Physics of Flow in Random Media

Publications/Collaborators:

1) "Postbreakthrough behavior in flow through porous media"
E. López, S. V. Buldyrev, N. V. Dokholyan, L. Goldmakher,
S. Havlin, P. R. King, and H. E. Stanley, Phys. Rev. E **67**, 056314 (2003).

2) "Universality of the optimal path in the strong disorder limit" S.
V. Buldyrev, S. Havlin, E. López, and H. E. Stanley, Phys. Rev. E
70, 035102 (2004).

3) "Current flow in random resistor networks: The role of percolation in weak
and strong disorder" Z. Wu, E. López, S. V. Buldyrev, L. A. Braunstein, S. Havlin, and
H. E. Stanley, Phys. Rev. E **71**, 045101 (2005).

4) "Anomalous Transport in Complex Networks" E. López, S. V. Buldyrev,
S. Havlin, and H. E. Stanley, cond-mat/0412030 (submitted to *Phys. Rev. Lett.*).

5) "Possible Connection between the Optimal Path and Flow in Percolation Clusters"
E. López, S. V. Buldyrev, L. A. Braunstein, S. Havlin, and H. E. Stanley, submitted to
Phys. Rev. E.

Outline

- Network Theory: “Old” and “New”
- Network Transport: Importance and model
- Results for Conductance of Networks
- Simple Physical Picture
- Conclusions

Reference

“Anomalous Transport on Complex Networks”, López, Buldyrev, Havlin and Stanley, cond-mat/0412030.

Network Theory: "Old"

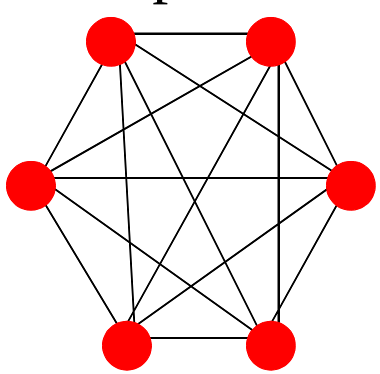
- Developed in the 1960's by Erd s and Rényi. **(Publications of the Mathematical Institute of the Hungarian Academy of Sciences, 1960).**

- *N* nodes and probability *p* to connect two nodes.

- Define *k* as the degree (number of links of a node), and ‹*k*› is average number of links per node.

