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Stability of directed Min-Max optimal paths

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Abstract – The stability of directed Min-Max optimal paths in cases of change in the random media is studied. Using analytical arguments it is shown that when small perturbations ϵ are applied to the weights of the bonds of the lattice, the probability that the new Min-Max optimal path is different from the original Min-Max optimal path is proportional to $t^{1/\nu_\parallel}\epsilon$, where t is the size of the lattice, and $\nu_\parallel$ is the longitudinal correlation exponent of the directed percolation model. It is also shown that in a lattice whose bonds are assigned with weights which are near the strong disorder limit, the probability that the directed polymer optimal path is different from the optimal Min-Max path is proportional to $t^{2/\nu_\parallel}/k^2$, where k is the strength of the disorder. These results are supported by numerical simulations.

The directed polymer model [1] is a well-studied [2] model in the field of disordered systems. The model is concerned with directed optimal paths in random media, which are characterized by two growth rate exponents, ω and ν. The exponent ω determines the energy variability of a path of length t by the relation $\Delta E \sim t^{\omega}$, and the exponent ν determines the mean transversal distance of the optimal paths from the origin, through the relation $D \sim t^{\nu}$. These two growth rate exponents are connected by the Huse-Henley scaling relation: $\omega = 2\nu - 1$ [3]. There are two cases in which the space exponent ν has the value of the directed percolation model [4], rather than its value in the regular case. In the first case [5,6], the bonds of the lattice are assigned with values taken from a bimodal (0,1) distribution, and the probability to have a zero-valued bond is p_c, the critical probability of directed percolation. In the second case, the bonds are assigned with values taken from a strong disordered distribution, and thus the energy of each path, which is defined as the sum of its bonds' values, is mainly determined by the value of the maximal bond along that path. In the strong disorder limit, the optimal paths are identical to the optimal Min-Max paths, which are characterized by the directed percolation exponent ν [7,8].

Directed Min-Max optimal paths are the subject of the present article, which studies two cases in which there is a small probability for a change in the position of the optimal path. In the first case, small perturbations are applied to the weights of the bonds of the lattice, and the new Min-Max optimal path might be different from the original one. This case was numerically studied in [9] for directed polymer (regular) optimal paths (which are determined by the minimal sum of bond values), and a general theoretical discussion was presented in [10]. Explanations to the numerical results presented in [9] were given in [10–13]. The second case is the one of strong disorder: While in the strong disorder limit the regular optimal path is identical to the Min-Max optimal path, as the strength of the disorder decreases, a strong disorder - weak disorder transition occurs. This transition was studied for the ordinary (non-directed) lattice in [14,15], and for the directed case in [16]. The present study shows that in the case of small perturbations, the probability to depart from the original Min-Max optimal path is

$$P_j(\epsilon, t) \sim t^{1/\nu_\parallel}\epsilon, \tag{1}$$

where t is the size of the lattice, $\nu_\parallel$ is the longitudinal correlation exponent of the directed percolation model, and ϵ is the strength of the perturbation. For the case of strong disorder - weak disorder transition, it is shown that the probability to depart from the optimal Min-Max path is

$$P_j(k, t) \sim t^{2/\nu_\parallel}/k^2, \tag{2}$$

where k is the strength of the disorder.

The first part of the article presents an explicit description of the lattice and of the variables studied in the two cases. The second part presents the theoretical analysis, and the third part presents numerical results which support the validity of the analysis.

The directed polymer model is usually implemented on a square lattice rotated 45° so that its apex is the origin of the paths going down to its main diagonal, which is the base of a 90° triangle. The bonds of the lattice are assigned with random numbers, and the energy of a path is defined as the sum of the bonds' values along that path. The paths are constrained to move only downwards from the origin to the base, and thus in a lattice of size t, all the permitted paths are of length t. In the Min-Max version of the model, the optimal path is the one whose maximal bond, denoted by b_{max}, is minimal. If there are several paths with the same b_{max}, the second highest bond of each path is considered, etc. Thus, though there are usually many paths with the same (lowest) b_{max}, the optimal path is uniquely defined.

