the entries in columns 2, 3, and 4. Draw a line through each set of points and verify that a line, square, and cube are 1-, 2-, and 3-dimensional, respectively.

Table 2.4: Verifying dimensions of a line, a square, and a cube.

| $\ell$ | Number of segments of length $d$ needed to span line of length $L = 100$ | Number of squares of side $d$ needed to cover square of area $L^2 = 10,000$ | Number of cubes of edge $d$ needed to fill cube of volume $L^3 = 10^6$ |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 5 | | | |
| 10 | | | |
| 20 | | | |
| 50 | | | |
| 100 | | | |

Note that in Figure 2.6 the dimension is the exponent in the equation for length or area or volume, as a function of $L$.

END ACTIVITY

### HandsOn 7: Circle Measurement of Dimension

One method we have been using to measure fractal dimension is the method of *covering*. There are other methods for measuring dimension.

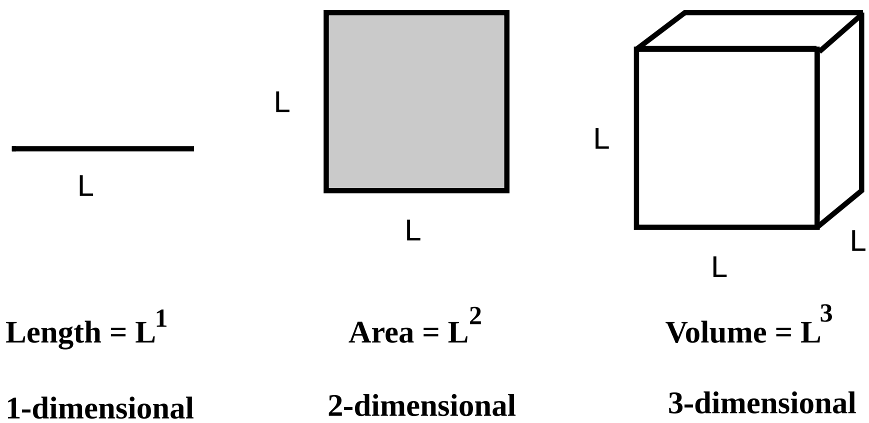



Figure 2.6: Examples of objects with 1, 2, and 3 dimensions.

One that is useful for objects with dimension less than or equal to two involves measuring the total area of the object present inside circles of different radii. For a surface in a plane, the area increases as the square of the radius: $A = \pi R^2$. The power 2 of $R$ tells us—again!—that a disk is 2-dimensional.

Check results of the formula $A = \pi R^2$ by counting the number of cells inside circles of different radii in Figure 2.7. For each circle, count all the cells inside that circle, not just those between that circle and the next smaller circle. Enter your results in a copy of Table 2.5.

Table 2.5: For use with Figure 2.7.

| $r$ (radius) | $N$ (count) |
|---|---|
| 0.5 | |
| 1.5 | |
| 3.5 | |
| 7.5 | |
| 15.5 | |

Plot the results from your copy of Table 2.5 on a log-log graph, the radius scale along the horizontal axis and $N$ along the vertical axis. From the measurement of the slope, calculate the dimension of the circular surface in Figure 2.7.