Construction

a) Complete network

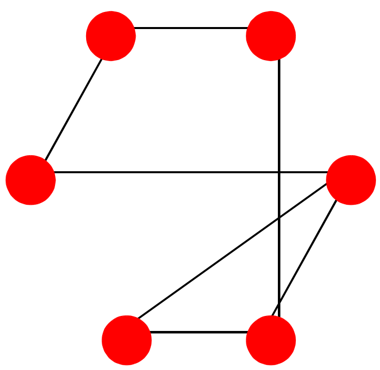


b) Annihilate links with probability

$$1-p$$

$$\left[p=\frac{\langle k\rangle}{N-1}\right]$$

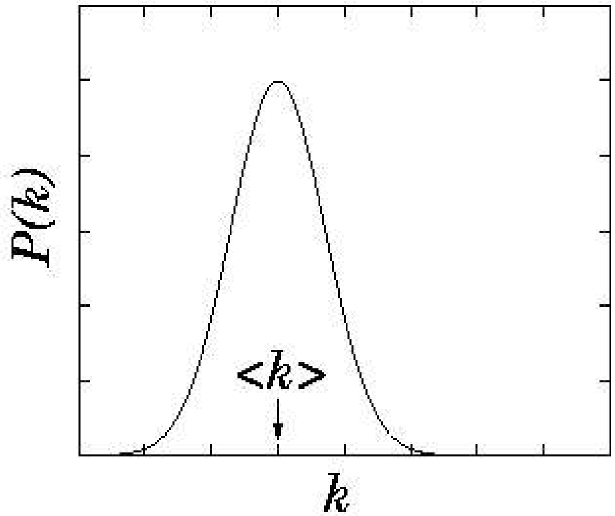
c) Realization of network



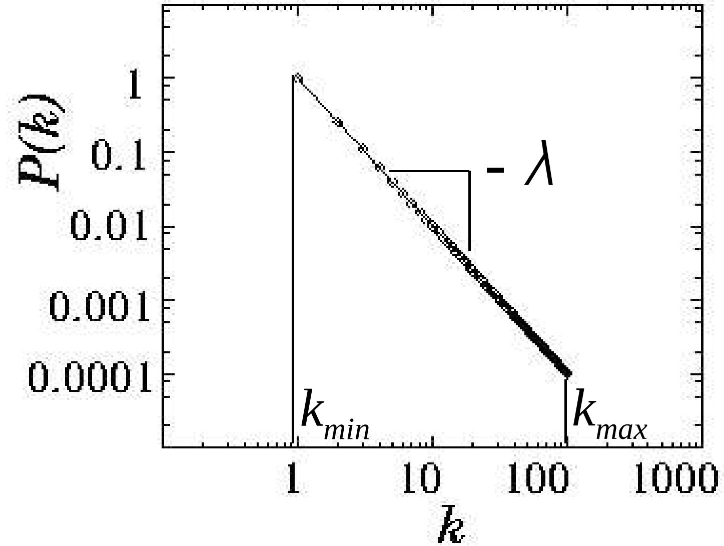
- Distribution of degree is Poisson-like (exponential) $P(k)=e^{-\langle k\rangle}\frac{\langle k\rangle^{k}}{k!}$

