Transport and Percolation in Complex Networks

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Outline

- Part I: Towards design principles for optimal transport networks G. Li, S. D. S. Reis, A. A. Moreira, S. Havlin, H. E. Stanley and J. S. Andrade, Jr., PRL 104, 018701 (2010); PRE (submitted)

  Motivation: e.g. Improving the transport of New York subway system

  Questions: How to add the new lines?

  → How to design optimal transport network?

- Part II: Percolation on spatially constrained networks

  D. Li, G. Li, K. Kosmidis, H. E. Stanley, A. Bunde and S. Havlin, EPL 93, 68004 (2011)

  Motivation: Understanding the structure and robustness of spatially constrained networks

  Questions: What are the percolation properties? Such as thresholds . . .

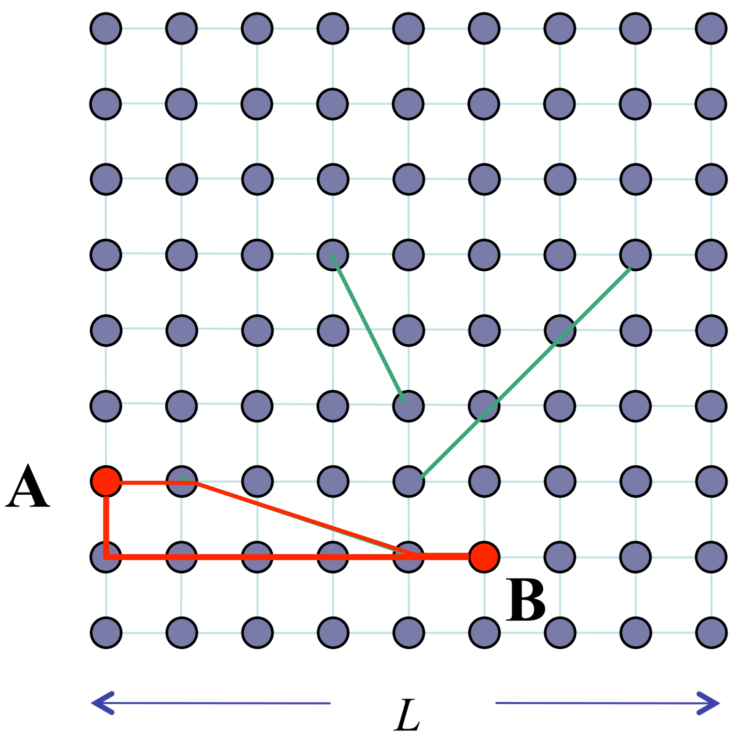
  What are the dimensions in percolation?

  → How are spatial constraints reflected in percolation properties of networks?

Part I: Towards design principles for optimal transport networks

G. Li, S. D. S. Reis, A. A. Moreira, S. Havlin, H. E. Stanley and J. S. Andrade, Jr., PRL 104, 018701 (2010); PRE (submitted)

Grid network



Shortest path length $\ell_{AB} = 6$

$$\langle \ell \rangle \sim L$$

Randomly add some long-range links:

small-world network

$$\langle \ell \rangle \sim \log L$$

$\ell_{AB} = 3$

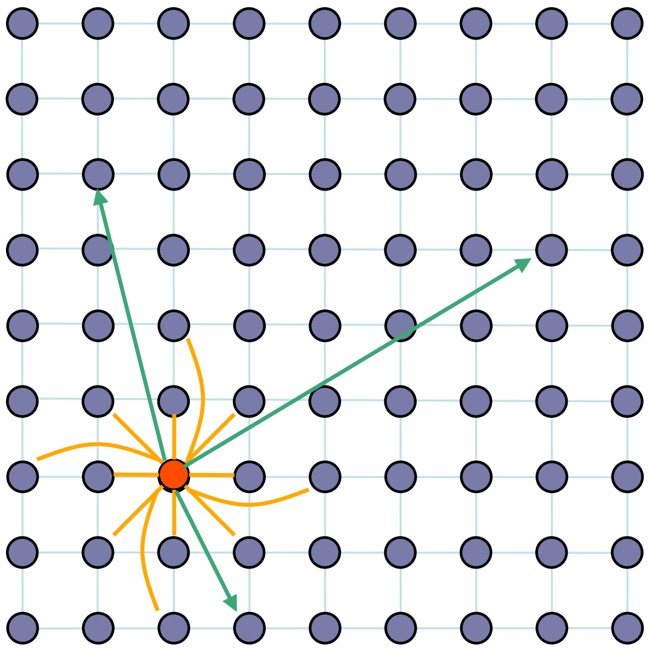
How to add the long-range links?

D. J. Watts and S. H. Strogatz, *Collective dynamics of "small-world" networks*, Nature, **393** (1998).

Part I

How to add the long-range links?

