A Letters Journal Exploring the Frontiers of Physics

August 2015

EPL, **111** (2015) 46001 www.epljournal.org
doi: 10.1209/0295-5075/111/46001

Synchronization in interacting scale-free networks

M. F. Torres[1], M. A. Di Muro[1], C. E. La Rocca[1] and L. A. Braunstein[1,2]

[1] *Instituto de Investigaciones Físicas de Mar del Plata (IFIMAR)-Physics Department, Universidad Nacional de Mar del Plata-CONICET - Funes 3350, (7600) Mar del Plata, Argentina*
[2] *Center for Polymer Studies, Boston University - Boston, MA 02215, USA*

received 8 May 2015; accepted in final form 11 August 2015
published online 4 September 2015

PACS 68.35.Ct – Interface structure and roughness
PACS 05.45.Xt – Synchronization; coupled oscillators
PACS 89.75.Da – Systems obeying scaling laws

Abstract – We study the fluctuations of the interface, in the steady state, of the Surface Relaxation Model (SRM) in two scale-free interacting networks where a fraction q of nodes in both networks interact one to one through external connections. We find that as q increases the fluctuations on both networks decrease and thus the synchronization reaches an improvement of nearly 40% when $q = 1$. The decrease of the fluctuations on both networks is due mainly to the diffusion through external connections which allows to reducing the load in nodes by sending their excess mostly to low-degree nodes, which we report have the lowest heights. This effect enhances the matching of the heights of low- and high-degree nodes as q increases reducing the fluctuations. This effect is almost independent of the degree distribution of the networks which means that the interconnection governs the behavior of the process over its topology.

Introduction. – In the last decades the study of complex networks has been growing strongly due to the large number of systems that exhibit this type of structures. A complex network is a set of nodes that are connected by internal links and the most fundamental property that characterizes its topology is the degree distribution $P(k)$, which represents the probability that a node has k neighbors or connectivity k. It has been found that many real systems such as social, communication and biological networks present a degree distribution given by $P(k) \sim k^{-\lambda}$, where λ is the exponent of the power law and $k_{min} \leq k \leq k_{max}$, where k_{min} and k_{max} are the minimum and maximum degree of the network. These kind of networks are called Scale Free (SF) and one of its most important features is that are in general very heterogeneous, *i.e.* most nodes of the network have a low connectivity while only a few have a high connectivity (hubs). In recent years the study of synchronization processes in isolate complex networks has been increasing because of its importance in neurobiology [1–5] and population dynamics [6,7]. A common theoretical approach to study synchronization in complex networks is to map this process onto an interface growth model by assigning to each node a scalar field h_i, with $i = 1, \cdots, N$, where N is the size of the network. This scalar field could represent, for example, the amount of load on a node in the problem of distributed parallel computing on processors. Without loss of generality we will relate the scalar field to a set of heights on the interface. The most relevant magnitude that characterizes the interface is $W(t) \equiv W$, which represents the fluctuations of the scalar field around its mean value on the network, given by

$$W = \left\{ \frac{1}{N} \sum_{i=1}^{N} (h_i - \langle h \rangle)^2 \right\}^{1/2}, \qquad (1)$$

where $\langle h \rangle = \frac{1}{N} \sum_{i=1}^{N} h_i$ is the average of the scalar field over the nodes at time t and $\{\ \}$ is the average over different network realizations. The roughness evolves in time until it saturates at the steady state. In the saturation regime W_s is a constant that depends only on λ. In complex networks the synchronization of the system is related to the roughness in the saturation regime [8–16].

One of the most simple and used models to study synchronization in complex networks is the Surface Relaxation Model (SRM) [8–12]. In this model, at each time step, a node i is randomly chosen and the node with the lowest height between the chosen node and all its neighbors evolves increasing its height. It has been found that for isolated SF networks, with $\lambda < 3$, $W_S \sim \ln N$ [8,10,17]

until a critical value $N = N^*$, after which W_s becomes independent of N, which means that the system becomes scalable [9,17]. Although it was an interesting result, many real systems are not isolated but interacting with other systems instead. This means that a process that develops in one network can be affected by a process developing in another and vice versa [18–34]. Such is the case of epidemic models where the interaction between networks make it very harmful for the healthy populations because the interaction increases the theoretical risk of infection compared with the same process in isolated networks [35–40].