New Type of Networks

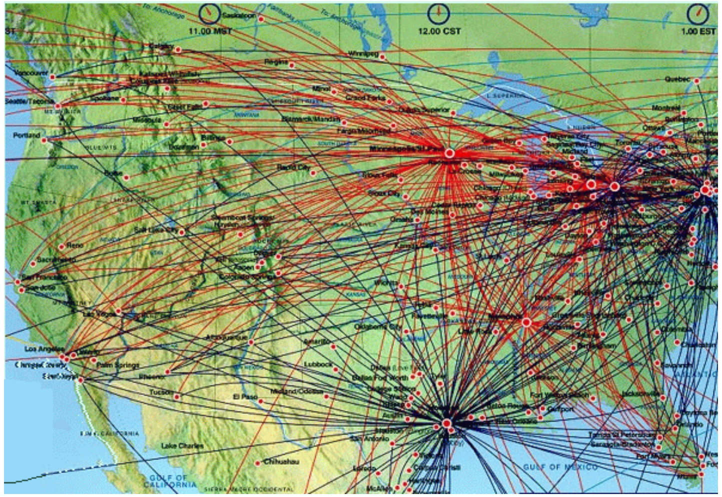
Old Model:
Poisson distribution



New Model:
Scale-free distribution



Erds-Rényi Network

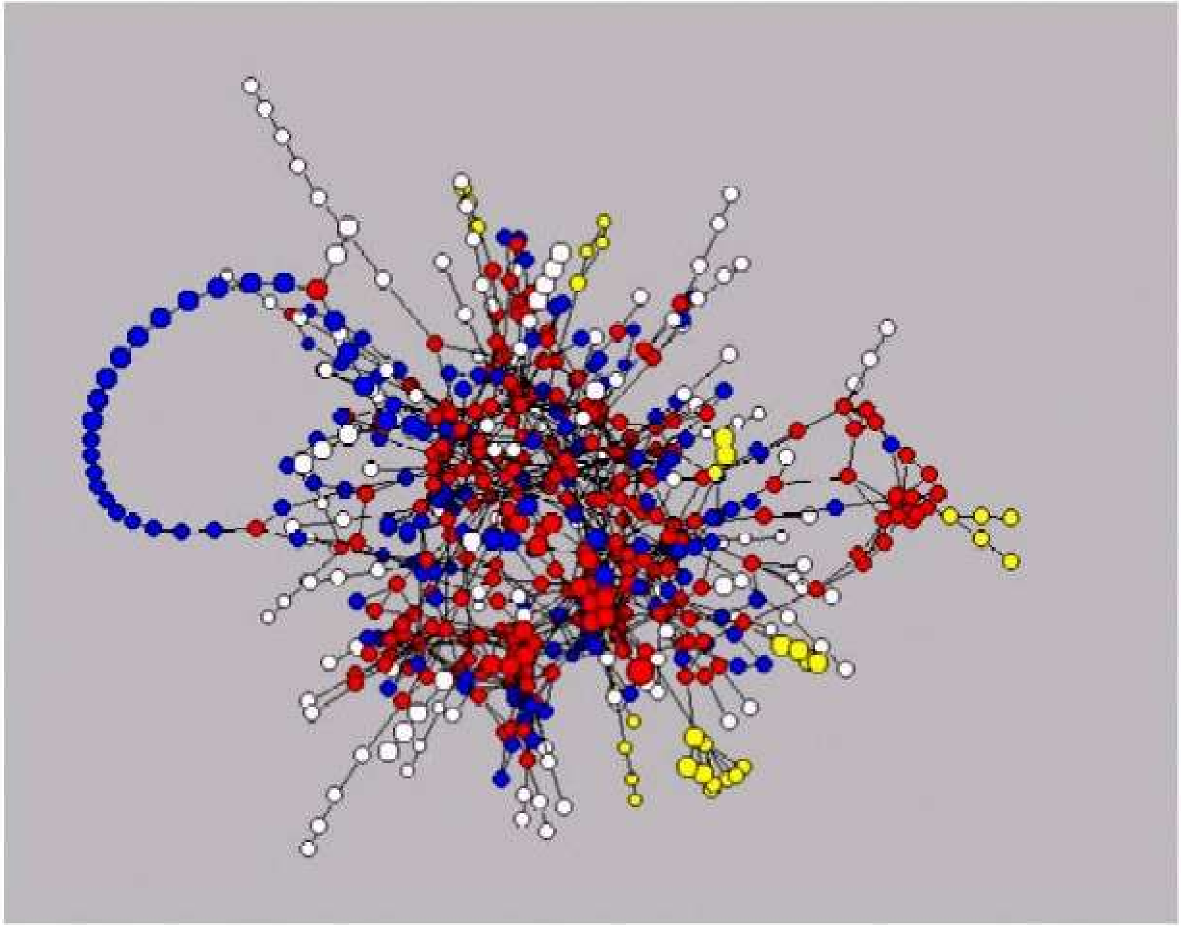


Scale-free Network

Example: “New’’ Network model

Metabolic Network

- **Nodes**: chemicals (substrates)
- **Links**: bio-chemical reactions



Jeong et al. Nature 2000

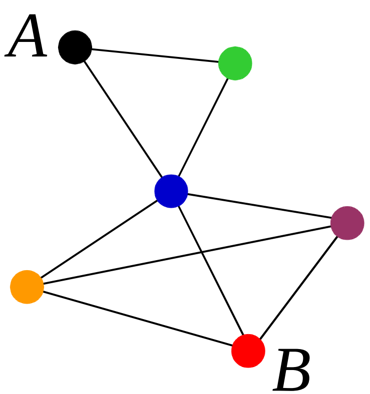
Why Transport on Networks?

1) Most work done studies ***static properties*** of networks.

2) No general theory of transport properties of networks.