Kleinberg model of social interactions



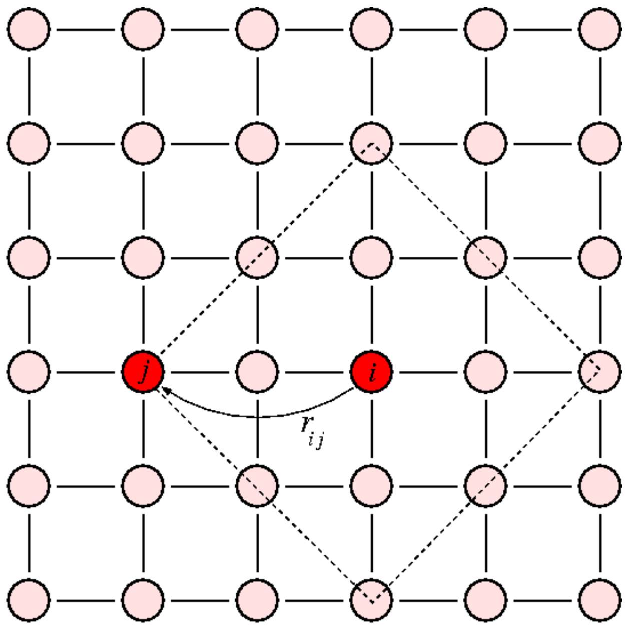
Rich in short-range connections

A few long-range connections

J. Kleinberg, *Navigation in a Small World*, Nature **406**, 845 (2000).

Part I

Kleinberg model continued



$$P(r_{ij}) \sim r_{ij}^{\ -\alpha}$$

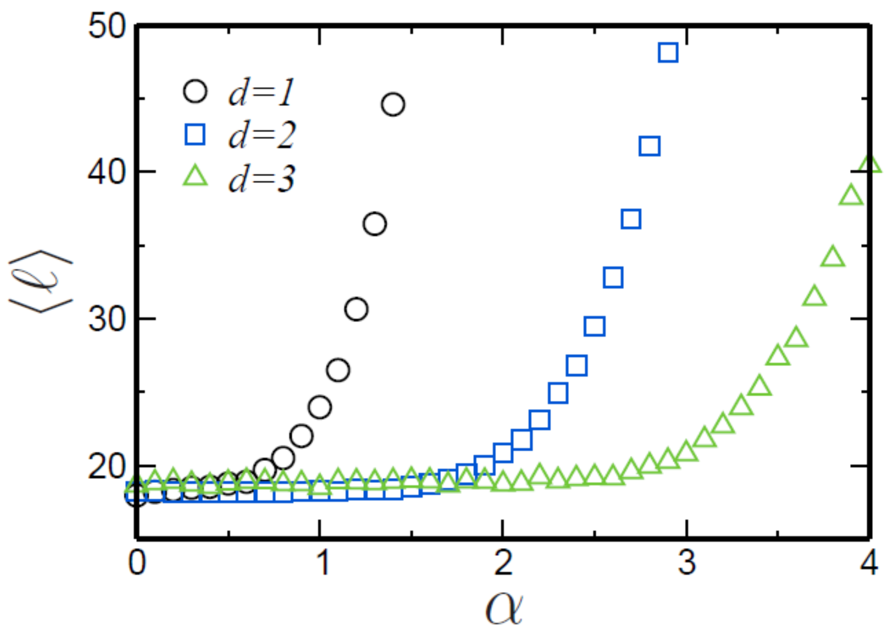
where $\alpha \geq 0$, and r_{ij} is lattice distance between node i and j

Steps to create the network:

- Randomly select a node i
- Generate r_{ij} from $P(r_{ij})$, e.g. $r_{ij} = 2$
- Randomly select node j from those 8 nodes on dashed box
- Connect i and j

Q: Which α gives minimal average shortest distance $\langle \ell \rangle$?

Optimal α in Kleinberg's model



d is the dimension of the lattice

Minimal $\langle \ell \rangle$ occurs at α=0

Without considering the cost of links

Considering the cost of links

- Each link has a cost $\propto$ length r
  (e.g. airlines, subway)
- Have budget to add long-range links
  (i.e. total cost Λ is usually $\propto$ system size N)
- Trade-off between the number N_l and length of added long-range links

From $P(r) \sim r^{-\alpha}$:

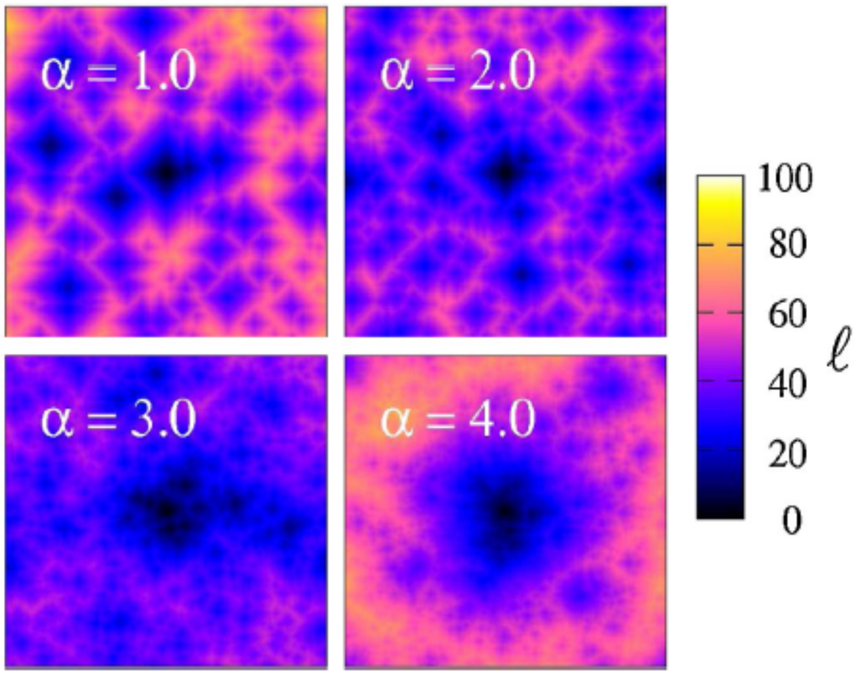
$\alpha=0$, $\langle r \rangle$ is large, $N_l = \Lambda / \langle r \rangle$ is small