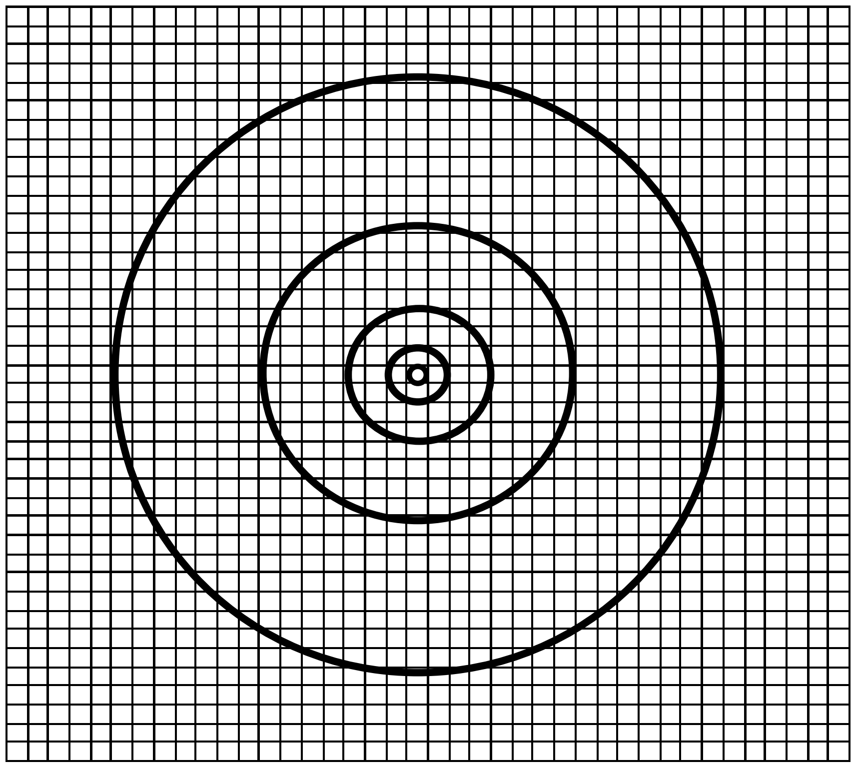


Figure 2.7: Count the cells inside each circle to discover that the surface of the figure is 2-dimensional.

Now measure the dimension of the line in Figure 2.8 using a similar procedure. First, count the number $N$ of short line segments inside each circle of radius $r$. Enter the results in a copy of Table 2.6. Second, plot $N$ versus $r$ on a log-log graph and measure the slope. What is the value of the dimension of the line?

Q2.11: Suppose the line in Figure 2.8 were not not straight, but curved. What dimension would it have, according to the circle method? Cut a piece of string to the length of the line in Figure 2.8, lay it down curved on Figure 2.8 and check its dimension with the circle method.

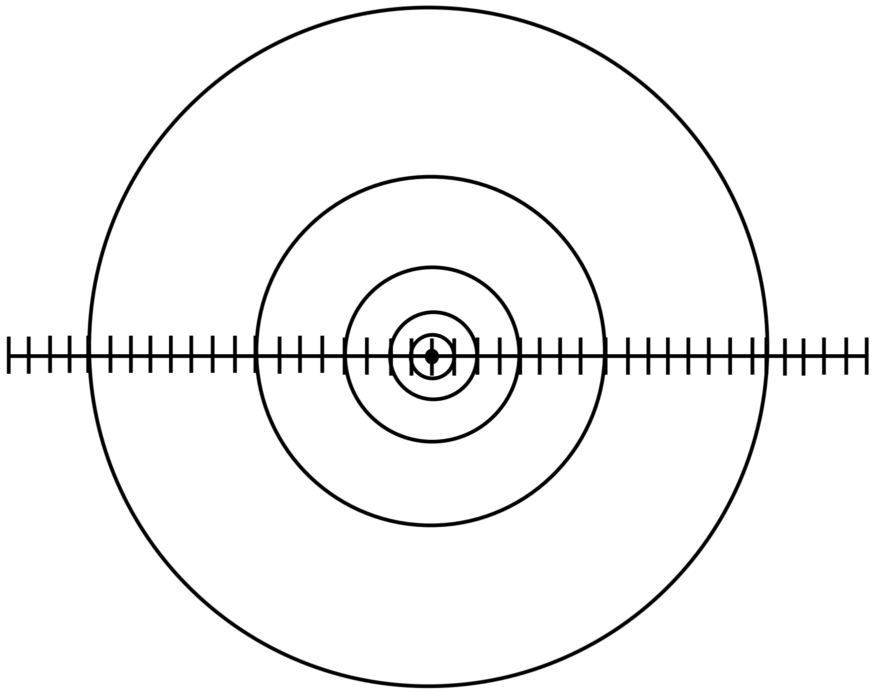
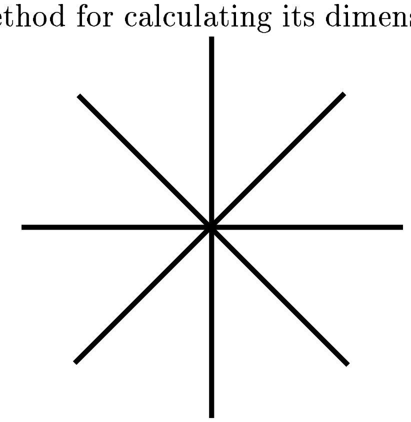