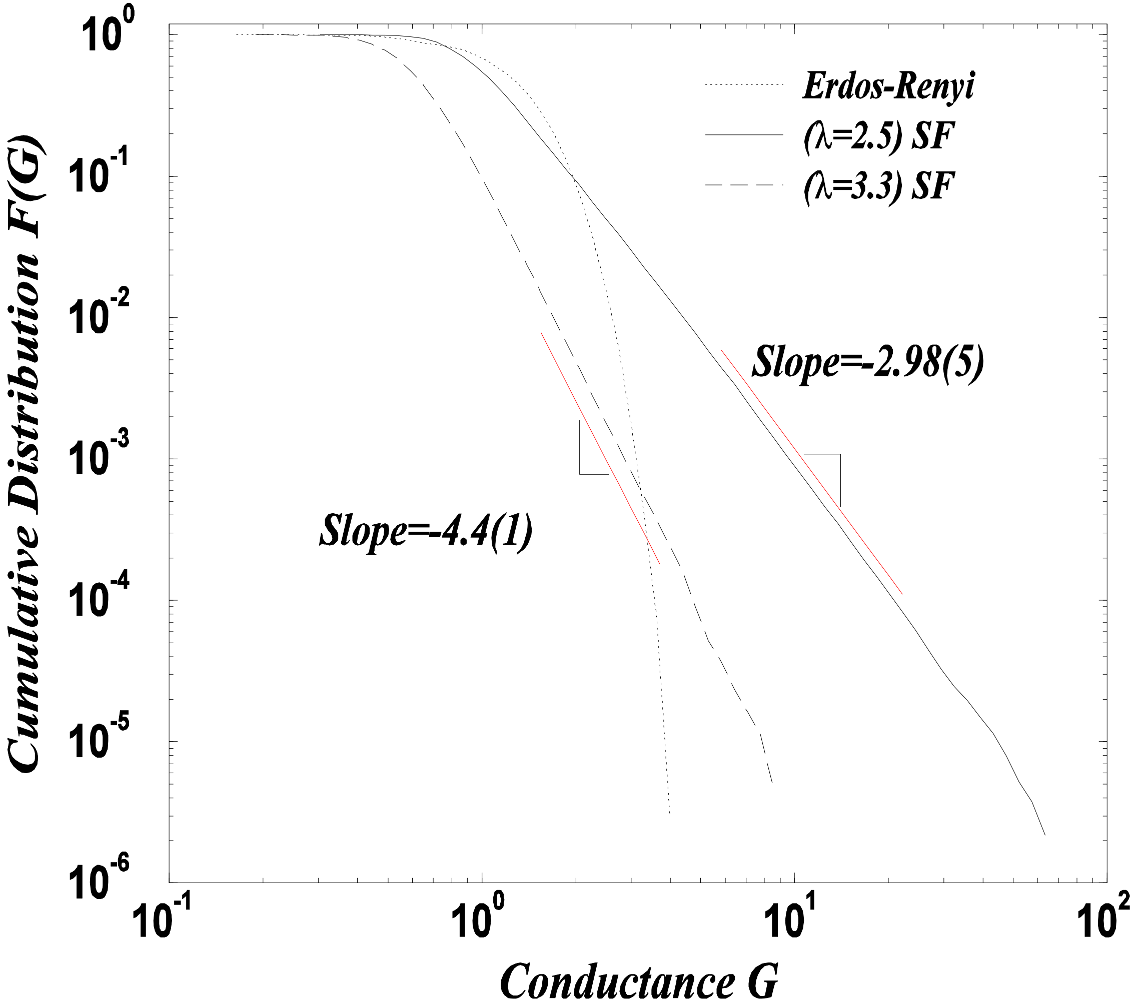
3) Many networks contain flow, e.g., emails over internet, epidemics on social networks, passengers on airline networks, etc.

Consider network links as equal resistors *r=1*



- Choose two nodes *A* and *B* as source and sink.
- Establish potential difference $V_A - V_B = 1$
- Solve Kirchhoff equations for current *I*, equal to ***conductance G=I***.
- Perform many realizations (minimum 10^6) to determine distribution of *G*, .

Results I:
Scale-Free and Erdos-Renyi Networks



- Cumulative distribution:

$$F(G)=\int_G^{\infty} \phi(G)\,dG$$

- Erds-Rényi
  narrow shape.

