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Synchronization in scale-free networks: The role of finite-size effects

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Abstract – Synchronization problems in complex networks are very often studied by researchers due to their many applications to various fields such as neurobiology, e-commerce and completion of tasks. In particular, scale-free networks with degree distribution $P(k) \sim k^{-\lambda}$, are widely used in research since they are ubiquitous in Nature and other real systems. In this paper we focus on the surface relaxation growth model in scale-free networks with $2.5 < \lambda < 3$, and study the scaling behavior of the fluctuations, in the steady state, with the system size N. We find a novel behavior of the fluctuations characterized by a crossover between two regimes at a value of $N = N^*$ that depends on λ: a logarithmic regime, found in previous research, and a constant regime. We propose a function that describes this crossover, which is in very good agreement with the simulations. We also find that, for a system size above N^*, the fluctuations decrease with λ, which means that the synchronization of the system improves as λ increases. We explain this crossover analyzing the role of the network's heterogeneity produced by the system size N and the exponent of the degree distribution.

Since a great variety of systems can be represented by complex networks, over the last decades many researchers have studied both the topology and processes that evolve on top of these networks. Systems such as neural networks, the Internet and airlines networks [1–3] can be described by a set of nodes connected by links that represent a relationship between them, such as an electric impulse, friendship or air traffic. Many of these real networks were found to be characterized by a scale-free (SF) topology, given by a degree distribution

$$P(k) \sim k^{-\lambda}, \tag{1}$$

where k is the degree of the nodes and $m \leq k \leq k_{max}$, where m and k_{max} are the minimum and maximum degree, respectively, and λ represents the broadness of the distribution. On most real systems, such as the World Wide Web or metabolic networks, it was found that $2 < \lambda < 3$ [1,2].

More recently, research has focused on dynamical processes taking place on the underlying network [4–10]. Particularly, many mathematical and numerical models have been elaborated to study the problem of synchronization [11–16], a phenomenon present in the behavior of many collective systems. In these processes the state of the system evolves to a synchronized state, where the coupled units adjust their dynamics with one another. Examples of synchronization can be seen in brain processes [17] or data distribution [18–20]; in a network made up of processors that distribute the task load, the system is best synchronized when the process minimizes the waiting time in each processor. For these kinds of systems, a scalar field h is usually defined on the network and it is of interest to measure the fluctuations of h. This problem can be studied mapping it into a non-equilibrium surface growth problem, where the scalar field h_i, with $i = 1, \ldots, N$ and N the system size, represents the "height" of the node i, and the fluctuations, also called roughness of the system, are given by

$$W(t) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} [h_i(t) - \langle h(t) \rangle]^2}, \tag{2}$$

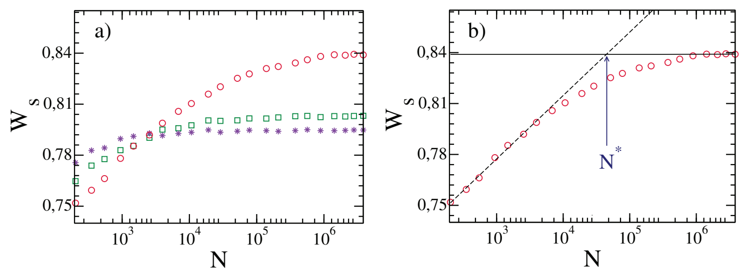

Fig. 1: (Colour on-line) (a) W_s as a function of N in a linear-log scale, with $\lambda = 2.6$ ($\circ$), 2.8 ($\Box$), 3.0 ($*$). We observe two regimes so that W_s behaves as $\sim \ln N$ for N smaller than a certain characteristic size N^*, and it increases at a slower rate for $N > N^*$, finally becoming constant for a sufficiently large N. (b) W_s as a function of N in a linear-log scale for $\lambda = 2.6$. It can be seen how we obtain the characteristic system size N^* for the intersection of the two regimes.

and

$$\langle h(t) \rangle = \frac{1}{N} \sum_{i=1}^{N} h_i(t) \tag{3}$$

is the mean value of h at time t. The roughness has two regimes, one where $W(t)$ increases until it saturates at a constant value W_s in the second regime, which depends on the topology of the system. In Euclidean lattices one of the most studied equations on surface growth is the Edwards-Wilkinson (EW) equation [21], which belongs to the same universality class as that of the stochastic growth model of surface relaxation to the minimum (SRM) [22]. In this model, a node of the network is randomly selected and a "particle" is placed on the node with the lowest height among the selected node and its neighbors. According to this rule, the nodes with higher heights distribute their excess of particles to their neighbors with lower heights. Pastore y Piontti *et al.* [12] studied the SRM model on SF networks through numerical simulations and found that, on the steady state, the behavior of the fluctuations with the system size N is given by

$$W_s \sim \begin{cases} \text{const}, & \text{for } \lambda \geq 3; \\ \ln N, & \text{for } \lambda < 3. \end{cases} \tag{4}$$

