Transport and Percolation Theory in Weighted Resistor Networks

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Why transport on networks?

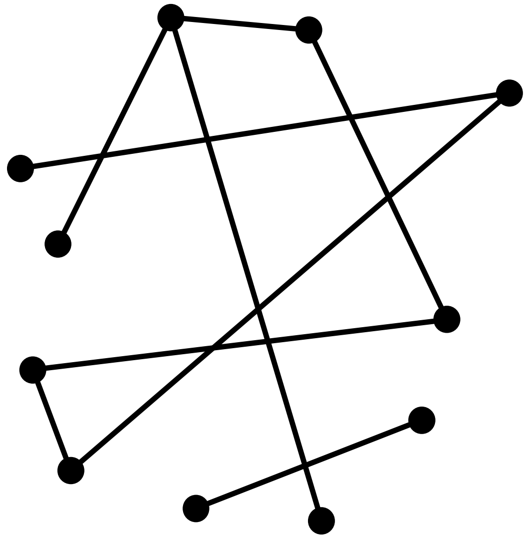
1) Many networks contain flow:

- emails over internet
- epidemics on human networks
- passengers on airline networks, etc.

2) Most work done studies ***static properties*** of networks.

3) No general theory of transport properties of networks.

Network models



Definitions:

A network with N nodes,

each node has k links.

k: Degree of the node

$P(k)$: Degree distribution

Two types of networks:

1) Erdős-Rényi networks (ER)

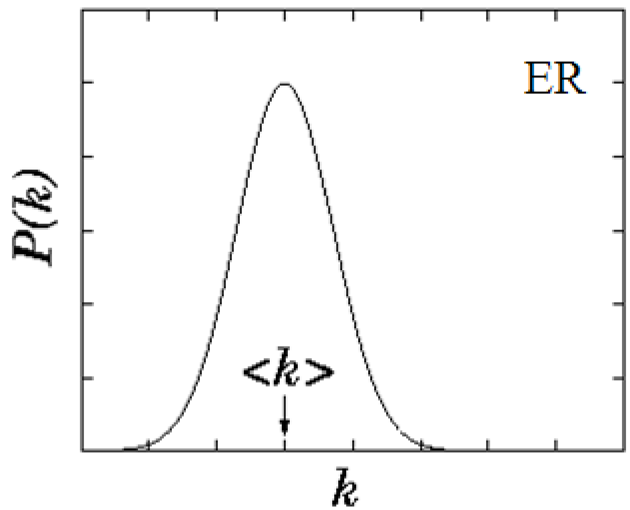
$P(k)$: Poisson distribution

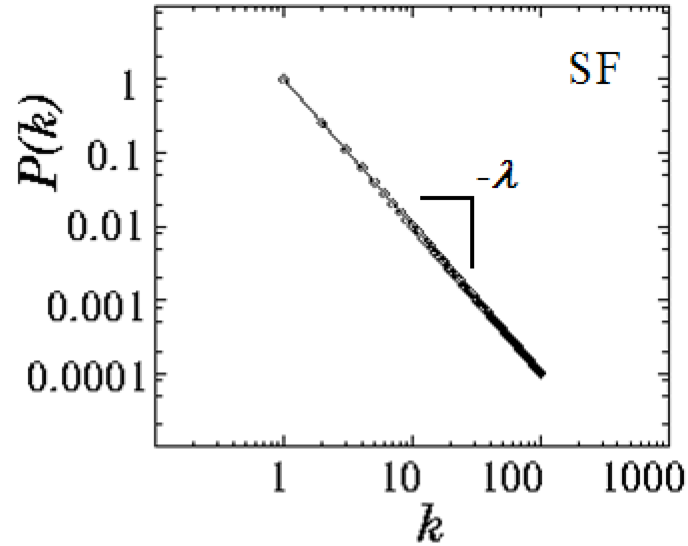
$$P(k) \sim \frac{\langle k \rangle^{k} e^{-\langle k \rangle}}{k!}$$

2) Scale-Free networks (SF)