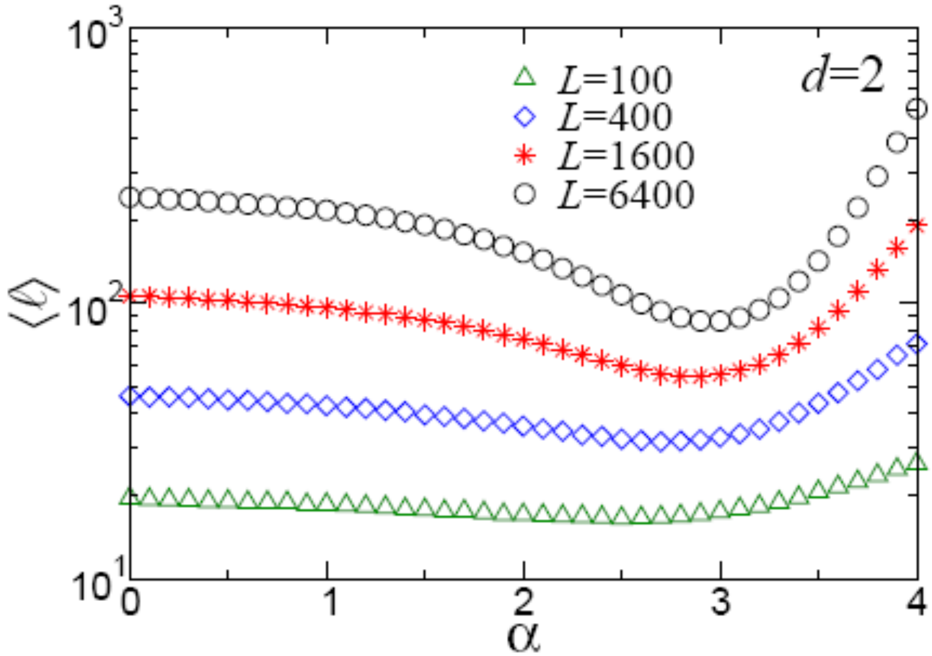
α large, $\langle r \rangle$ is small, N_l is large

Q: Which α gives minimal $\langle \ell \rangle$ with cost constraint?

