Percolation analysis on scale-free networks with correlated link weights

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Z. Wu, L. A. Braunstein, V. Colizza, R. Cohen, S. Havlin, H. E. Stanley, *Phys. Rev. E* **74**, 056104 (2006)

General question

Do link weights affect the network properties?

Outline

1. Motivation
2. Modeling approach
   - q_c: definition and simulations in the model
   - p_c: percolation threshold (define λ_c and T_c)
3. Numerical results
4. Summary

Part 1. Motivation:

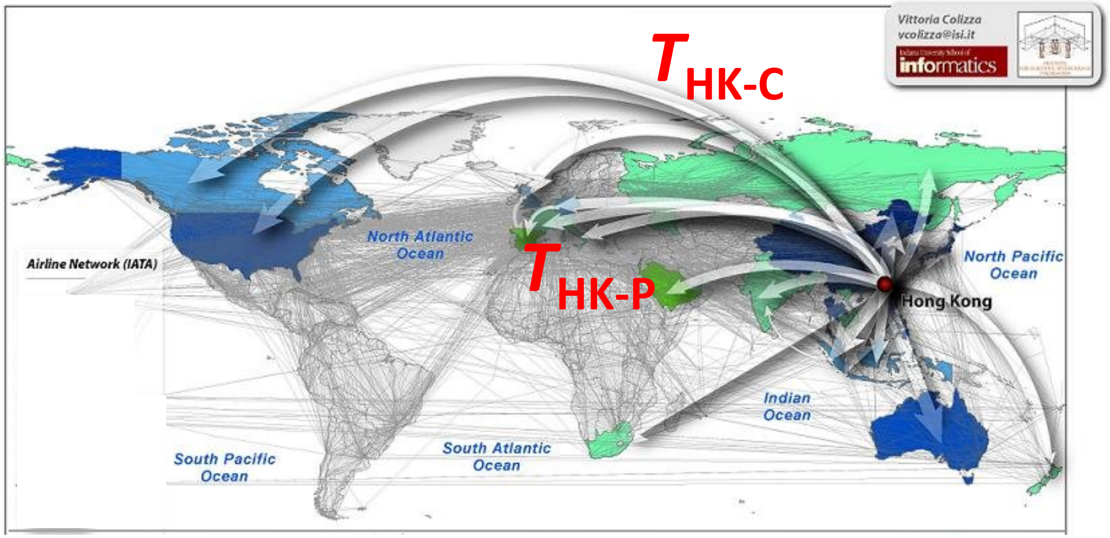
Properties of real-world networks

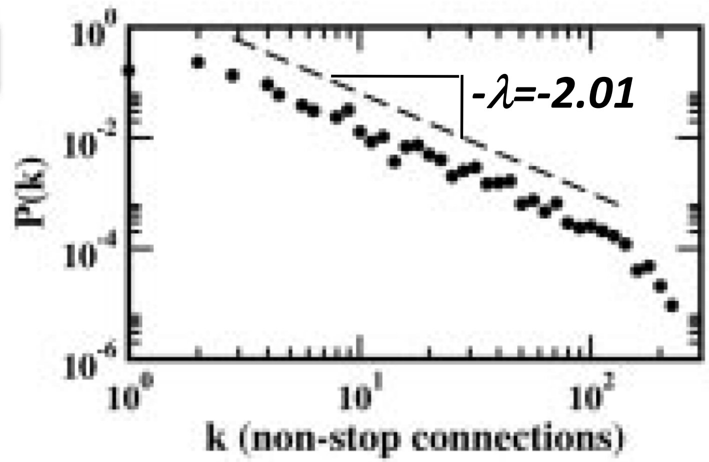
Example: world-wide airport network (WAN)

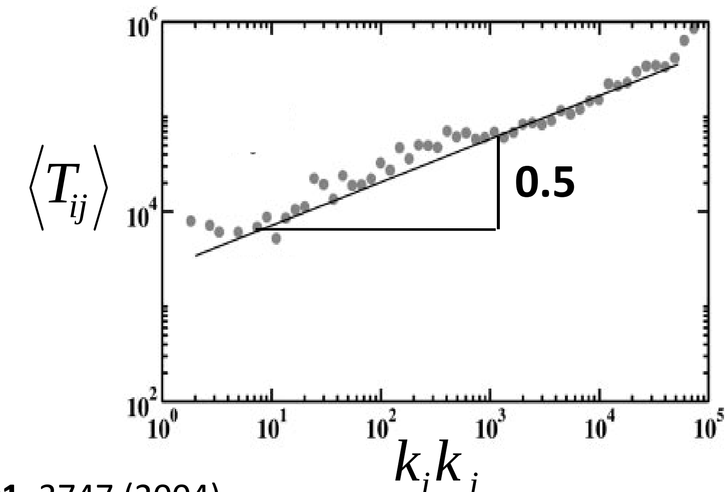
Large cities (hubs) have many routs k (degree)

Link weight T_{ij} is "# of passengers"