These interacting systems can be modeled as networks that interact through external links that connect nodes that belongs to different networks. Now the question is, does synchronization in interacting networks performs worse or better than synchronization in isolated networks? In order to answer this question, in this letter we study the synchronization of two SF networks with the same size N that interact through a fraction q of nodes connected, one by one, between them. The model used is the SRM model which we adapt to interacting networks and study the effect of the interaction parameter q on the fluctuations in both networks.

Model. – In our model, two uncorrelated SF networks, called A and B, with the same size N and exponent λ_A and λ_B, respectively, are built using the Molloy-Reed algorithm [41] disallowing self-loops and multiple connections. As we need a single interface on each network, in order to ensure that we have a single component we use $k_{min} = 2$ [42]. Each node $i \in \alpha$, with $\alpha = A, B$, has a connectivity k_i^α and we denote the set of its neighbors by v_i^α. In order to build the external connections between the networks we connect by simplicity, the first $q\,N$ nodes in A one by one with the first $q\,N$ in B, where q is the interaction parameter with $0 < q \leq 1$. Notice that if both networks are uncorrelated this procedure is the same as connecting a fraction q of nodes at random. We define the vector M, where M_i, with $i = 1, \ldots, N$, is equal to 1 if the node i in A has an external connection with i in B, and $M_i = 0$ otherwise. In order to simplify the growth rules we choose random initial conditions for the scalar field in the interval $[0, 1]$, hence, we avoid the cases in which different nodes have equal heights, as we are only interested in the saturation regime on both networks where the initial condition plays no role.

The evolution rules of the interface growth are given as follows:

1) A network α (with $\alpha = A, B$) is chosen with probability 1/2 and then a "particle", which represents the load, is dropped in a node i selected randomly in α.

2) The particle diffuses to the node ϵ that is the node with the lowest height between the node i and its neighbors v_i^α.

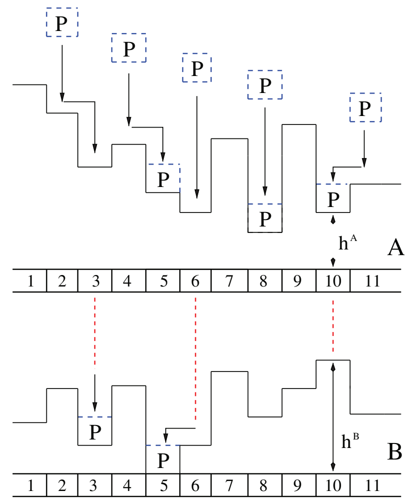

Fig. 1: (Colour on-line) Schematic of the rules of the model in a one-dimensional Euclidean lattice. The particles P are only dropped on the network A in this scheme. The numbers represent different nodes in each network and the red dotted lines represent the external connections between nodes of different networks. In this case $N = 11$ and $q = 3/11$. The arrows indicate the path that the particles follow, which goes from the node where the particle was originally dropped, to the node where the particle gets finally deposited. The height of the nodes is measured from the upper line of the boxes that represent the numbers assigned to the nodes in each network.

3) If $M_\epsilon = 0$ or $M_\epsilon = 1$ and $h_\epsilon \in \alpha < h_\epsilon \in \beta$ (with $\beta \neq \alpha$) the particle is deposited in $\epsilon \in \alpha$. Otherwise the particle diffuses to the network β and is deposited in the node with the lowest height between ϵ and its neighbors v_ϵ^β.

Thus if we denote $\ell \in \alpha$ as the node where the particle is finally deposited, then $h_\ell^\alpha = h_\ell^\alpha + 1$. At each Monte Carlo step the time is increased by $1/2N$. In fig. 1 we show a schematic of the rules of the process for the case of a one-dimensional lattice.