Figure 2.8: Count the number of line segments inside each circle to discover again that a line is 1-dimensional.

Q2.12: What is the dimension of this pattern? Is this a fractal? What would be the best method for calculating its dimension ?



END ACTIVITY

Table 2.6: For use with Figure 2.8.

| $r$ (radius) | $N$ (count) |
|---|---|
| 1 | |
| 2 | |
| 4 | |
| 8 | |
| 16 | |

## 2.3 Mathematical Fractals

We learned previously that there are two types of fractals: natural and mathematical. Unlike a natural fractal, a mathematical fractal is constructed according to a set of fixed rules which do not involve any random processes. Given the fixed rules, the resulting structures are always identical to one another.

Now we create a coastline that is a mathematical fractal, using a different set of rules—rules that have no randomness. You won't be needing any dice to throw or coins to flip. Since the rules have no randomness, this time you and your classmates should get the identical result.

### HandsOn 8: Creating Your Own Mathematical Fractal

The mathematical fractal coastline we are about to create is called the "Koch curve." You will need:

- a piece of elastic or a rubber band,
- a ruler, and
- a handful of thumb tacks.

In this procedure, we follow the same rule over and over. The central third of a rubber band segment gets turned into a V, like this:

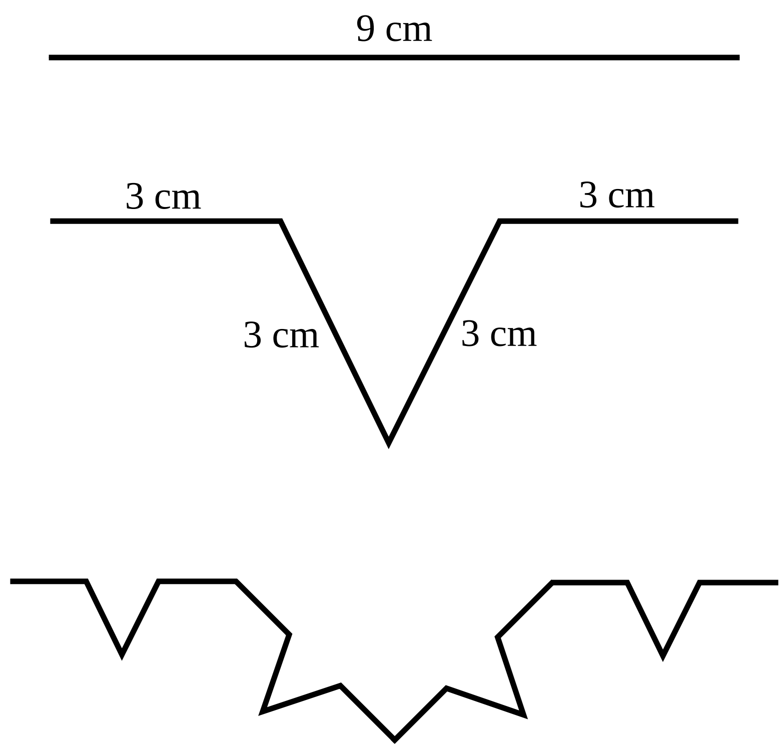



Figure 2.9: Beginning the construction of the mathematical fractal called the Koch curve.