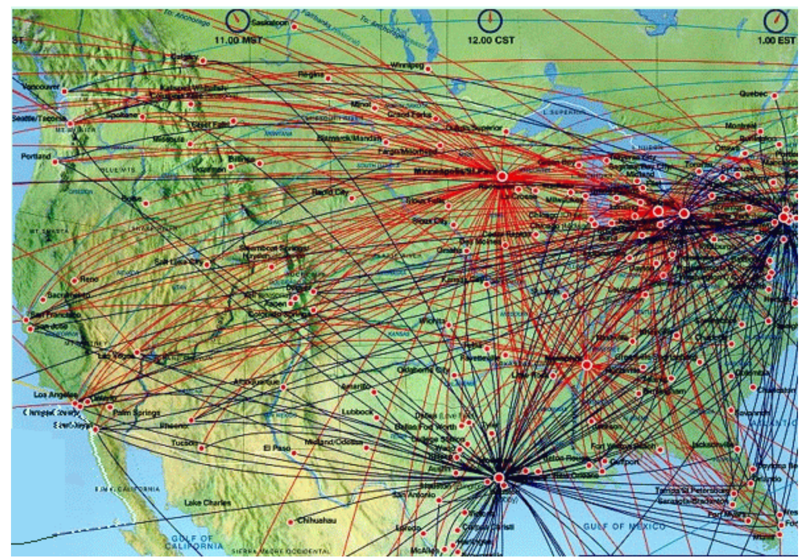
$P(k)$: Power-law distribution

$$P(k) \sim k^{-\lambda}$$





US highway network

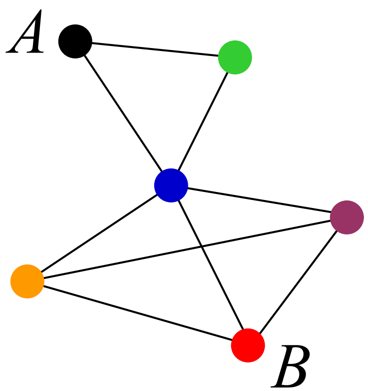


US airline network

by A. Barabási 1999

Resistor network

Each link is a resistor. Assign each link i with conductance $\sigma_i = \exp(- a\, x_i)$ (*)



$a \geq 0$: strength of disorder, if $a >> 0$, strong disorder

$0 < x_i < 1$: uniformly distributed random number

We study conductance distribution $P(\sigma)$ as the transport property:

1) Randomly choose Nodes A and B as source and sink

2) Establish potential difference $V_A - V_B = 1$

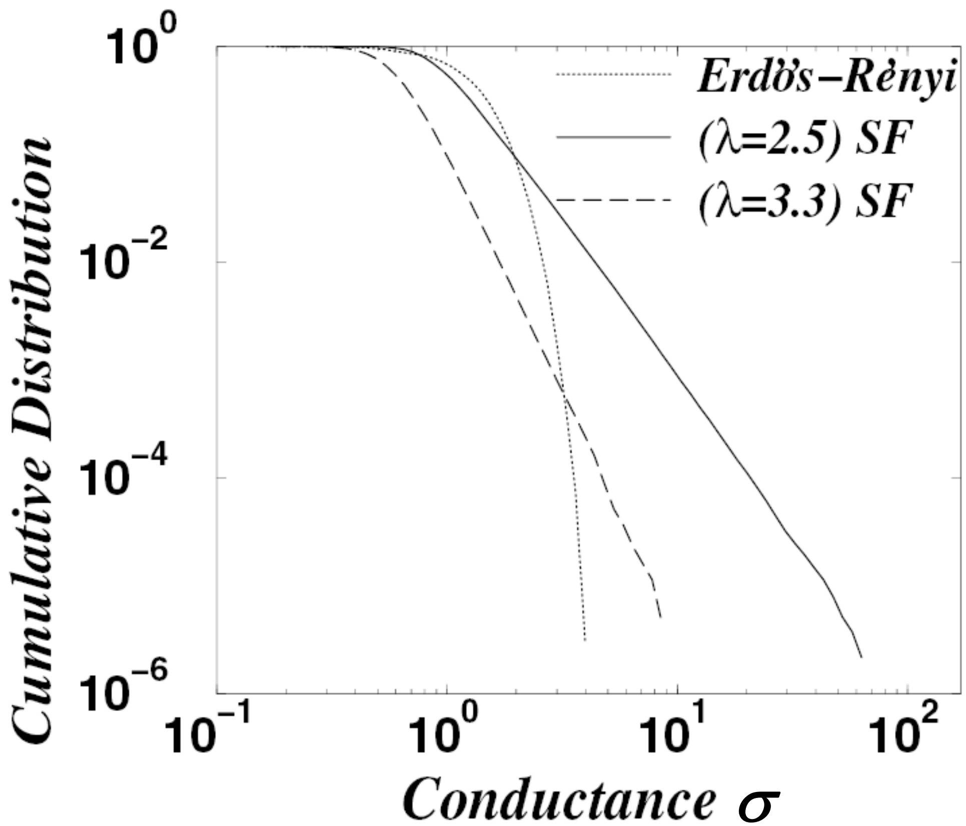
3) Solve Kirchhoff equations for current I, conductance $\sigma_{AB} = I/(V_A - V_B) = I$