Unlike Euclidean lattices, in complex networks one cannot extend the discrete nature of the network to a continuum, and thus the dynamics of the network are not well represented by a continuum equation such as the EW equation. However, using a discrete Laplacian and a mean-field approximation, Korniss *et al.* [13] and Guclu *et al.* [23] found that in the limit $N \to \infty$, W_s increases with λ. Solving the discrete EW equation numerically for finite-size systems, in [12] the authors found that W_s decreases with N, which is not representative of any growth model. With a different approach, La Rocca *et al.* [14] developed a Langevin stochastic equation that describes the evolution of the interface, and solved it up to second order by numerical integration for finite system sizes, recovering eq. (4).

In this letter we mainly consider SF networks with $\lambda < 3$ because they are representative of abundant systems in Nature. We find that, for the SRM model, W_s has a crossover from a logarithmic regime to constant regime, at a characteristic value of N that depends on the topology of the network. Also, we find that, for system sizes above this characteristic size, W_s increases as λ decreases.

By stochastic numerical simulations, we study the SRM process on SF networks. To generate the network, we use the Molloy Reed (MR) algorithm or configurational model [24] and we use a minimum degree $m = 2$ because for this value we have a high probability of obtaining only one component and thus a single interface [25]. As for the maximum degree, the network has a natural cut-off given by $k_c \sim N^{\frac{1}{\lambda-1}}$ [24] and no structural cut-off is imposed, although this alternative will be discussed later. We define a scalar field h on the network, which represents the system's feature we want to study, so that each node is assigned a height h_i, with $i = 1, \ldots, N$. At $t = 0$, we allocate the nodes with a random height between 0 and 1. This initial condition does not affect the scaling behavior of the roughness in its steady state. At each time step, we deposit a particle on a node randomly selected with a probability $1/N$. Denoting the selected node by i and the set of its k_i neighbors by v_i, we simulate the SRM process according to the following rules:

1) if $h_i < h_j \, \forall \, j \, \epsilon \, v_i \Rightarrow h_i = h_i + 1$,

2) else, $l \in v_i : \; h_l < h_i$ and $h_l < h_j \, \forall \, j \neq l, \quad j \, \epsilon \, v_i \Rightarrow h_l = h_l + 1$,

and compute the roughness at the saturation as a function of the system size.

In fig. 1(a) we plot the fluctuations W_s as a function of N for different values of λ. We can see that for $\lambda = 3$ we obtain the behavior predicted by eq. (4), *i.e.*, the fluctuations go rapidly to a constant when the system size increases. For $\lambda < 3$ and for a range of system sizes ($N \lesssim 10^3$), the behavior of W_s is logarithmic, which agrees with eq. (4), but when N increases, the fluctuations increase slower than a logarithmic, reaching a constant that is independent of N and only depends on λ. The scaling

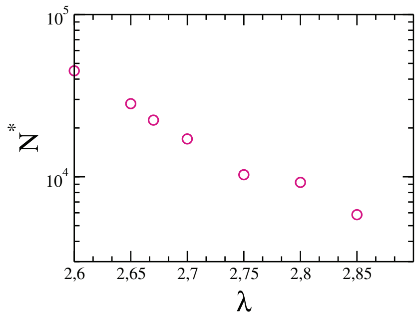

Fig. 2: (Colour on-line) N^* as a function of λ, for SF networks with $2.6 \leq \lambda < 2.9$ with $m = 2$.

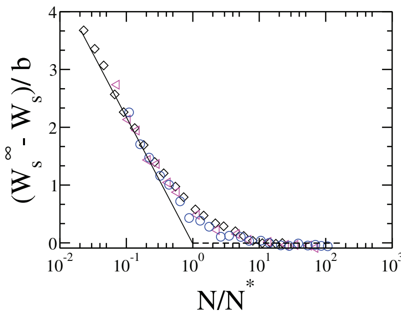

Fig. 3: (Colour on-line) $(W_s^\infty - W_s)/b$ as a function of N/N^* in a linear-log scale for $\lambda = 2.6$ ($\diamond$), 2.7 ($\triangleleft$) and 2.8 ($\circ$). The curves overlap, which proves the scaling hypothesis given by eq. (5) correct.