1. On a bulletin board, stretch a piece of elastic or a rubber band between two thumb tacks a distance 9 centimeters apart (Figure 2.9). What is the initial length of your straight “coastline?” If this were a map, a giant could span the entire coast in one step.

2. Place thumb tacks into the elastic 3 centimeters from each end, so that it is divided into 3 equal segments.

3. Take the midpoint of the middle part and pull it perpendicular to the original line to form a V, with each side of the V being 3 centimeters long. Insert a thumb tack to hold it in place. How many segments does your coastline have now – that is, how many straight-line steps does it take to walk exactly along the coastline? What is its total length now? Complete the second row of your

copy of Table 2.7.

4. There are now four segments, each 3 centimeters long. In each segment, place two thumb tacks so that the segment will be divided into three 1-centimeter parts; then turn the middle part of each segment into a V by pulling it perpendicular to the line and inserting a thumb tack. Each side of each V should be 1 cm. long. How many straight-line steps does it take now to follow the coastline exactly? What is its total length? Add this information to your copy of Table 2.7.

5. Now think what the result of the next stage will be. How long will each segment (step) be? How many steps to follow the coastline exactly? What is the total length? Add this information to the copy of Table 2.7.

6. Complete the final row of your copy of Table 2.7 by looking at the pattern of other entries.

Table 2.7: Properties of a Koch Curve.

| Length of each step (cm.) | Number of steps to follow coastline exactly | Total length (cm.) |
|---|---|---|
| 9 | 1 | 9 |
| 3 | 4 | |
| | | |
| | | |
| | | |

Q2.13: Imagine carrying out the process of turning the middle third of each segment into a V again and again an *infinite* number of times. How would the curve look? How long would each step be? How many steps would it take to follow the coastline exactly? And what would be the total length of the coastline?

| Q2.14: How does this mathematical fractal differ in appearance from the random fractal coastlines you created in the exercises above? |
|---|

Plot the table entries on a copy of the log-log graph paper of Figure 2.2. (Note: Recall that you can multiply all entries on both axes by 10 or 1/10 to accommodate the range of numbers to be plotted.) The length of each segment is plotted along the horizontal axis and the number of steps along the vertical axis. Is the result a straight line? If so, what is the slope of this line? The accepted value of the dimension of a Koch curve is 1.26. How does the magnitude of your measured slope compare with this number?

END ACTIVITY

## 2.4 Dimensions and Logarithms (Advanced)

Let's derive the equations that justify the measurement of the dimension of an object as the magnitude of the slope of a straight line on a log-log graph. Think of a pattern that has a fixed area and fixed overall width $L$. We are going to cover this pattern with square boxes of width $d$ and count the number $N$ of the boxes needed to cover it. For a solid area, we have a dimension of 2 and the general formula

$$(\text{Area}) = (\text{Constant})L^2.$$

The constant depends on the shape. As examples, Figure 2.10 shows shapes with three different values of this constant.

Now we cover any of these shapes with little boxes of width $d$ and area $d^2$. How many boxes $N$ does it take? The following formula is approximately correct:

$$(\text{Area}) = Nd^2 = (\text{Const})L^2 = (\text{Const})\left(L\frac{d}{d}\right)^2 = (\text{Const})\left(\frac{L}{d}\right)^2 d^2,$$

or

$$Nd^2 = (\text{Const})\left(\frac{L}{d}\right)^2 d^2.$$

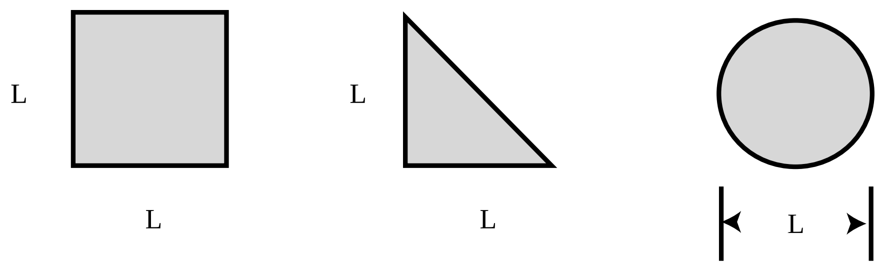



Figure 2.10: The area of three 2-dimensional figures, showing three values of the constant in the area formula (Area) = (Constant)$L^2$.