4) Perform many realizations ($>10^6$) to determine $P(\sigma)$

(*) Y. M. Strelniker et al. Phys. Rev. E **69**, 065105R 2004

Unweighted case

Each link has conductance $\sigma_i = \exp(-a\, x_i) = 1$, $\boldsymbol{a = 0}$



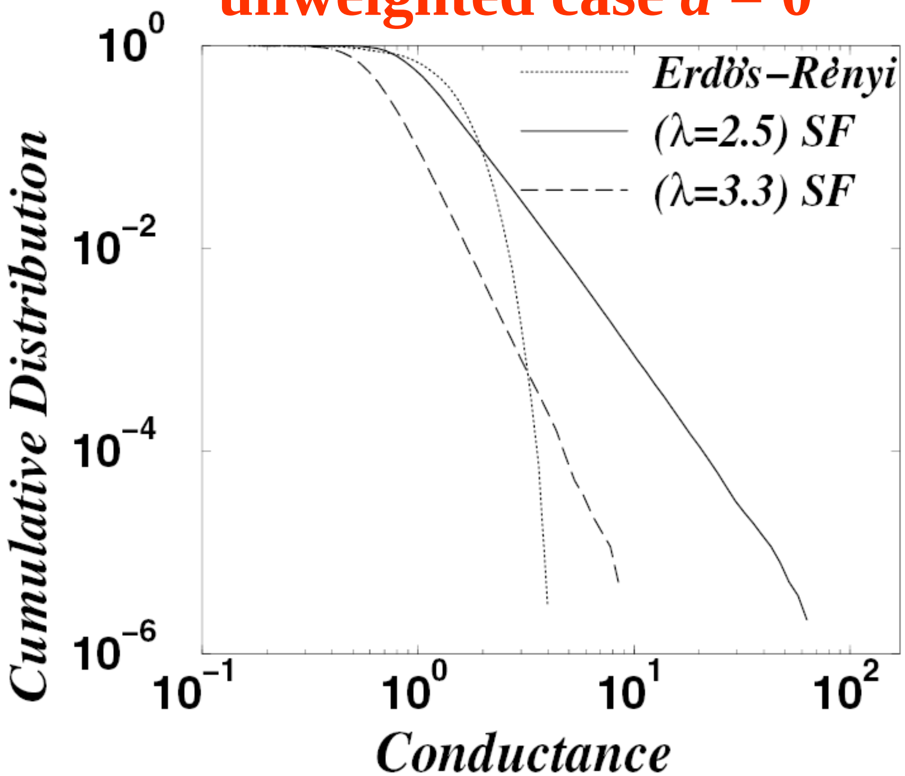
- Erdős-Rényi narrow shape (exponential tail).
- Scale-free wide range (power law tail).
- SF networks exhibit larger values of conductance than ER networks, thus making SF networks better for transport.