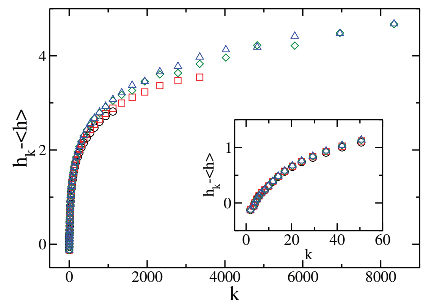

Fig. 4: (Colour on-line) $h_k - \langle h \rangle$ as a function of k for $N = 6044\,(\circ), 32768\,(\square), 500000\,(\diamond)$ and $1000000\,(\triangle)$, and $\lambda = 2.6$. The inset is an enlargement for small values of k; it can be seen that $h_k - \langle h \rangle$ does not depend on N in this region.

behavior of W_s suggests that above a certain system size $N = N^*$ (which depends only on λ), the fluctuations become independent of N and therefore the system has the same degree of synchronization. It is worth noticing that the second regime was not seen in [12] since in their research the authors simulated systems smaller than N^*.

We then estimate the system size for which the behavior of the fluctuations changes from a logarithmic regime to a constant, *i.e.*, the crossover between these two regimes. In fig. 1(b) we plot W_s as a function of N only for $\lambda = 2.6$ in order to show how N^* is determined. We compute N^* for different values of λ, and we see that N^* decreases with λ for $\lambda < 3$, and $N^* \to 0$ for $\lambda \geq 3$, as can be seen in fig. 2.

The behavior of W_s with N can be described as follows:

$$W_s \sim \begin{cases} b \ln(N), & \text{for } N < N^*; \\ W_s^\infty, & \text{for } N > N^*, \end{cases}$$

where $b \equiv b(\lambda)$ and $W_s^\infty \equiv W_s^\infty(\lambda)$ is the roughness value in the thermodynamic limit (above N^*). Thus, we propose a scaling function $f(N/N^*)$ where

$$f(N/N^*) \sim \begin{cases} \ln(N/N^*), & \text{for } N/N^* < 1; \\ 0, & \text{for } N/N^* > 1. \end{cases} \tag{5}$$

Then, the behavior of W_s for all the values of N can be expressed as

$$W_s = W_s^\infty + b\ f(N/N^*).$$

To lose all dependence on λ, we work with the expression $({W_s}^\infty - W_s)/b$ so that

$$(W_s^\infty - W_s)/b \sim -f(N/N^*) \sim \begin{cases} -\ln(N/N^*), & \text{for } N/N^* < 1; \\ 0, & \text{for } N/N^* > 1. \end{cases}$$

In fig. 3 we plot $(W_s^\infty - W_s)/b$ as a function of N/N^*. From the plot we can see that the curves indeed overlap, which shows that our scaling hypothesis is correct.

To understand the dynamics of the system as N increases, we study the behavior of the fluctuations relative to the topology of the network, specifically the degree of the nodes. We compute the mean height of the nodes with degree k, denoted by h_k. In fig. 4 we show $h_k - \langle h \rangle$ as a function of k for $\lambda = 2.6$ and different values of N, below and above N^*. We see that, in average, the difference between the height of the nodes and the mean value of the entire network, increases with k. Thus, for the SRM model, nodes with high degree worsen synchronization while small-degree nodes improve it. This is because the nodes with high degree receive the excess of particles of their low-degree neighbors, which are the majority in SF networks. In order to understand the effect of the high-degree nodes on the behavior of W_s, we study the SRM process for a network with a structural cut-off of $k_s \sim N^{1/2}$ [26,27], which is smaller than k_c for $\lambda < 3$ (not shown here). For this case, we also find a crossover between two regimes at a characteristic size, although it is larger than the one found with no structural cut-off. This means that the hubs contribute to the finite-size effects for $N \ll N^*$ with $\lambda < 3$.

We also notice that the behavior of $h_k - \langle h \rangle$ does not depend on N for small k, which means that low-degree nodes do not contribute to finite-size effects on W_s. However, for larger k, $h_k - \langle h \rangle$ increases with N. As k increases, the rate of increase of $h_k - \langle h \rangle$ decreases, so that the nodes have heights more similar to one another. This behavior, combined with the fact that the probability of high connectivities is very low, determines that nodes with high degree do not contribute to an increase in the fluctuations for $N \gg N^*$. Due to this combined effect W_s approaches to a constant for $N \gg N^*$. On the other hand, as λ increases, the ratio of small-degree nodes to hubs increases. This explains why, around the value of N^*, the synchronization improves as λ increases. This can be seen from fig. 1, where N^* decreases as $\lambda \to 3$ and, consequently, the constant value to which W_s approaches gets smaller, which means that the system synchronizes better for increasing λ.