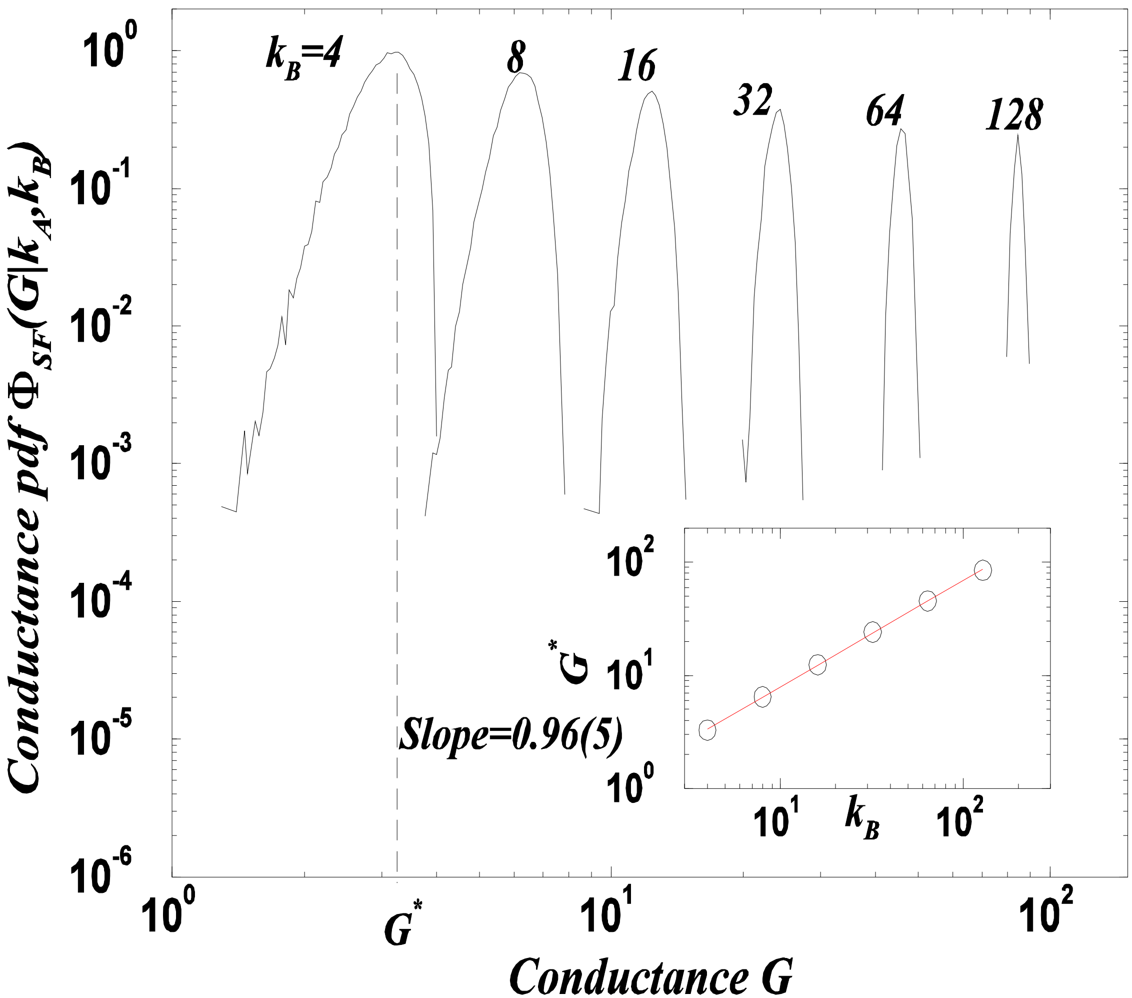
- Scale-free
  wide range
  (power law).

- Power law
  λ -dependent.

- Large *G* suggests
  dependence on degree
  distribution.

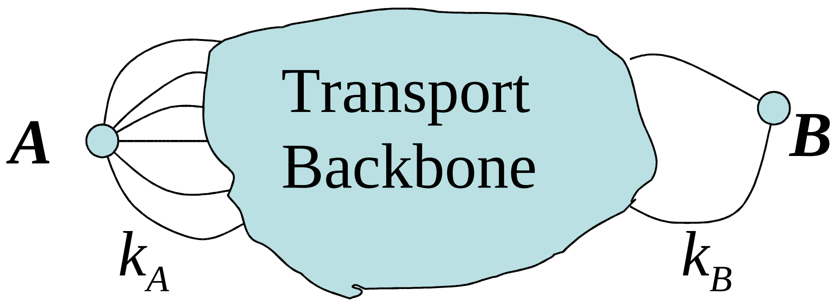
Results II:

Probability of G for k_A and vary k_B



- Fix $k_A=750$
- $\Phi(G|k_A,k_B)$ narrow well characterized by most probable value $G^*(k_A,k_B)$
- $G^*(k_A,k_B)$ proportional to k_B