E. López et al. Phys. Rev. Lett. **94**, 248701 2005

Questions about weighted case

Each link has conductance σ_i=exp(- a x_i), a > 0

unweighted case *a* = 0



Comparing to unweighted case:

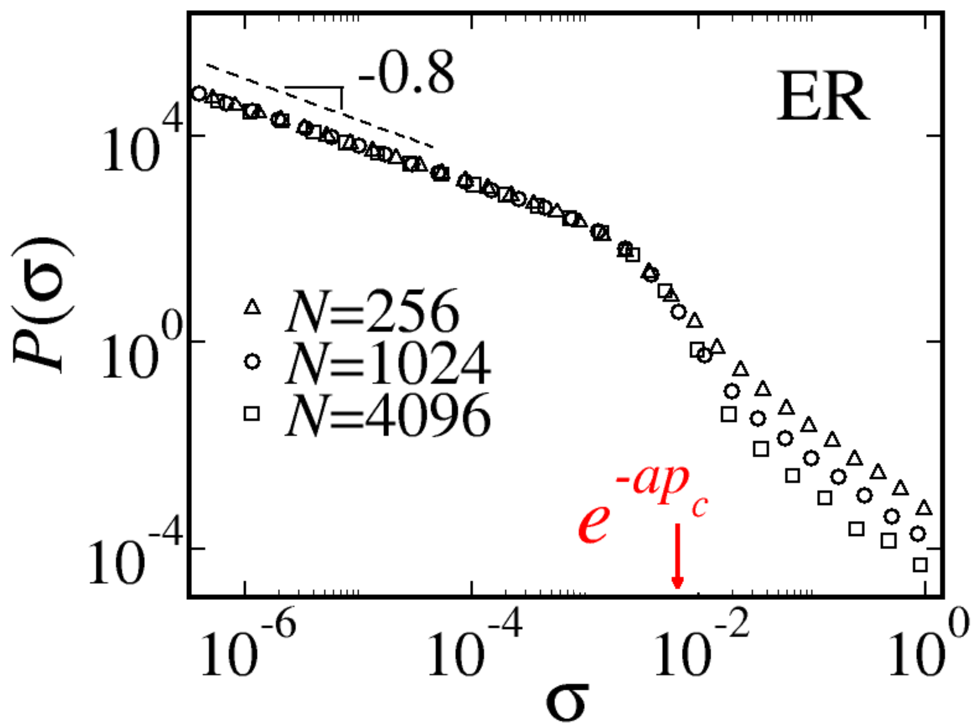
1) Is a high conductance regime expected?

2) How is $P(\sigma)$ related with system size N (Number of nodes)?

3) How about the shape of the tail, still exponential for ER and power law for SF?

$P(\sigma)$ for weighted case (ER)

Strong disorder, a =15 and $\langle k \rangle$=3, so $\langle k \rangle / a$=0.2



- For $\sigma < e^{-a\,p_c}$, where p_c=1/$\langle k \rangle$ is the critical percolation threshold:

$P(\sigma) \sim \sigma^{-\delta}$, where δ=1- $\langle k \rangle / a$

Two regimes:

1) low conductance regime independent of N

(In this figure, - δ= - (1-0.2)= - 0.8)

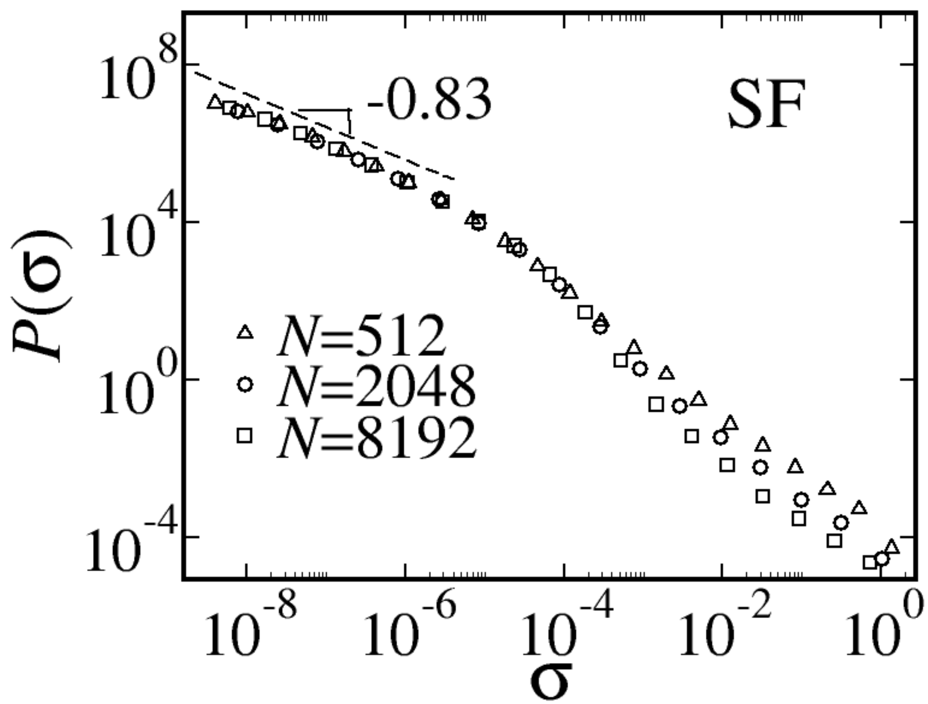
2) high conductance regime with strong N dependence