Part I

With cost constraint



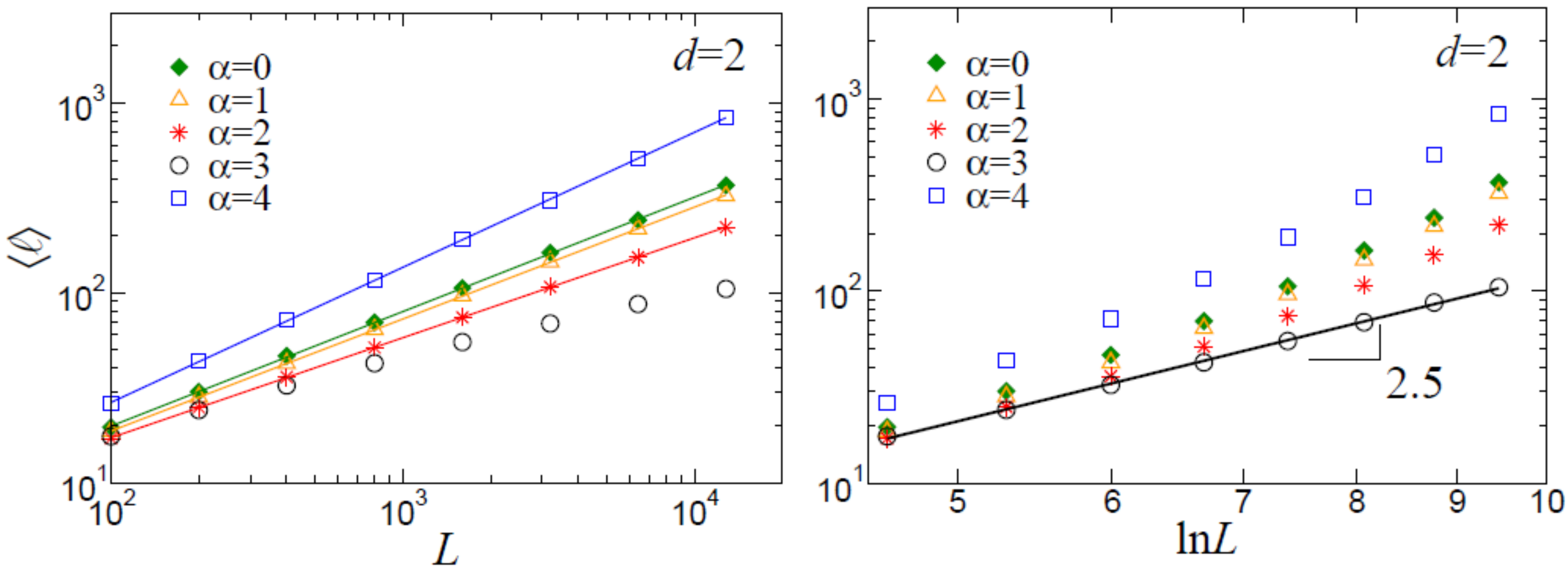


ℓ is the shortest path length from each node to the central node

Minimal $\langle \ell \rangle$ occurs at α=3

Part I With cost constraint

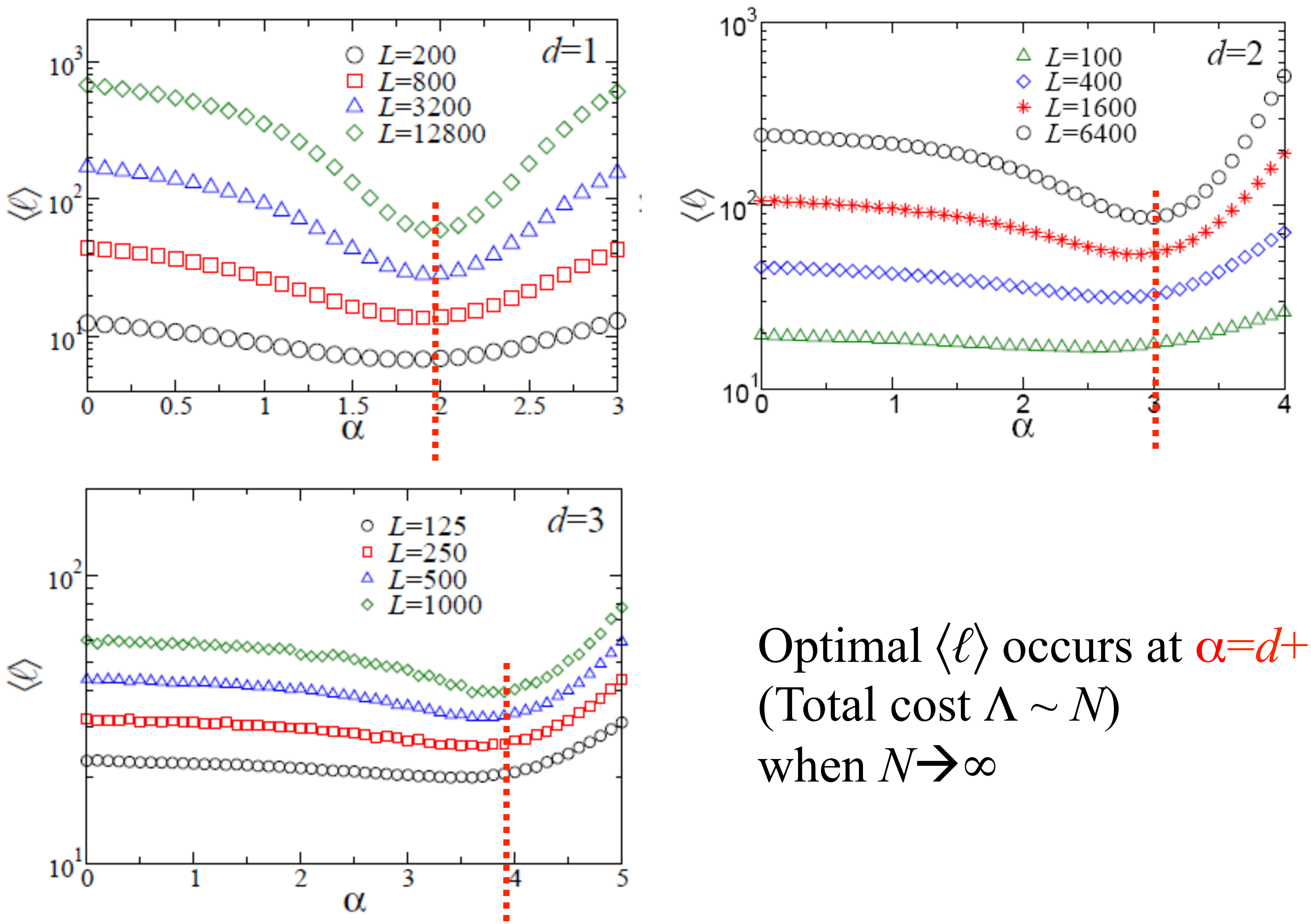
$\langle \ell \rangle$ vs. L (Same data on log-log plots)



Conclusion:

For $\alpha \neq 3$, $\langle \ell \rangle \sim L^{\delta}$ For $\alpha = 3$, $\langle \ell \rangle \sim (\ln L)^{\gamma}$

Different lattice dimensions



Optimal $\langle \ell \rangle$ occurs at $\alpha = d+1$
(Total cost $\Lambda \sim N$)
when $N \to \infty$

Conclusion, part I

For regular lattices, d=1, 2 and 3, optimal transport occurs at $\alpha=d+1$

More work can be found in thesis (chapter 3):

1. Extended to fractals, optimal transport occurs at $\alpha= d_f+1$
2. Analytical approach

Empirical evidence

1. Brain network:

L. K. Gallos, H. A. Makse and M. Sigman, PNAS, **109**, 2825 (2011).

d_f=2.1±0.1, link length distribution obeys $P(r) \sim r^{-\alpha}$ with α=3.1±0.1