Results and discussions. – We are interested in the behavior of the fluctuations in the steady state of both networks with the system size above N^* [17], value for which the system is scalable. For isolated SF networks this regime for $\lambda < 3$ is close to $N^* \approx 2 \times 10^5$ [17]. We check that the nature of this regime is due to the distribution of internal connectivities and that it is almost not affected by the interaction parameter q. Thus in our research we use $N = N_A = N_B = 3 \times 10^5$ in order to ensure that we are in the scalable regime. We will show our results only for $\lambda_A = 2.6$ and $\lambda_B = 3$, because all the other combinations of the exponents λ in $2.5 < \lambda \leq 3$ give qualitatively the same results. We compute the fluctuations in the saturation regime of both networks $W_s^\alpha = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(h_i^\alpha - \langle h^\alpha \rangle)^2}$, with $\alpha = A, B$ and in fig. 2 we show W_S^α as a function of q. It is clear that

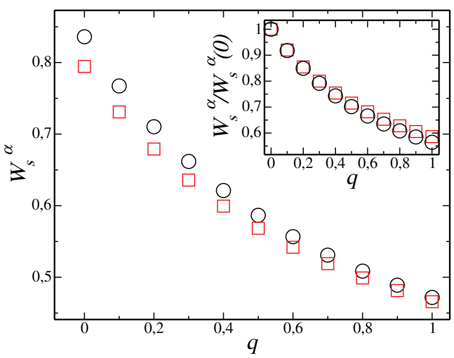

Fig. 2: (Colour on-line) W_s^α as a function of q for A ($\bigcirc$) and B ($\square$) with $\lambda_A = 2.6$ and $\lambda_B = 3$. In the inset $W_S^\alpha / W_S^\alpha(q=0)$ as a function of q.

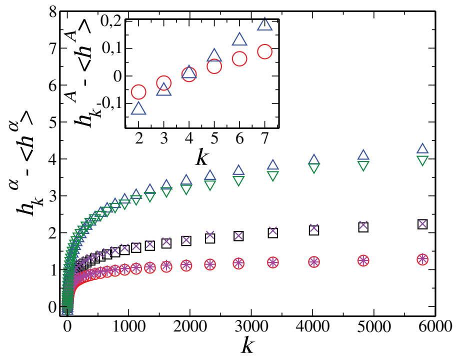

Fig. 3: (Colour on-line) $h_k^\alpha - \langle h^\alpha \rangle$ as a function of k for $q = 0$ in A ($\triangle$) and B ($\bigtriangledown$), $q = 0.5$ in A ($\times$) and B ($\square$) and $q = 1$ in A ($\bigcirc$) and B ($*$). The inset is an amplification of $h_k^\alpha - \langle h^\alpha \rangle$ for the lowest values of k for $q = 0$ in A ($\triangle$) and $q = 1$ in A ($\bigcirc$).

as the interaction parameter q increases, W_S^α decreases in both networks, which implies that the synchronization improves. From the plot we can also observe that as q increases, the difference between W_S^A and W_S^B becomes smaller, which means that the synchronization in each network becomes mainly controlled by q and not by the internal degree distributions. In the inset of fig. 2 we show $W_S^\alpha / W_S^\alpha(q=0)$ as a function of q. It can be seen how is the rate of improvement in the synchronization with the increment of q. For example, for $q = 0.3$ the synchronization enhances approximately 20% and for $q = 1$ around 40%. It is worth pointing out that this rate decreases as q increases, and this means that the effect of the optimization gets less significant as the networks have more interconnections between them.

To understand the effect of the interaction parameter q on the optimization of the process we compute the difference between the average height of nodes with degree k, denoted by h_k^α and the mean value of the height $\langle h^\alpha \rangle$ as a function of k. In fig. 3 we show $h_k^\alpha - \langle h^\alpha \rangle$ as a function of k for different values of q. We can see that, for $q = 0$ the heights of low-connectivity nodes, which are the majority in SF networks, are closer to the average height of the network than the heights of high-degree nodes, which are above $\langle h^\alpha \rangle$. This means that hubs are usually overloaded because all their neighbors send them their excess of load, affecting negatively the synchronization of the system. However, as the factor q increases the height of the hubs decreases, approaching to $\langle h^\alpha \rangle$ and becoming independent of k for $q = 1$. In the inset of fig. 3, we can see an amplification of the behavior of h_k^α for the nodes with the lowest connectivity. We can see that as q increases their heights also approach to the average value. These results imply that as we increase the factor q hubs are no longer overloaded and therefore their excess of load is now absorbed by low-connectivity nodes.