In order to find the response of Min-Max optimal paths to small perturbations, the bonds of the lattice were assigned with random numbers taken from uniform distribution in the range (0,1). After the endpoint of the Min-Max optimal path was identified, small perturbations in the form of random numbers taken from uniform distribution in the range $(0,\epsilon)$ were added to the original random numbers, and the endpoint of the new Min-Max optimal path was identified. Denote by $P_j(\epsilon,t)$ the probability that the new endpoint is different from the original endpoint, its dependence on ϵ and t is studied.

In the strong disorder case, the values of the random bonds were defined by $B_i = e^{kb_i}$, where b_i is taken from uniform distribution in the range (0,1), and k is a constant. On this lattice, the endpoint of the optimal Min-Max path and the endpoint of the optimal path which minimizes the sum of bonds' values are identified. Denote by $P_j(k,t)$ the probability that these two endpoints are different, its dependence on k and t is studied.

Starting with the response to small perturbations, denote by $O(h)$ the value of the optimal Min-Max path at height h above the base, where this value is defined as the value of the maximal bond between the base and the site at height h. Denote by $A(h)$ the value of the best alternative path which splits from the optimal path at height h, and leads towards a different endpoint on the base. $A(h)$ is also measured between the branching point and the base. The difference between these two values is denoted by $\Delta b(h) \equiv A(h) - O(h)$, and it is clear that after the perturbations in the range $(0,\epsilon)$ are applied, the probability to jump at height h, denoted by $P_j(\epsilon,h)$, has the same dependence on h as the probability that $\Delta b(h) < \epsilon$. The total probability to jump from the original optimal path is $\int_1^t P_j(\epsilon,h)\mathrm{d}h$, and since this study is concerned with cases in which the total probability to jump is very small, there is no need to worry about cases of more than one jump along the optimal path.

As mentioned above, the probability to jump in the regular case was studied in [13], and the present analysis follows (in part) the analysis presented there: The optimal paths which lead from the origin to the sites of the base form an ultrametric tree structure. There are $\sim \ln t$ branches which split from the optimal path on its way down from the origin to the base, and the probability that the site at height h is a branching point is $\sim 1/h$ [13,17]. The jumps from the original optimal path occur mainly to the best paths of the branches, and thus the probability to jump at height h is the product of the probability to have a branch at this height, which is $\sim 1/h$, and the probability to jump in case that there is such a branch, which is proportional to the probability that at the branching point $\Delta b(h) < \epsilon$.

For optimal Min-Max paths of length h, in a lattice whose bonds are assigned with random numbers taken from uniform distribution in the range $(0,1)$, the probability that $b_{max}(h)$ is lower than p_c decreases at a rate $\sim h^{-\delta}$ [8], where δ is the decay exponent of the directed percolation model. However, since the branching point at height h is in the bulk of the lattice, both the optimal path and the best path of the branch are freely chosen as the best paths from many ordinary optimal paths of length h. As a result of this freedom of choice, the characteristics of these paths are different from those of ordinary optimal paths. The following scaling rule was verified in the present numerical study for optimal paths of length $t > h$:

The probability that $O(h)$ is lower than p_c is a function of the ratio h/t.

In order to understand the significance of this scaling rule, consider a site at height h above the base. If this site is chosen at random, the probability that it can be connected to the base with a path whose highest bond is lower than p_c is determined by h alone (it is proportional to $h^{-\delta}$). But if the site belongs to an optimal Min-Max path which starts at height $t > h$ above the base, this probability is not only a function of h, but also a (increasing) function of t. If h is doubled and t is also doubled, this probability does not change. At any height $h > 100$, for $t = 2h$ this probability is $\simeq 0.59$, and for $t = 4h$ this probability is $\simeq 0.78$. The validity of this scaling rule is the cornerstone of the present analysis.