2. Airport network:

G. Bianconi, P. Pin and M. Marsili, PNAS, **106**, 11433 (2009).

d=2, distance distribution obeys $P(r) \sim r^{-\alpha}$ with α=3.0±0.2

Part II: Percolation on spatially constrained networks

D. Li, G. Li, K. Kosmidis, H. E. Stanley, A. Bunde and S. Havlin, EPL 93, 68004 (2011)

$$P(r) \sim r^{-\alpha}$$

α controls the strength of spatial constraint

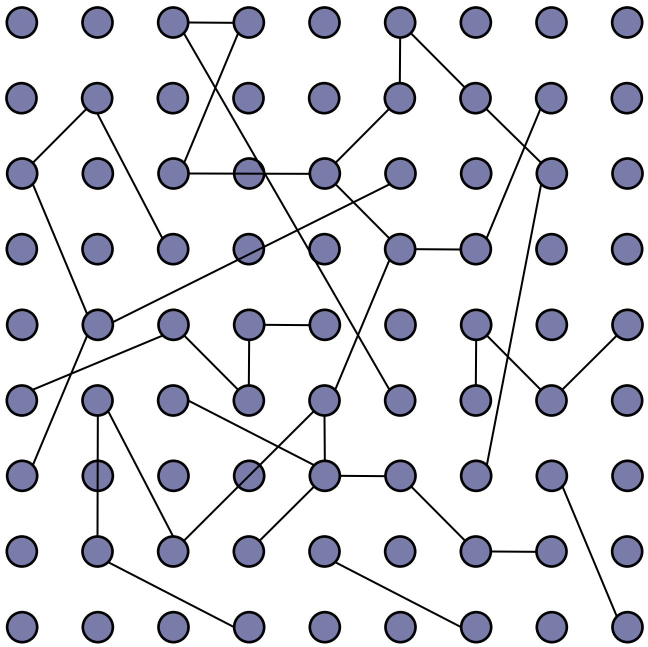
- α=0, no spatial constraint → ER network
- α large, strong spatial constraint → Regular lattice

Questions:

- What are the percolation properties? Such as the critical thresholds, etc.
- What are the fractal dimensions of the embedded network in percolation?

Part II

The embedded network



Start from an empty lattice

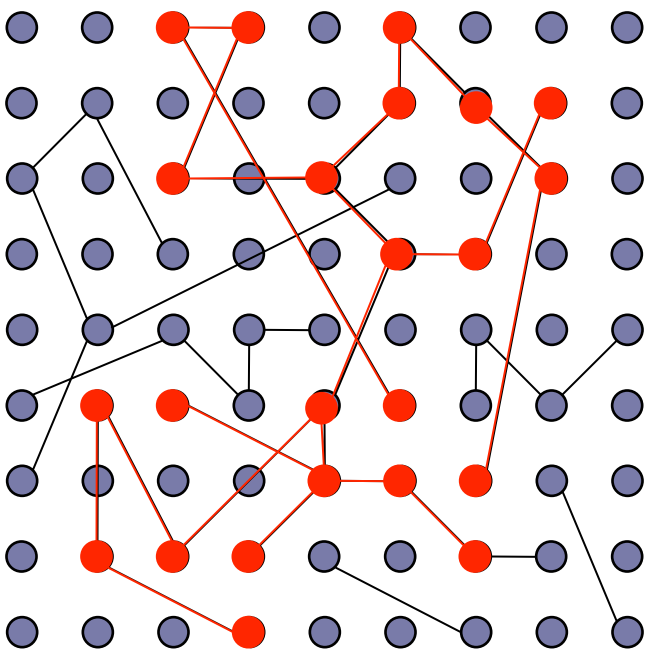
Add long-range connections with $P(r) \sim r^{-\alpha}$ until $\langle k \rangle \sim 4$

Two special cases:

α=0 → Erdős-Rényi(ER) network

α large → Regular 2D lattice

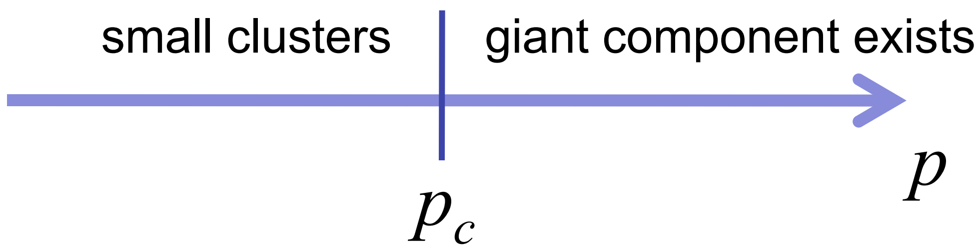
Percolation process



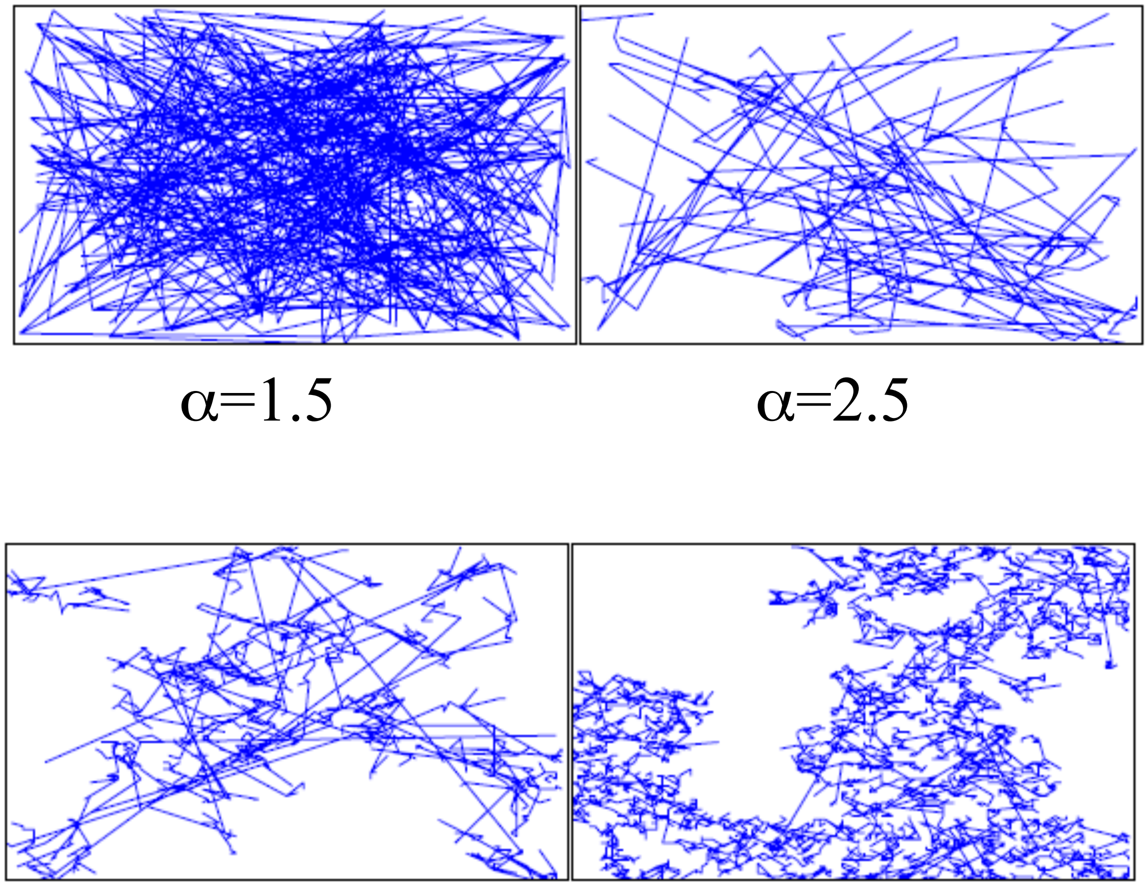
Start randomly removing nodes

Remove a fraction q of nodes, a giant component (red) exists

Increase q, giant component breaks into small clusters when q exceeds a threshold q_c, with $p_c=1-q_c$ nodes remained