Simple Physical Picture



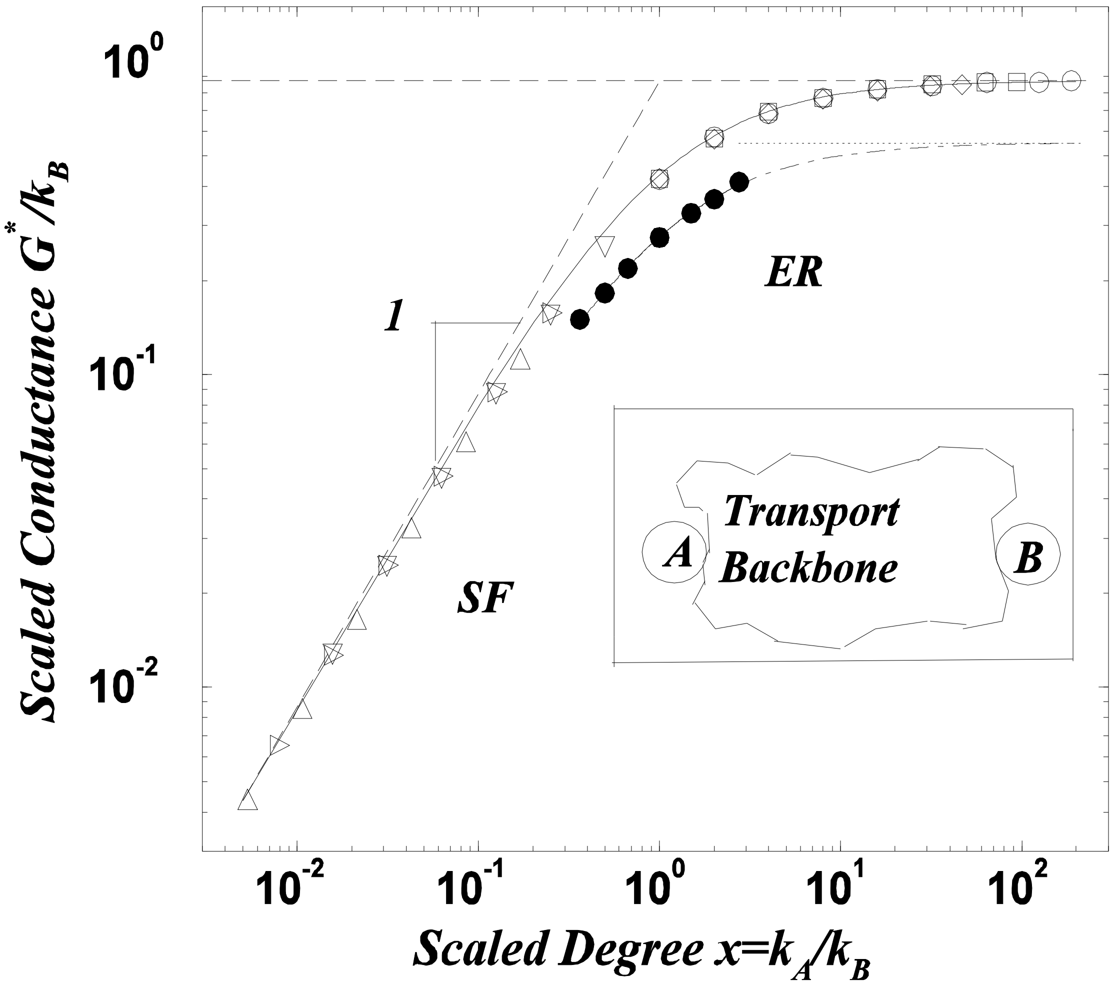
- Network can be seen as series circuit.

- Conductance G^* is related to node degrees k_A and k_B through a network dependent parameter c.

- To first order (conductance of "transport backbone" >> ck_Ak_B)

$$G^* ¿ c \frac{k_A k_B}{k_A + k_B}$$

Results III (Model):
G^*/k_B vs. k_A/k_B



- From series circuit expression

$$\frac{G*¿}{k_B} = c\,\frac{x}{1+x}\ ; \qquad \left[x \equiv \frac{k_A}{k_B}\right]$$