Since the characteristics of the best paths of the branches are similar to those of the optimal paths, the probability that $A(h)$ at the branching point is lower than p_c should also be a function of h/t. On the other hand, from the discussion presented in [8] it follows that $O(h)$ is bounded below by $p_c - f(h)$, $f(h) \sim h^{-1/\nu_\parallel}$, where $\nu_\parallel$ is the longitudinal correlation exponent of directed percolation. Denote by $A_{br}(h)$ the value of $A(h)$ at a branching point, remember that $A_{br}(h) > O(h)$, and conclude that in the majority of the branching points, there is a significant probability that $O(h)$ and $A_{br}(h)$ are in the $\sim h^{-1/\nu_\parallel}$ vicinity of p_c, and the probability that their difference is lower than ϵ is $\sim \epsilon\, h^{1/\nu_\parallel}$. The total probability to jump is thus $\sim \epsilon \int_1^t 1/h * h^{1/\nu_\parallel}\mathrm{d}h \sim \epsilon\, t^{1/\nu_\parallel}$, eq. (1). Since the value of $\nu_\parallel$ is $\simeq 1.734$ [18], and $1/\nu_\parallel \simeq 0.577$, the above analysis implies that $P_j(\epsilon,t) \sim \epsilon\, t^\alpha$, where $\alpha \simeq 0.577$.

The above discussion indicates that the total probability to jump from the optimal path is similar to the probability

to jump at the highest branching points whose $h \simeq t$. Of course, it is the $\sim 1/h$ probability to have a branch at height h which guarantees this similarity, which also characterizes the regular case [13].

Moving to the strong disorder - weak disorder transition, remember that the bonds' values are $B_i = e^{kb_i}$, and that for high values of k, $\Sigma_i B_i \simeq e^{kb_{max}}$. Since only the highest valued bonds along the paths are relevant, the jumping condition is $e^{kO(h)} + e^{kb_2(h)} > e^{kA_{br}(h)}$, where $b_2(h)$ is the second highest bond of the optimal Min-Max path between the branching point and the base. Divide the jumping condition by $e^{kO(h)}$, define $\Delta b'(h) \equiv O(h) - b_2(h)$, and remember that $\Delta b(h) \equiv A_{br}(h) - O(h)$, the jumping condition becomes: $1 + e^{-k\Delta b'(h)} > e^{k\Delta b(h)}$. In the majority of cases, this condition is fulfilled when both $\Delta b(h)$ and $\Delta b'(h)$ are very small, of the order of $1/k$. Thus, the probability to fulfill the jumping condition is proportional to the probability that both $\Delta b(h)$ and $\Delta b'(h)$ are small, which is the product of the probabilities that each one of them is small. Using similar arguments to those presented above for the perturbation problem, it is easy to verify that each probability is $\sim h^{1/\nu_\parallel}/k$, and their product is $\sim h^{2/\nu_\parallel}/k^2$. The total probability to jump is thus $\sim 1/k^2 \int_1^t 1/h * h^{2/\nu_\parallel} \mathrm{d}h \sim t^{2/\nu_\parallel}/k^2$, eq. (2). This analysis implies that the exponent which characterizes the strong disorder - weak disorder transition is twice its value in the small perturbations case.

In both cases studied above it is possible to define correlation lengths: $\xi(\epsilon) \equiv \epsilon^{-\nu_\parallel}$ and $\xi(k) \equiv k^{\nu_\parallel}$, and in both cases the jumping condition is a scaling function of t/ξ: In the first case $P_j(\epsilon, t) \sim (t/\xi(\epsilon))^{1/\nu_\parallel}$, while in the second case $P_j(k, t) \sim (t/\xi(k))^{2/\nu_\parallel}$. The difference between these two scaling functions is an outcome of the fact that in the first case the value of only one other bond should be in the close vicinity of $O(h)$, while in the second case the values of two other bonds should be in that vicinity.

The results of the numerical simulations performed for the small perturbations case indicate that for fixed values of ϵ, $P_j(\epsilon, t) \sim t^\alpha$. In order to estimate α, its local values computed by $\log_2(P_j(\epsilon, t)/P_j(\epsilon, t/2))$ are shown in fig. 1 for the cases $\epsilon = 0.00002$ and $\epsilon = 0.00001$. As can be seen, the two sets of local values are quite close, and both of them approach the value of 0.577 derived from the theoretical analysis, eq. (1). Since the theoretical analysis presented above leads to exact results, there is no need to estimate the error range, and the data presented in the figures is used only to support the validity of the analytical derivation.