In order to understand the diffusion process between networks we want to know how frequently the load spreads from one network to another and how it gets distributed after diffusion. Hence we measure the rates $\nu_{\alpha\beta}$ at which a particle dropped in the network α gets finally deposited in the network β. In fig. 4 we plot $\nu_{\alpha\beta}$ as a function of q. When $\alpha = \beta$ the rate always decreases with q, because the particles have more chances to cross to the other network. We can see that the rate is always much bigger when $\alpha = \beta$ than in the case of $\alpha \neq \beta$, which means that the particles tend to diffuse in the same network most of the time. The fact that a small portion of load that crosses between networks is enough to enhance the synchronization is due to the fact that the dynamics of the SRM in both networks are coupled due to the interaction. Another observation is that all the rates do almost not depend on the degree distribution of each network. The majority of nodes with $M_i \neq 0$ have an external connection with low-connectivity nodes, due to the fact that these are selected at random and are the most common ones in SF networks. The fact that the amount of nodes with low connectivity does not change much with variations in λ_α for $2.5 < \lambda_\alpha \leq 3$, and also because the diffusion between networks depends directly on the externally connected nodes whether a particle crosses to another network or remains in

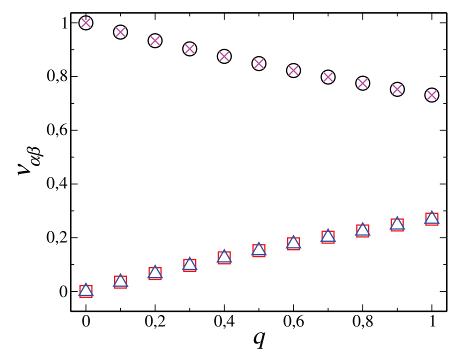

Fig. 4: (Colour on-line) Rates $\nu_{\alpha\beta}$ at which a particle spreads from the network α to the network β as a function of q. With ν_{AA} ($\bigcirc$), ν_{AB} ($\square$), ν_{BB} ($\times$) and ν_{BA} ($\triangle$). From the definition it is clear that $\nu_{AA} + \nu_{AB} = 1$ and $\nu_{BA} + \nu_{BB} = 1$.

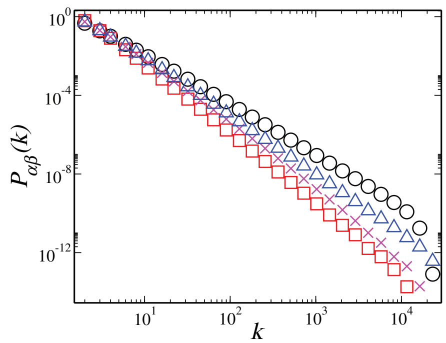

Fig. 5: (Colour on-line) Degree distributions $P_{\alpha\beta}(k)$ of the particles which get deposited on a node with connectivity k on the network β after being dropped on the network α for $q = 0.5$. With P_{AA} (○), P_{AB} (□), P_{BA} (×) and P_{BB} (△).

Table 1: Dispersion $\sigma_{\alpha\beta}$ of the distributions $P_{\alpha\beta}$ for different values of q.

q	$\lambda_A = 2.6$ $\lambda_B = 3.0$			
	σ_{AA}	σ_{BB}	σ_{AB}	σ_{BA}
0	22.1	7.5	–	–
0.1	22.4	7.7	2.2	3.3
0.5	23.9	8.2	2.7	4.5
1.0	25.7	8.6	3.0	5.7

the same, make that the rates have almost no dependence on the exponents of the original degree distributions.