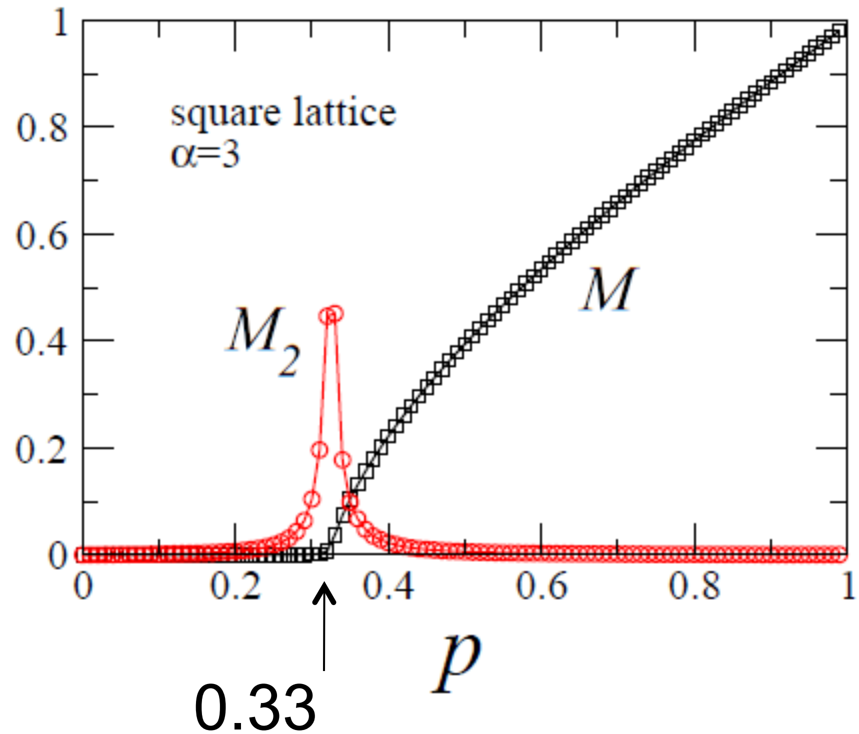
Giant component in percolation



α=1.5 α=2.5

α=3.5 α=4.5

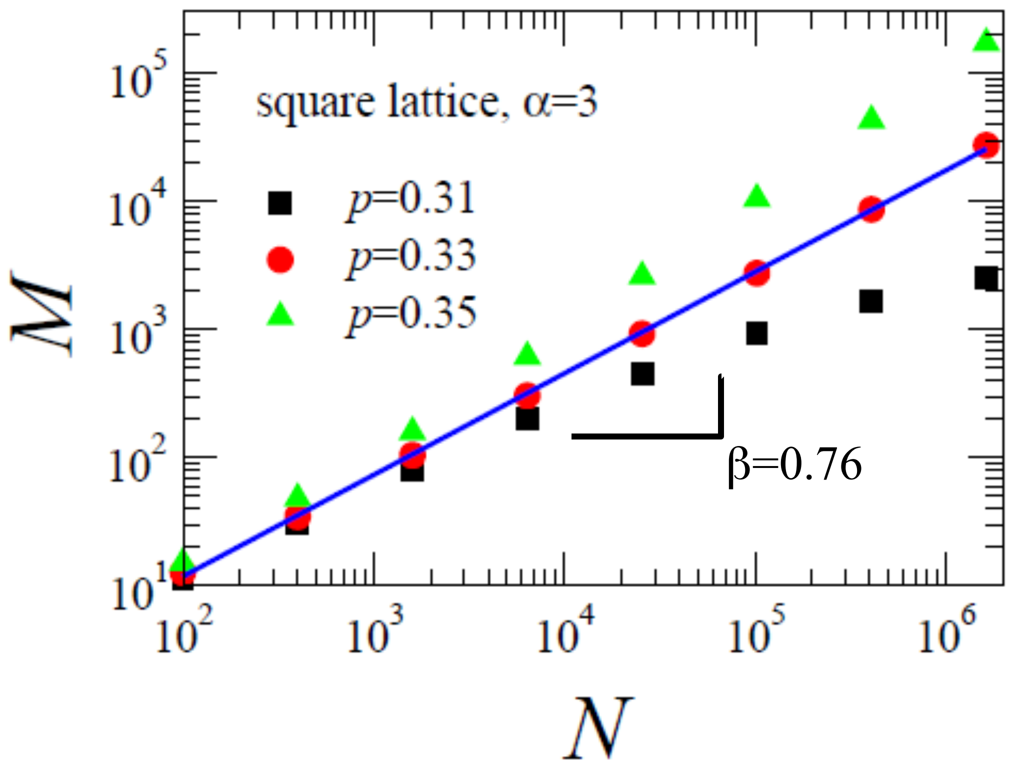
Result 1: Critical threshold p_c



α	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**	**5**
p_c	0.25	0.25	0.27	0.33	0.41	0.49	0.57
	ER (Mean field) $p_c=1/\langle k\rangle=0.25$		Intermediate regime				Lattice $p_c \approx 0.59$

Part II

Result 2: Size of giant component: M



$M \sim N^{\beta}$ in percolation

β: critical exponent

α	1.5	2.0	2.5	3.0	3.5	4.0	5
β	0.67	0.67	0.70	0.76	0.87	0.93	0.94
	ER (Mean field) β=2/3		Intermediate regime				Lattice β=0.95

Result 3: Dimensions

$M \sim N^{\beta}$ in percolation

- ❑ Percolation ($p = p_c$): $\boldsymbol{M \sim R^{d_f}}$
- ❑ Embedded network ($p = 1$): $\boldsymbol{N \sim R^{d_e}}$

⇨ $\boldsymbol{M \sim N^{d_f / d_e}}$ ⇨ $\boldsymbol{\beta = d_f / d_e}$

Examples:

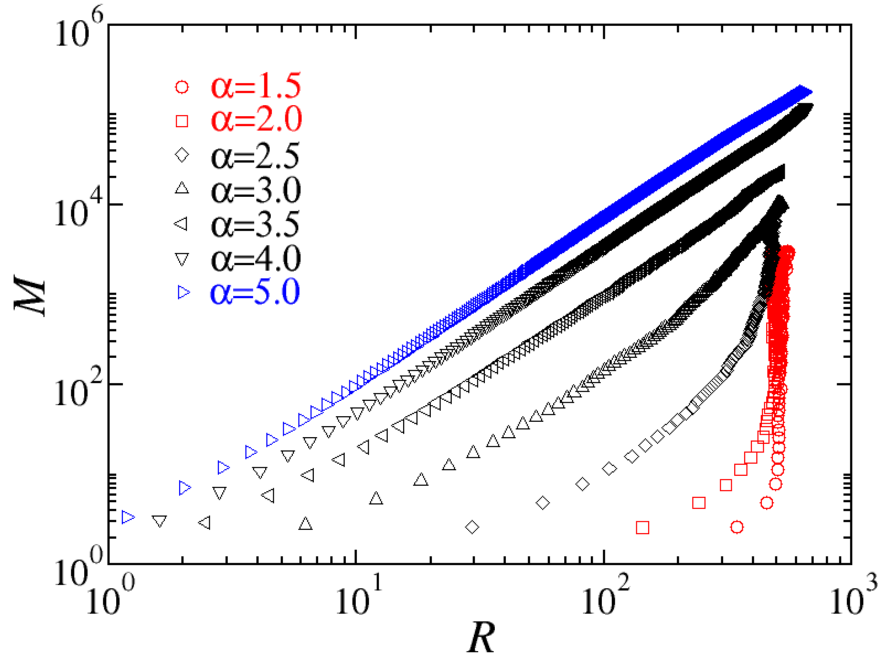
- ER: $d_f \rightarrow \infty$, $d_e \rightarrow \infty$, but $\beta = 2/3$
- 2D lattice: $d_f = 1.89$, $d_e = 2$, $\beta = 0.95$