The results of the numerical simulations performed for the strong disorder - weak disorder transition indicate that for fixed values of k, $P_j(k, t) \sim t^\beta$. In order to estimare β, its local values computed for the cases $k = 32768$ and $k = 16384$ are presented in fig. 2. The data presented in this figure indicates that the asymptotic value of β should be in the vicinity of 1.16. A comparison with the results presented in fig. 1 indicates that $\beta = 2\alpha$, and support the theoretical analysis presented above.

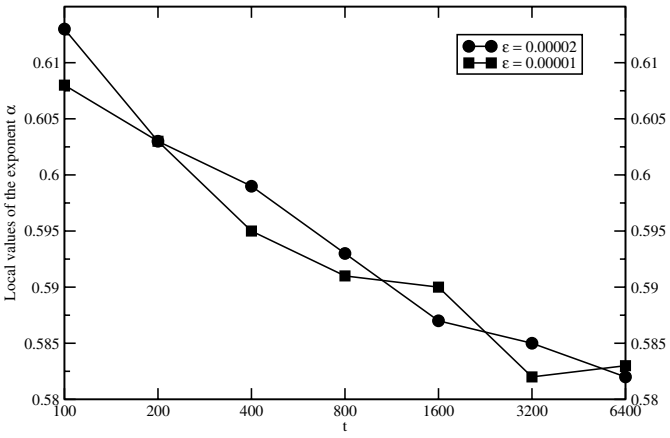

Fig. 1: The local values of the exponent α computed for $\epsilon = 0.00002$ (circles), and for $\epsilon = 0.00001$ (squares).

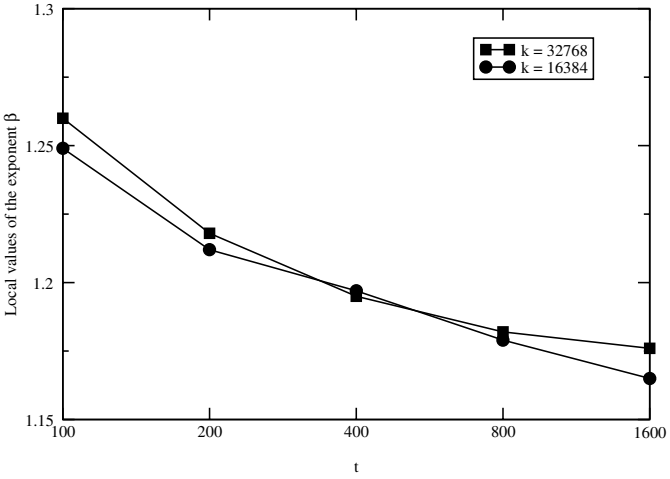

Fig. 2: The local values of the exponent β computed for $k = 32768$ (squares), and for $k = 16384$ (circles).

In the small perturbation case, the theoretical analysis implies that for fixed values of t, $P_j(\epsilon, t) \sim \epsilon$. This linear dependence on ϵ was verified by the numerical study. In the strong disorder - weak disorder transition, the theoretical analysis implies that for fixed values of t, $P_j(k, t) \sim 1/k^2$, and this fact was also verified in the present numerical study.

The crucial role played in the above analysis by the directed percolation characteristics is another evidence to the relevance of the idea of "self-organized critical phenomena" introduced in [8]. This idea defends the study of systems near the critical probability against claims that no real system can be found exactly at this probability. In [8] and also in the present study it is shown that though in the structure of the system no p (or p_c) is involved, its behaviour is determined by rules which are similar to the rules which determine the behaviour of other systems near p_c.

In conclusion, the present study shows that there is a close connection between the response of optimal Min-Max paths to small perturbations, and the response of strong disordered optimal paths to a decrease in the strength of the disorder. In both cases, the behavior near the critical probability of directed percolation dictates this response, in accordance with the idea of "self-organized critical phenomena" introduced in [8].

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