Link weight T_{ij} depends on degree k_i and k_j of airports *i and j*







Real-world networks:

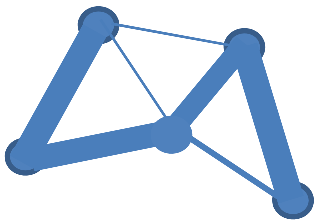
1. Heterogeneous connectivity
2. Heterogeneous weights
3. Correlation between connectivity and weights

A. Barrat, M. Barthélemy, R. Pastor-Satorras, and A. Vespignani, *PNAS*, **101**, 3747 (2004).

Part 1. Motivation:

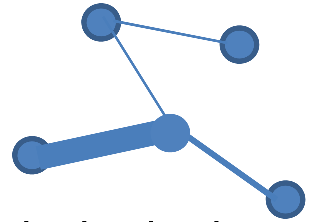
What part of network is more important for traffic?

Thickness of links: Traffic, ex: number of passengers

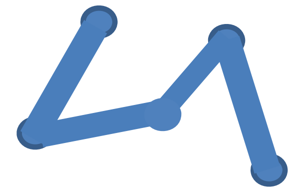


How to choose the most import links?

Bad choice



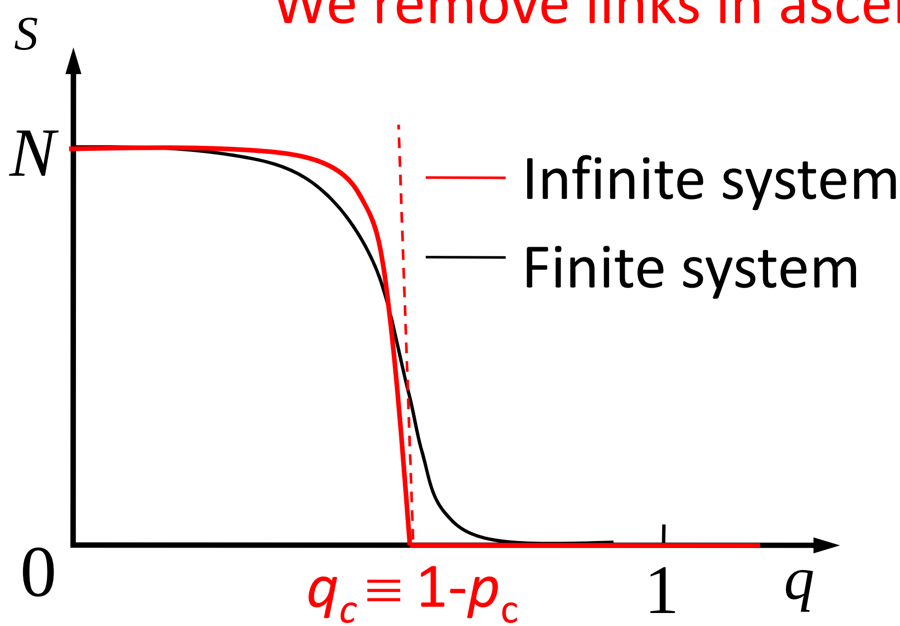
Good choice



Choose the links with the highest traffic

Introduce rank-ordered percolation

We remove links in ascending order of weight, *T*



S: number of nodes in the largest connected cluster

q: fraction of links removed with lowest weights

q_c: critical *q* to break network

N: number of nodes

What is effect of weights & correlation on percolation?

Why is it important?

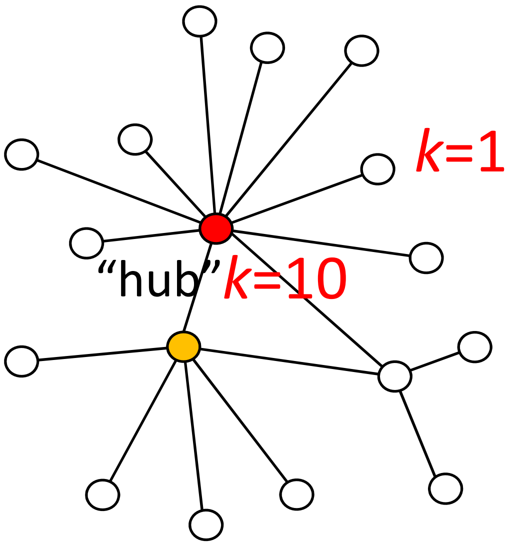
- World-wide airport network is closely related to epidemic spreading such as the case of SARS[1].
- Help to develop more effective immunization strategies.
- Biological networks such as the *E. coli* metabolic networks also has the same correlation between weights and nodes degree.[2]

[1] V. Colizza *et al*., *BMC Medicine* **5**, 34 (2007)
[2] P. J. Macdonald et al., *Europhys. Lett.* **72,** 308 (2005)