¿

- Parameter *c* characterizes network flow

- Erds-Rényi narrow range
  Scale-free wide range

Power law $\Phi(G)$ for scale-free networks

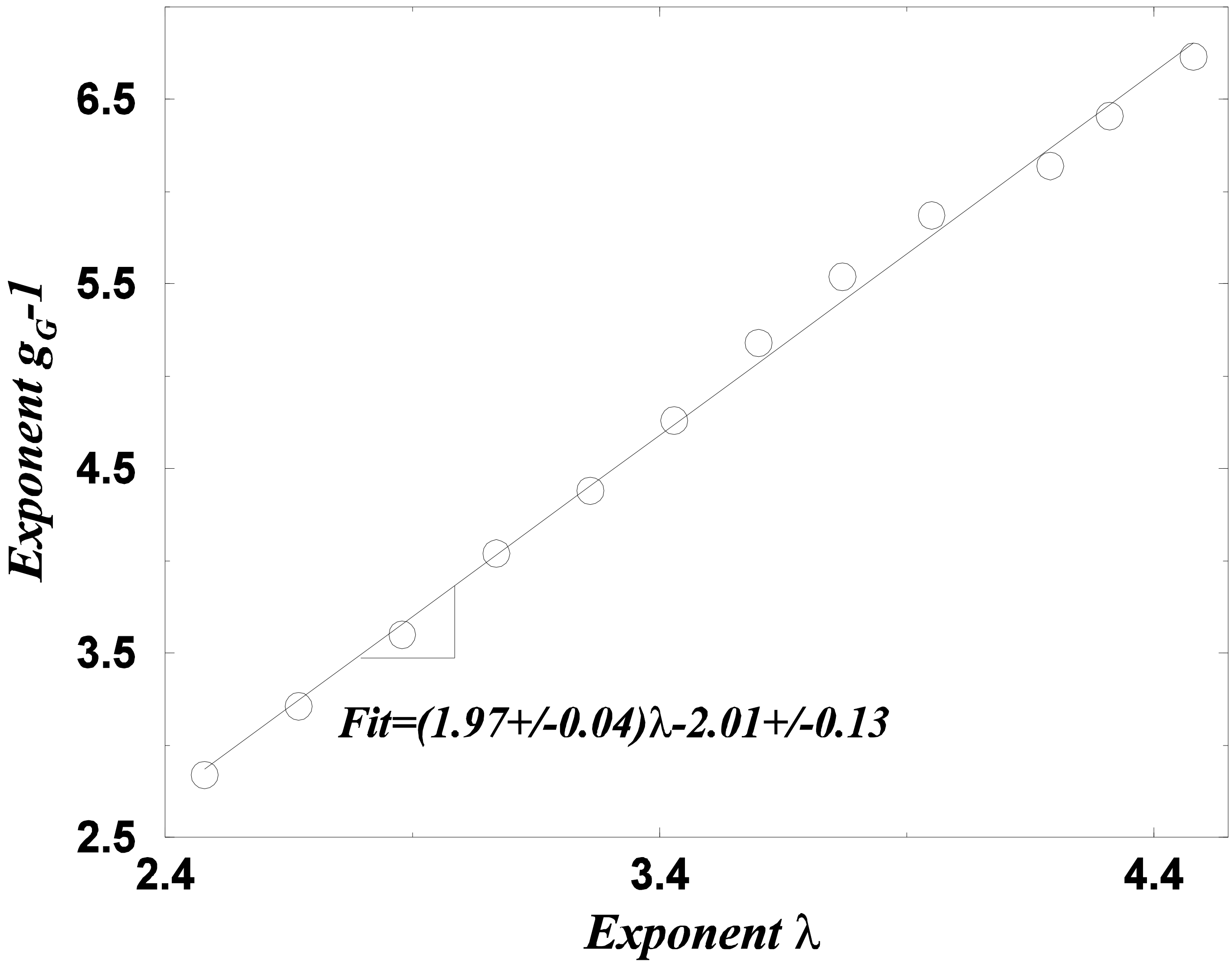
- Leading behavior for $\Phi(G)$

$$\Phi(G) \sim G^{-g_G}; \quad \left[g_G = 2\lambda - 1 \right]$$

- Cumulative distribution

$$F(G) \sim G^{-g_G+1} \sim G^{-(2\lambda-2)}$$

Results IV: Scale-free exponent g_G-1 vs. λ



Conclusions

- Scale-free networks exhibit larger values of conductance *G* than Erds-Rényi networks, thus making the scale-free networks better for transport.

- We relate the large *G* of scale-free networks to the large degree values available to them.

- Due to a simple physical picture of a source and sink connected to a transport backbone, conductance on both scale-free and Erd s-Rényi networks is given by $ck_Ak_B/(k_A+k_B)$. Parameter *c* can be determined in one measurement and characterizes transport for a network.

- The simple physical picture allows us to calculate the scaling exponent for $\Phi(G)$, $1\text{-}2\lambda$, and for $F(G)$, $2\text{-}2\lambda$.