To summarize, we studied the behavior of the fluctuations W_s with the system size N in the steady state for the SRM model in SF networks with $2.5 < \lambda < 3$, aiming to understand the role of finite-size effects in the system's synchronization. We found a crossover between two different regimes at a characteristic size N^* below which W_s has a logarithmic dependence on N and above which W_s is constant. We measured N^* for different values of λ and found that N^* decreases with λ. We also found that the synchronization enhances as λ increases. The behavior of W_s with N and λ is determined by the performance of high-degree nodes in the dynamics of the system, which reach heights above the mean value and worsen the synchronization. However, the rate of increase of the heights compared with the average height decreases with k and, given that the degree distribution also decreases with k, we conclude that high-degree nodes do not contribute to an increase of W_s for $N \gg N^*$. It is important to mention that even though high-degree nodes are the responsible for the finite-size effects observed for $N < N^*$, the explanation of the logarithmic behavior of W_s goes beyond the aim of our actual research. However, this will be the scope of future researches. As for the heterogeneity of the network, as λ increases, the proportion of high-degree nodes decreases and the previous effect is noted for smaller system sizes. In the limit $\lambda \to 3$, $N^* \to 0$ and W_s is constant for all N.

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REFERENCES

[1] Albert R. and Barabási A.-L., *Rev. Mod. Phys.*, **74** (2002) 47.

[2] Dorogovtsev S. N. and Mendes J. F. F., *Evolution of Networks* (Oxford University Press, Oxford) 2003.

[3] Verma T., Arajo N. A. M. and Herrmann H. J., *Sci. Rep.*, **4** (2014) 5495.

[4] Watts D. J. and Strogatz S. H., *Nature*, **393** (1998) 440.

[5] Newman M. E. J., *SIAM Rev.*, **45** (2003) 167.

[6] Li Q., Braunstein L. A., Wang H., Shao J., Stanley H. E. and Havlin S., *J. Stat. Phys.*, **151** (2013) 92.

[7] Valdez L. D., Macri P. A. and Braunstein L. A., *PLoS ONE*, **7** (2012) e44188.

[8] Braunstein L. A., Macri P. A. and Iglesias J. R., *Physica A*, **392** (2013) 1788.

[9] Motter A. E., *Phys. Rev. Lett.*, **93** (2004) 098701.

[10] Jost J. and Joy M. P., *Phys. Rev. E*, **65** (2002) 016201; Barahona M. and Pecora L. M., *Phys. Rev. Lett.*, **89** (2002) 054101; Jalan S., Amritkar R. E. and Hu Chin-Kun, *Phys. Rev. E*, **72** (2005) 016212; Nishikawa T. and Motter A. E., *Proc. Natl. Acad. Sci. U.S.A.*, **107** (2010) 10342; Motter A. E., Myers S. A., Anghel M. and Nishikawa T., *Nat. Phys.*, **9** (2013) 191.

[11] Arenas A., Daz-Guilera A., Kurths J., Moreno Y. and Zhou C., *Phys. Rep.*, **469** (2008) 93.

[12] Pastore y Piontti A. L., Macri P. A. and Braunstein L. A., *Phys. Rev. E*, **76** (2007) 046117.

[13] Korniss G., *Phys. Rev. E*, **75** (2007) 051121.

[14] La Rocca C. E., Braunstein L. A. and Macri P. A., *Phys. Rev. E*, **77** (2008) 046120.

[15] La Rocca C. E., Pastore y Piontti A. L., Braunstein L. A. and Macri P. A., *Physica A*, **388** (2009) 233.

[16] La Rocca C. E., Braunstein L. A. and Macri P. A., *Physica A*, **390** (2011) 2840.

[17] Li Xiaoli, Cui Dong, Jiruska Premysl, Fox John E., Yao Xin and Jefferys John G. R., *J. Neurophysiol.*, **98** (2007) 3341.

[18] Valverde S. and Sole R. V., *Eur. Phys. J. B*, **38** (2004) 245.

[19] Rabani Y., Sinclair A. and Wanka R., *Proceedings of the 39th Symposium on the Foundations of Computer Science (FOCS)* (IEEE) 1998, p. 694.

[20] Korniss G., Toroczkai Z., Novotny M. A. and Rikvold P. A., *Phys. Rev. Lett.*, **84** (2000) 1351.

[21] Edwards S. F. and Wilkinson D. R., *Proc. R. Soc. London, Ser. A*, **381** (1982) 17.

[22] Family F., *J. Phys. A*, **19** (1986) L441.

[23] Guclu H., Korniss G. and Toroczkai Z., *Chaos*, **17** (2007) 026104.

[24] Molloy M. and Reed B., *Algorithms*, **6** (1995) 161; *Combinatorics, Probab. Comput.*, **7** (1998) 295.

[25] Cohen R., Havlin S. and ben-Avraham D., *Handbook of Graphs and Networks*, edited by Bornholdt S. and Shuster H. G. (Willey-VCH, New York) 2002, sect. 4.

[26] Boguñá M., Pastor-Satorras R. and Vespignani A., *Eur. Phys. J. B*, **38** (2004) 205.

[27] Catanzaro M., Boguñá M. and Pastor-Satorras R., *Phys. Rev. E*, **71** (2005) 027103.