How the load gets distributed after the diffusion process? In order to answer this question we compute the probability $P_{\alpha\beta}(k)$, defined as the probability that a particle dropped in the network α gets deposited in a node with degree k in the network β. In fig. 5 we plot $P_{\alpha\beta}(k)$ with $\alpha, \beta = A, B$ for $q = 0.5$. Also in table 1 we report the dispersion $\sigma_{\alpha\beta}$ of these distributions, which quantifies the heterogeneity of the deposition process, for different values of q and for the isolated networks. From the plot we can see that the probabilities $P_{\alpha\alpha}(k)$ for $\alpha = A, B$ are very heterogeneous and have a similar dispersion to the dispersion in the isolated networks. This means that when a particle stays on the network where it was originally dropped finds an environment with a similar heterogeneity than in the isolate SF network. Moreover the distributions $P_{\alpha\beta}(k)$ for $\alpha \neq \beta$ are more homogeneous than in the case $\alpha = \beta$ and thus have a lower dispersion. This means that when a particle crosses from one network to another it finds a more homogeneous neighborhood than in the case in which stays in the same network, and as we will show below low-degree nodes are filled more often than high-degree nodes. Also we can see that the dispersion, for all cases, slightly grows with q, and this is due to the fact that hubs are less overloaded and can participate more often in the diffusion process.

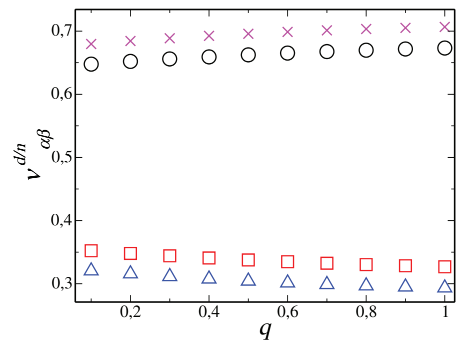

Fig. 6: (Colour on-line) Rates $\nu^d_{\alpha,\beta}$ and $\nu^n_{\alpha,\beta}$ at which a particle spreads from α to β network and gets directly deposited or deposited on a neighboring node, respectively. With ν^d_{AB} (○), ν^n_{AB} (□), ν^d_{BA} (×) and ν^n_{BA} (△).

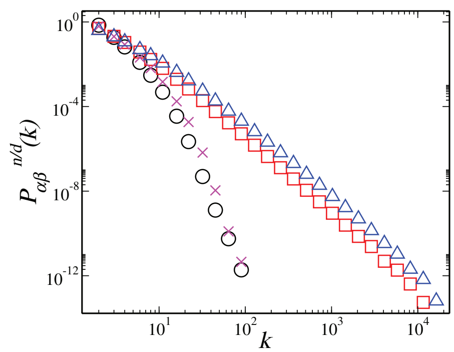

Fig. 7: (Colour on-line) Probabilities $P^d_{\alpha\beta}(k)$ and $P^n_{\alpha\beta}(k)$, that after an external diffusion, the particle gets deposited directly or gets deposited on a neighboring node with degree k, respectively, for $q = 0.5$. With P^d_{AB} (○), P^n_{AB} (□), P^d_{BA} (×) and P^n_{BA} (△).

We want to understand the reason that makes the load get rather deposited on low-connectivity nodes when it crosses to a different network. In order to explain this effect we study the diffusion process when the load crosses to the other network. When a particle spreads from one network to another, it can be directly deposited on a node connected by the external connection or it can be deposited in one of its neighbors. To understand which of these two scenarios is more probable, we compute $\nu^d_{\alpha,\beta}$ and $\nu^n_{\alpha,\beta}$, which are the rates at which a particle that spreads from one network to another gets deposited directly (d) or in a neighboring (n) node, respectively. Notice that $\nu^d_{\alpha,\beta} + \nu^n_{\alpha,\beta} = 1$. In fig. 6 we plot these rates as a function of q and we can see that $\nu^d_{\alpha,\beta}$ is more important than $\nu^n_{\alpha,\beta}$, which means that for any q, most of the times the particles that cross from one network to another get directly deposited.