Cancel the factor $d^2$ on both sides of the equation to obtain:

$$N = (\text{Const})\left(\frac{L}{d}\right)^2 = d^{-2} \times [(\text{Const}) \times L^2].$$

This result is for a 2-dimensional object, such as those shown in Figure 2.10. For a fractal, the dimension is not necessarily 2. Call the dimension $D$. Then the corresponding equation becomes:

$$N = d^{-D} \times [(\text{Const}) \times L^D].$$

Now take the logarithm (log) of both sides:

$$\log N = \log(d^{-D}) + \log[(\text{Const}) \times L^D].$$

Here $L$ is fixed; we are not changing the area or the overall width $L$ of the figure as we use boxes of different width $d$ to cover it. Therefore everything in the square bracket is a constant, and the log of the quantity in the square bracket is also a constant, which we can call "Constant" Using the property of logs, we have:

$$\log N = -D \log d + \text{Constant}.$$

Think of the variables as $\log N$ and $\log d$ rather than $N$ and $d$. Then this can be thought of as the equation of a straight line with slope $-D$. Hence our *log-log box-covering plot* will yield a straight line whose slope is the negative of the dimension $D$.

Take care when measuring the slope $D$ to not use the numbers along the logarithmic scales. Instead, measure this slope directly, that is, with an ordinary ruler, as shown in Figure 2.3, or use the formula:

$$\text{slope} = \frac{\log N_2 - \log N_1}{\log d_2 - \log d_1}.$$

## 2.5 What Do You Think?

Q2.15: What does it mean to you that an object has a fractional dimension? (HINT: The following questions could prompt their thinking: How does the object fill space? Is its use of space dense or sparse? Are its edges smooth or jagged? What is similiar throughout different parts of the object? What is random or different throughout different parts of the object? How does the whole object compare with individual parts of the object? What geometric shapes do you see in the object: circles, lines, ovals, spheres?)

Q2.16: Is there a quantitative difference between measuring the coastline by the "ruler method", "box method", and "circle method"? If so, can you explain why? Why would scientists need all three methods (i.e., why don't they just pick the "best one")? HINT: Think of some examples from nature which would be better suited towards a particular method.

Q2.17: In the beginning of Section 1.1 we talked about the length of the coastline as measured by various observers (e.g., an automobile, bicyclist, jogger and bird). Can you say whether the fractal dimension of the coastline measured by these observers would be the same or different? How does it change (or not change)? Which observer would get the most accurate fractal dimension?

## Research Projects

We encourage you to pursue independent research projects. Science moves forward through research! Try one of the suggestions below or design your own. Or, feel free to write an essay using any of the questions throughout this chapter as inspiration.

*Research projects can be published on our Web site.*

- **Working With Coastlines.** Obtain a printed map of your favorite coastline (a good source is `www.mapquest.com` on the Web). Using calipers or a pencil compass "walk" along the coastline with different step sizes. Construct a table similar to that produced in the **Coastline** program. Determine the fractal dimension by hand and using the computer program. Write a report which discusses the role of multiple measurements and a comparison of your hand measurements to the computer measurements.

- **Formation of Natural Fractals.** As a variation, trace the evolution in time of the fractal dimension of a natural fractal such as a coastline as it is formed. You are asking the question: what is the fractal dimension after each step in the creation process? Does this change in dimension in terms of the natural process of erosion.