- For $\sigma > e^{-a\,p_c}$, we find

$P(\sigma) \sim f(\sigma, ap_c/N^{1/3})$

$P(\sigma)$ for weighted case (SF)

a=20 and $\langle k \rangle \approx 3.33$, so $\langle k \rangle / a = 0.17$



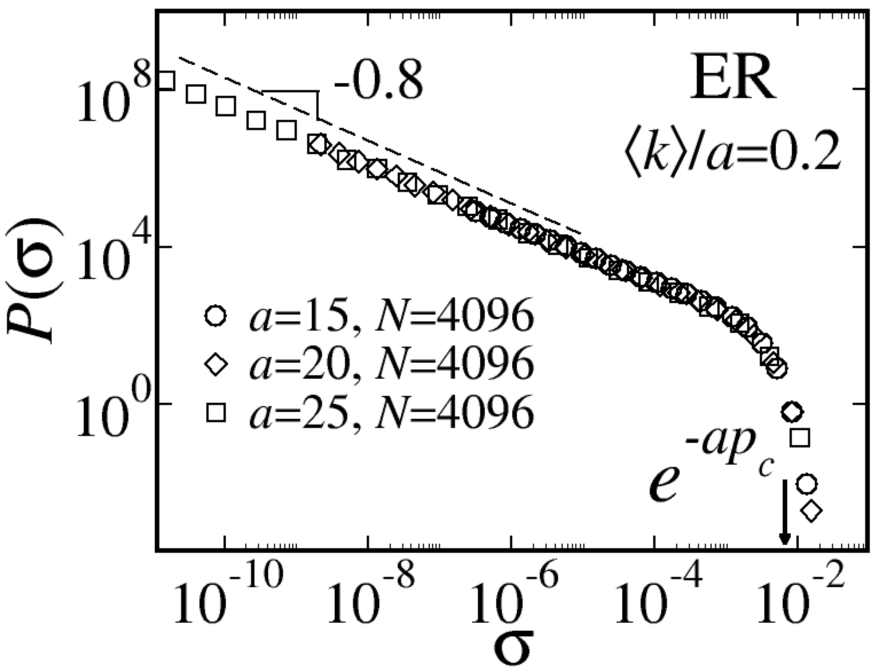
- For $\sigma < e^{-a p_c}$, where $p_c = 1/\langle k \rangle$ is the critical percolation threshold:

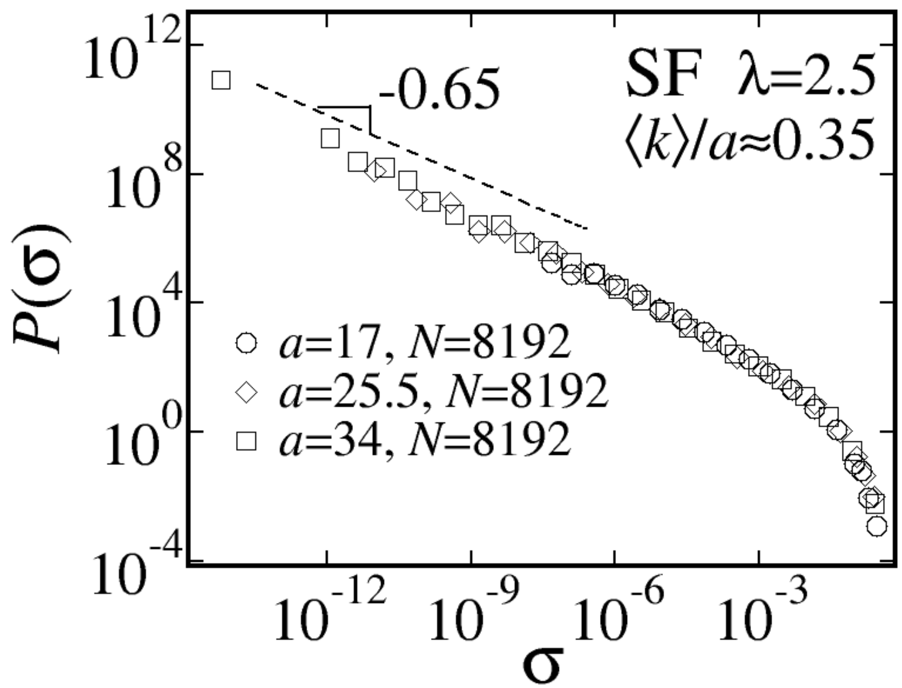
$P(\sigma) \sim \sigma^{-\delta}$, where $\delta = 1 - \langle k \rangle / a$

(In this figure, $-\delta = -(1-0.17) = -0.83$)

$P(\sigma)$ only depends on $\langle k \rangle / a$

$P(\sigma) \sim \sigma^{-\delta}$, where $\delta = 1 - \langle k \rangle / a$

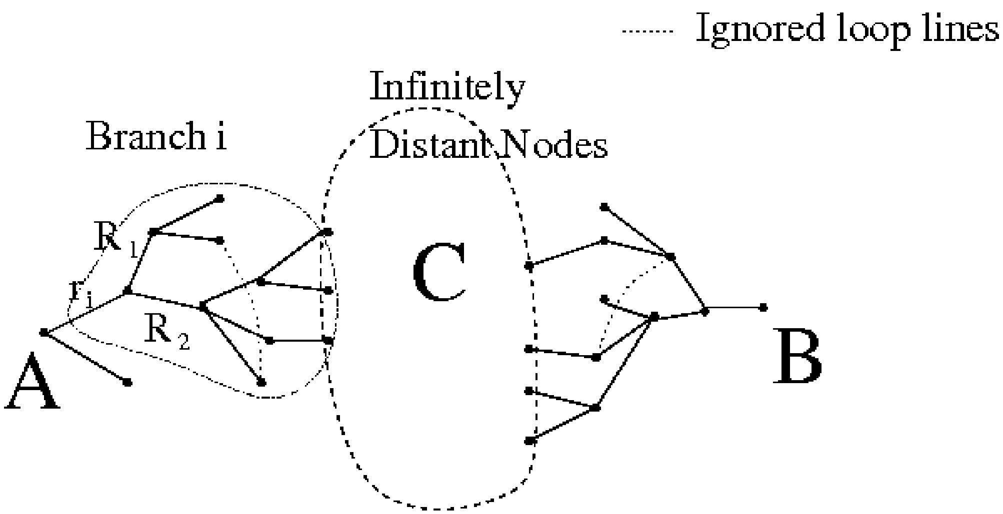




Iterative algorithm

1) Simple and fast

2) It gives us $N \to \infty$ result



Steps:

1) Ignore loops because of low probability when $N \to \infty$

2) Randomly select branch i connecting A with infinitely distant nodes C, then calculate the resistance R_i at step n+1 based on pre step:

$$R_i^{(n+1)} = r_i^{(n)} + \frac{1}{\sum_{j=1}^{k-1} 1/R_j^{(n)}}$$

where $R_i^{(0)}=0$

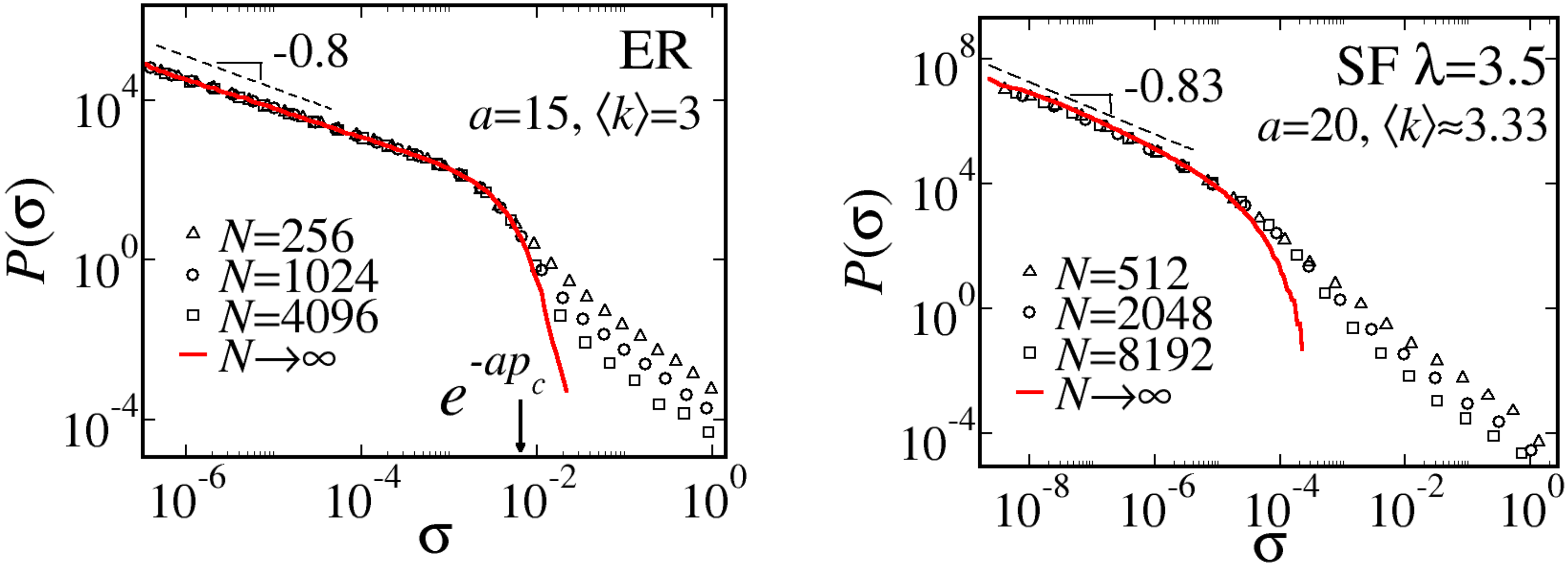
3) Calculate until R_i converges

($n > 10^6$)

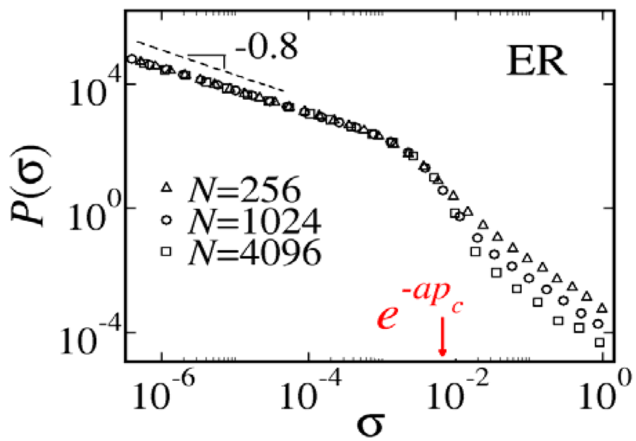
Results of iterative algorithm

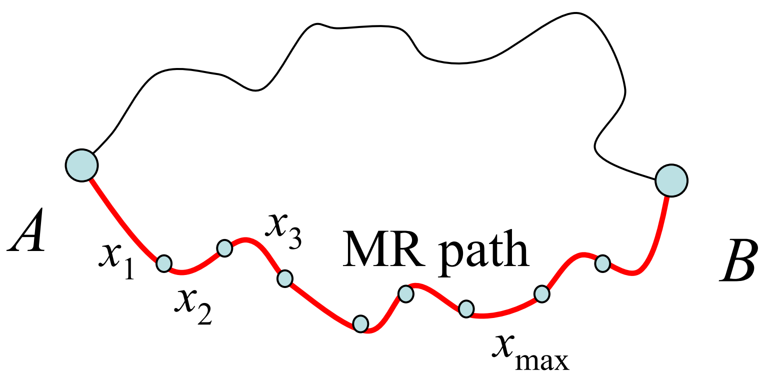
Red solid lines are iterative algorithm results.