In order to explain the last observation, we define $P^d_{\alpha\beta}(k)$ and $P^n_{\alpha\beta}(k)$ as the probabilities that, after an external diffusion, the particle gets directly deposited or gets deposited on a neighboring node with degree

k, respectively. In fig. 7 we plot these probabilities for $q = 0.5$. We can see that the probabilities $P^d_{\alpha\beta}(k)$ are more homogeneous than $P^n_{\alpha\beta}(k)$ and that nodes with low degree are the ones which receive the majority of the particles that are directly deposited. We can also see that the probabilities $P^n_{\alpha\beta}(k)$, which contemplate the scenario of neighboring deposition, have a wider spectrum, which agrees with the fact that in SF networks there are a few nodes with high degrees that receive the load by diffusion from their neighbors. This mechanism reduces the fluctuations, due to a matching of the heights of low-degree and high-degree nodes that is more efficient as q increases. Finally the system is optimally synchronized for $q = 1$.

Conclusions. – We study the synchronization in two SF networks where the dynamic of growth is ruled by a modified SRM model. We study the fluctuations in the steady state of each network as a function of the interacting parameter q and we find that the synchronization of each network improves when the interconnection between them increases. This improvement in both networks is about a 40% better than in isolated networks when $q = 1$, which is an important value regarding the decrease of the fluctuations. However, we show that the rate of this improvement decrease with the number of interconnections and this can be an useful result in future research to determine if a larger interconnection between networks apart from $q = 1$, is worth it.

On the other hand, the improvement in the synchronization is due mainly to the diffusion through external connections. We also found that the majority of the particles that travel through the external connections are directly deposited in nodes with low connectivity, which are the majority in SF networks. We observe that in average these nodes usually have a the lowest heights in a SF network. Then when q increases, the height of the nodes with low connectivity increases compared to the mean value and the difference with the height of the hubs decreases. We also found various distinctive characteristics of the model, such as the fact that the synchronization and the heights of the hubs become almost independent of the internal degree distribution as q increase, or the fact that the percentage of particles that diffuse between networks only depends on q and not on the internal degree distribution of the networks. This result indicates that for high values of q the behavior of the system is ruled by the interconnection between networks and not by the topology of the system.

* * *

MADM, CELR and LAB want to thank UNMdP, FONCyT, Pict 0429/2013, CONICET, PIP 00443/2014 for financial support. MFT acknowledges CONICET, PIP 00629/2014 for financial support.

REFERENCES

[1] Netoff T. I., Clewley R., Arno S., Keck T. and White J. A., *J. Neurosci.*, **24** (2004) 8075.

[2] Grinstein G. and Linsker R., *Proc. Natl. Acad. Sci. U.S.A.*, **102** (2005) 9948.

[3] Izhikevich E., *SIAM Rev.*, **43** (2001) 315.

[4] Wang Q., Duan Z., Perc M. and Chen G., *EPL*, **83** (2008) 50008.

[5] Wang Q., Perc M., Duan Z. and Chen G., *Phys. Rev. E*, **80** (2009) 026206.

[6] Lusseau D., Wilson B., Hammond P. S., Grellier K., Durban J. W., Parsons K. M., Barton T. R. and Thompson P. M., *J. Anim. Ecol.*, **75** (2006) 14.

[7] Rabani Y., Sinclair A. and Wanka R., *Local Divergence of Markov Chains and the Analysis of Iterative Load-Balancing Schemes*, in *Proceedings of the 39th Annual Symposium on Foundations of Computer Science* (IEEE Computer Society, Washington, DC) 1998, pp. 694–702.

[8] Pastore y Piontti A. L., Macri P. A. and Braunstein L. A., *Phys. Rev. E*, **76** (2007) 046117.

[9] Korniss G., *Phys. Rev. E*, **75** (2007) 051121.

[10] La Rocca C. E., Braunstein L. A. and Macri P. A., *Phys. Rev. E*, **77** (2008) 046120.

[11] La Rocca C. E., Pastore y Piontti A. L. Braunstein L. A. and Macri P. A., *Physica A*, **388** (2009) 233.

[12] La Rocca C. E., Braunstein L. A. and Macri P. A., *Physica A*, **390** (2011) 2840.

[13] Guclu H., Korniss G. and Toroczkai Z., *Chaos*, **17** (2007) 026104.

[14] Pastore y Piontti A. L., La Rocca C. E., Toroczkai Z., Braunstein L. A., Macri P. A. and López E. D., *New J. Phys.*, **10** (2008) 093007.

[15] Toroczkai Z., Kozma B., Bassler K. E., Hengartner N. W. and Korniss G., *J. Phys. A: Math. Theor.*, **41** (2008) 155103.