So we compare β with $\boldsymbol{d_f / d_e}$

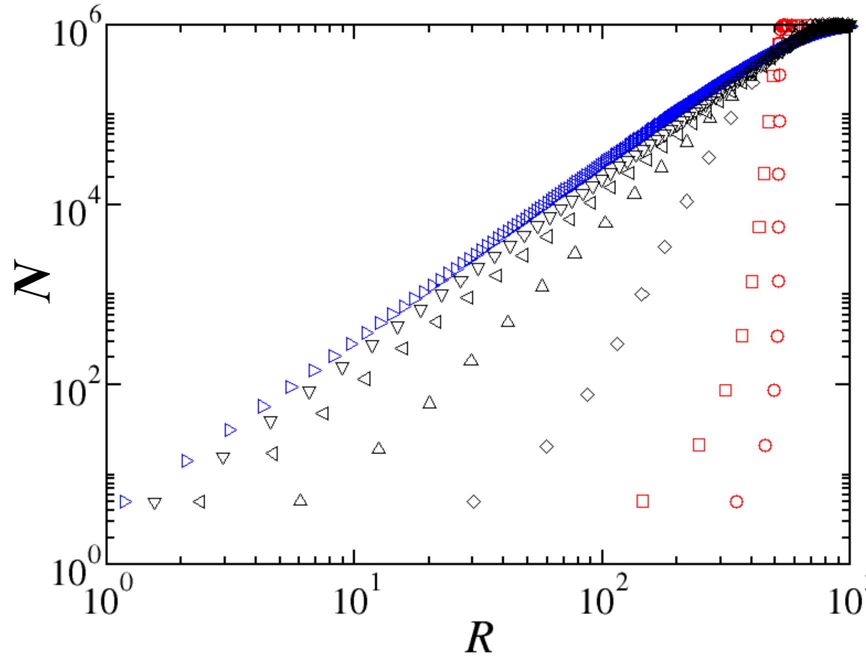
Part II

compare β with d_f / d_e

In percolation

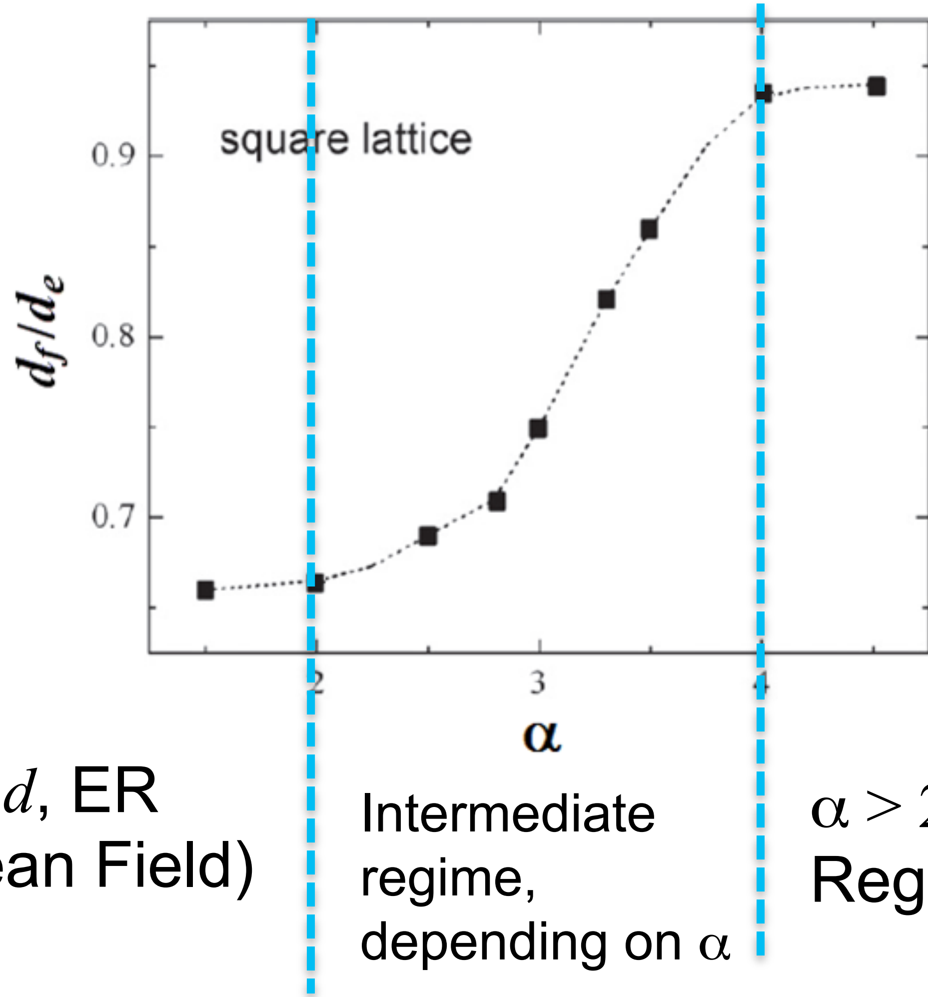


Embedded network



α	1.5	2.0	2.5	3.0	3.5	4.0	5
d_f	∞	∞	3.92	2.12	1.92	1.89	1.87
d_e	∞	∞	5.65	2.76	2.18	2.00	1.99
d_f/d_e			0.69	0.76	0.88	0.94	0.94
β			0.70	0.76	0.87	0.93	0.94

Conclusion, part II: Three regimes



$\alpha \leq d$, ER
(Mean Field)

Intermediate
regime,
depending on α

$\alpha > 2d$,
Regular lattice

Transport properties $\langle \ell \rangle$ still show three regimes.

K. Kosmidis, S. Havlin and A. Bunde, EPL **82**, 48005 (2008)

Summary

- For cost constrained networks, optimal transport occurs at $\alpha = d+1$ (regular lattices) or d_f+1 (fractals)

- The structure of spatial constrained networks shows three regimes:

1. $\alpha \leq d$, ER (Mean Field)
2. $\mathrm{d} < \alpha \leq 2d$, Intermediate regimes, percolation properties depend on α
3. $\alpha > 2d$, Regular lattice