Agrees with solving Kirchhoff eqs. method.



Theoretical approach





$$r_i = e^{a\, x_i}$$

Why does $P(\sigma)$ exhibit two conductance regimes?

For strong disorder $a >> 0$, current I follows the minimal resistance path (MR path, the red line),

the resistance between A and B is

dominated by $e^{a\, x_{max}}$, where x_{max} is the max x in the MR path.

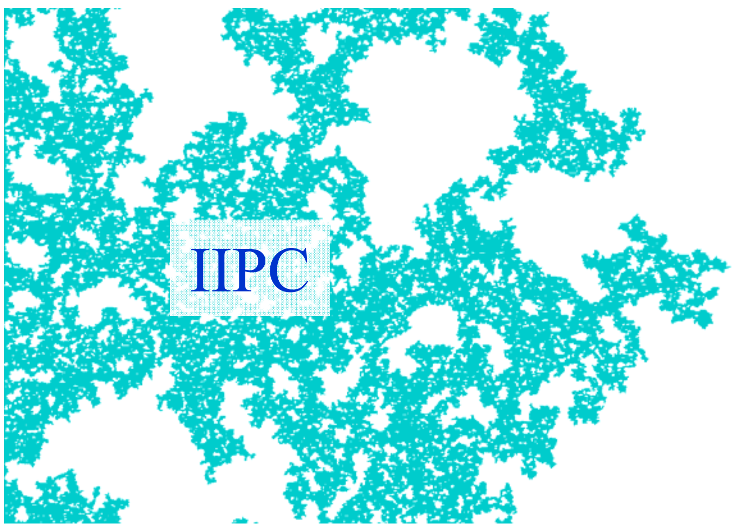
The MR path problem can be mapped onto a percolation problem (*)

(*) by L. A. Braunstein et al. 2005

Percolation process

Start with isolated nodes, randomly connect any two nodes with probability p ($0<p<1$), when $p=p_c$, a giant component cluster appears.

This giant component cluster is called Incipient Infinite Percolation Cluster (IIPC)



If both nodes belong to IIPC, we can always find a minimal resistance path in which $x_{\max} < p_c$, these nodes contribute to high conductance regime.

Since size of IIPC $\propto N^{2/3}$ (*), the probability of both nodes inside IIPC is $(N^{2/3}/N)^2=N^{-2/3}$, we find that indeed:

$$\int_{e^{-ap_c}}^{\infty} P(\sigma)d\sigma \sim N^{-2/3}$$

(*) by L. A. Braunstein et al. 2005

Analytical results (strong disorder)

For $e^{-a} \leq \sigma << e^{-a\,p_c}$ with $p_c = 1/\langle k \rangle$

$$P(\sigma) \approx 2\frac{\langle k \rangle}{a}\sigma^{\langle k \rangle / a - 1}$$

For $\sigma > e^{-a\,p_c}$

$$\langle \sigma \rangle P_p(\sigma) = f\left(\frac{\sigma}{\langle \sigma \rangle}, \frac{a p_c}{N^{1/3}}\right)$$

where $P_p(\sigma)$ is the conductance distribution for the pairs on the IIPC

Conclusions

1) Both analytically and numerically, $P(\sigma)$ exhibit two regimes:

(i) A low conductance regime $\sigma < e^{-a\, p_c}$, $P(\sigma) \sim \sigma^{-\delta}$, where $\delta = 1 - \langle k \rangle / a$

(ii) A high conductance regime $\sigma > e^{-a\, p_c}$, $P(\sigma) \sim f(\sigma, ap_c/N^{1/3})$

2) We developed an iterative algorithm to give $N \to \infty$ result.

3) Compared to the unweighted resistor networks, the conductance is much smaller, and both ER and SF networks exhibit similar distributions.