[16] Toroczkai Z. and Bassler K. E., *Nature*, **428** (2004) 716.

[17] Torres D., Di Muro M. A., La Rocca C. E. and Braunstein L. A., *EPL*, **110** (2015) 66001.

[18] Buldyrev S. V., Parshani R., Paul G., Stanley H. E. and Havlin S., *Nature*, **464** (2010) 1025.

[19] D'Agostino G. and Scala A. (Editors), *Networks of Networks: The Last Frontier of Complexity* (Springer, Rome) 2014.

[20] Gao J., Buldyrev S. V., Havlin S. and Stanley H. E., *Phys. Rev. Lett.*, **107** (2011) 195701.

[21] Gao J., Buldyrev S. V., Stanley H. E. and Havlin S., *Nat. Phys.*, **8** (2012) 40.

[22] Dong G., Gao J., Du R., Tian L., Stanley H. E. and Havlin S., *Phys. Rev. E*, **87** (2013) 052804.

[23] Valdez L. D., Macri P. A. and Braunstein L. A., *Phys. Rev. E*, **88** (2013) 050803(R).

[24] Baxter G. J., Dorogovtsev S. N., Goltsev A. V. and Mendes J. F. F., *Phys. Rev. Lett.*, **109** (2012) 248701.

[25] Brummitt C. D., D'Souza R. M. and Leicht E. A., *Proc. Natl. Acad. Sci. U.S.A.*, **109** (2012) 680.

[26] Brummitt C. D., Lee K.-M. and Goh K.-I., *Phys. Rev. E*, **85** (2012) 045102(R).

[27] Gómez S., Díaz-Guilera A., Gómez-Gardeñes J., Pérez-Vicente C. J., Moreno Y. and Arenas A., *Phys. Rev. Lett.*, **110** (2013) 028701.

[28] Lee K.-M., Kim Jung Yeol, Cho W. K., Goh K.-I. and Kim I.-M., *New J. Phys.*, **14** (2012) 033027.

[29] Kim J. Y. and Goh K.-I., *Phys. Rev. Lett.*, **111** (2013) 058702.

[30] Kaluza P., Kólzsch A., Gastner M. T. and Blasius B., *J. R. Soc. Interface*, **7** (2010) 1093.

[31] Cozzo E., Arenas A. and Moreno Y., *Phys. Rev. E*, **86** (2012) 03811.

[32] Cardillo A., Gómez-Gardeñes J., Zanin M., Romance M., Papo D., Del Pozo F. and Boccaletti S., *Sci. Rep.*, **3** (2013) 1344.

[33] Son S. W., Bizhani G., Christensen C., Grassberger P. and Paczuski M., *EPL*, **97** (2012) 16006.

[34] Boccaletti S., Bianconi G., Criado R., del Genio C. I., Gómez-Gardees J., Romance M., Sendia-Nadal I., Wang Z. and Zanin M., *Phys. Rep.*, **544** (2014) 1.

[35] Dickison M., Havlin S. and Stanley H. E., *Phys. Rev. E*, **85** (2012) 066109.

[36] Yagan O., Qian D., Zhang J. and Cochran D., *IEEE J. Sel. Areas Commun.*, **31** (2013) 1038.

[37] Buono C., Alvarez-Zuzek L. G., Macri P. A. and Braunstein L. A., *PLOS ONE*, **9** (2014) e92200.

[38] Sanz J., Xia C., Meloni S. and Moreno Y., *Phys. Rev. X.*, **4** (2014) 041005.

[39] Buono C. and Braunstein L. A., *EPL*, **109** (2015) 26001.

[40] Sahneh F. D. and Scoglio C., *Phys. Rev. E*, **89** (2014) 062817.

[41] Molloy M. and Reed B., *Random Struct. Algorithms*, **6** (1995) 161; *Combin. Probab. Comput.*, **7** (1998) 295.

[42] Cohen R., Havlin S. and ben-Avraham D., *Handbook of Graphs and Networks*, edited by Bornholdt S. and Shuster H. G. (Willey-VCH, New York) 2002, sect. 4.