Molloy-Reed Algorithm for scale-free Networks

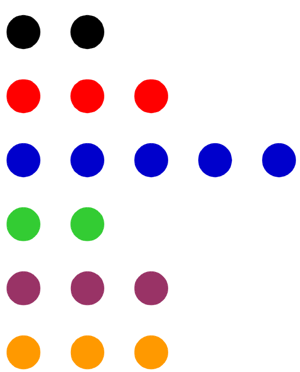
Create network with pre-specified degree distribution *P(k)*

Example:

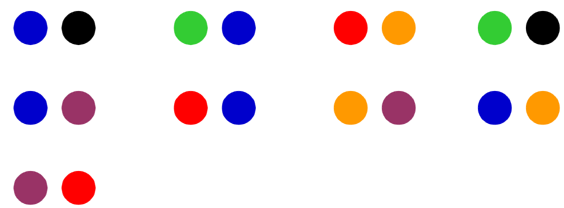
1) Generate set of nodes with pre-specified degree distribution from $P(k) \sim k^{-\lambda}$

Degree: 2 3 5 2 3 3

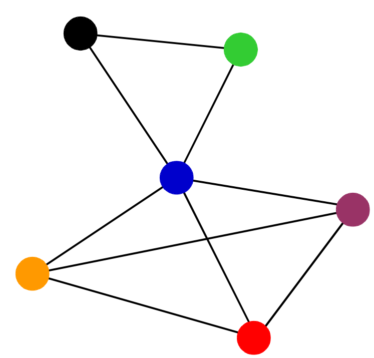
2) Make k_i copies of node *i*:



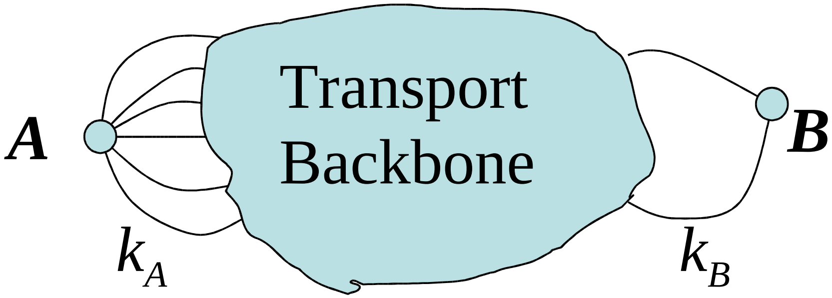
3) Randomly pair copies excluding self-loops and double connections:



4) Connect network:



Simple Physical Picture



- Network can be seen as series circuit.

- Conductance G^* is proportional to node degrees k_A and k_B.

- Conductance given by

$$\frac{1}{G*¿} = \frac{1}{ck_A} + \frac{1}{ck_B} + \frac{1}{G_{tb}} \Rightarrow G*¿c\frac{k_A k_B}{k_A + k_B + ck_A / G_{tb}}$$

¿

- To first order

$$G*¿c\frac{k_A k_B}{k_A + k_B}$$

Power law *Φ(G)* for scale-free networks

- Probability to choose k_A and k_B

$$P(k_A)P(k_B) \sim k_A^{-\lambda} \; k_B^{-\lambda}$$

- *Φ(G)* given by convolution

$$\Phi(G) \sim \int_{k_{min}}^{k_{max}} k_A^{-\lambda} dk_A \int_{k_{min}}^{k_{max}} k_B^{-\lambda} \, \delta\left(G - c\frac{k_A k_B}{k_A + k_B}\right) dk_B$$

- Leading behavior for *Φ(G)*

$$\Phi(G) \sim G^{-g_G}; \quad \left[g_G = 2\lambda - 1\right]$$

- Cumulative

$$F(G) \sim G^{-(2\lambda - 2)}$$

Conclusions

- Scale-free networks exhibit larger values of conductance *G* than Erds-Rényi networks, thus making the scale-free networks better for transport.

- We relate the large *G* of scale-free networks to the large degree values available to them.

- Due to a simple physical picture of a source and sink connected to a transport backbone, conductance on both scale-free and Erd s-Rényi networks is characterized by a single parameter *c*. Parameter *c* can be determined in one measurement.

- The simple physical picture allows us to calculate the scaling exponent for $\Phi(G)$, $1-2\lambda$, and for $F(G)$, $2-2\lambda$.