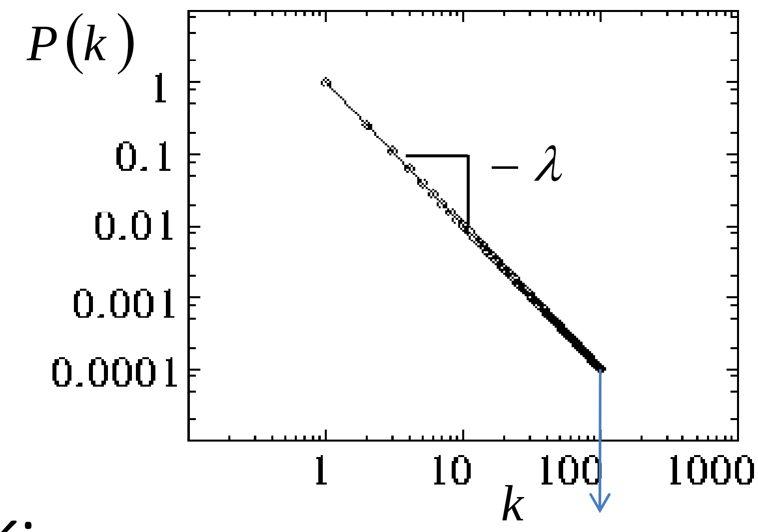
Part 2. Model:

Weighted scale-free networks

Scale-free (SF)



Power-law distribution



k_{max}: maximum degree

Define the weight on each link:

$$T_{ij} \equiv x_{ij} \left(k_i k_j \right)^{\theta}$$

θ: controls correlation

Definitions:

- x_{ij}: Uniform distributed random numbers $0 < x_{ij} < 1$.
- k_i: Degree of node *i*.
- T_{ij}: Weight, ex, traffic, number of passengers in WAN.
  For WAN, $\theta = 0.5$

Part 2. Model:

Effect of θ

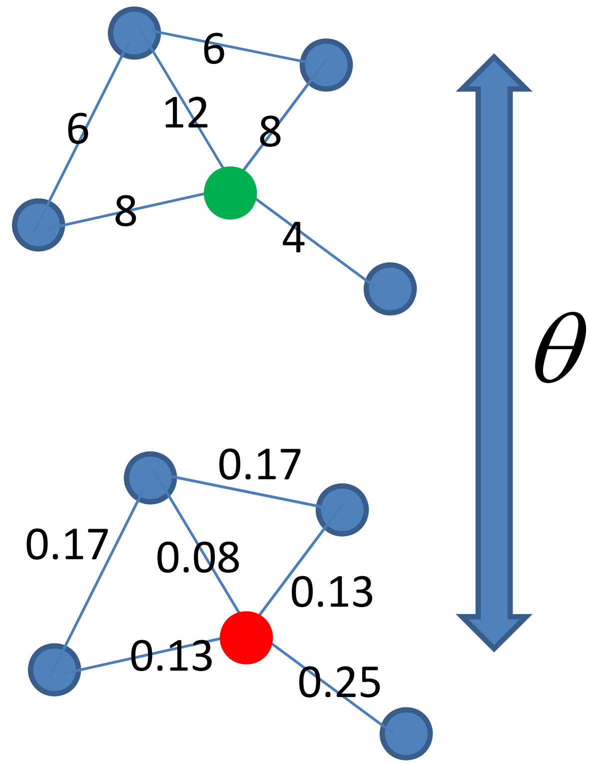
$$T_{ij} \equiv x_{ij}\left(k_i k_j\right)^{\theta}$$

$$\theta \begin{cases} > 0 & \text{hub-philic} \\ = 0 & \text{uncorrelated} \\ < 0 & \text{hub-phobic} \end{cases}$$

For $x_{ij} = 1$:

Ex: $\theta = 1$

Ex: $\theta = -1$



The sign of θ determines the nature of the hubs

Part 2. Model:

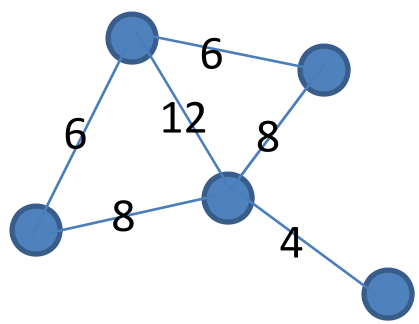
Percolation properties only depend on the sign of θ

$$T_{ij} \equiv x_{ij}\left(k_i k_j\right)^{\theta} \qquad 0 < x_{ij} < 1 \qquad -\infty < \theta < \infty$$

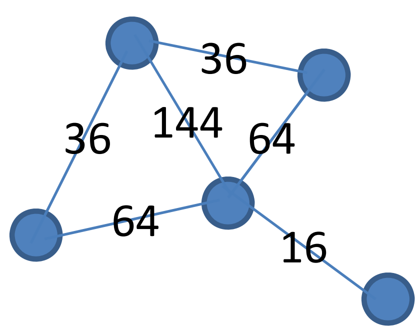
In studies of percolation properties,
what matters is the rank of the links according their weight.

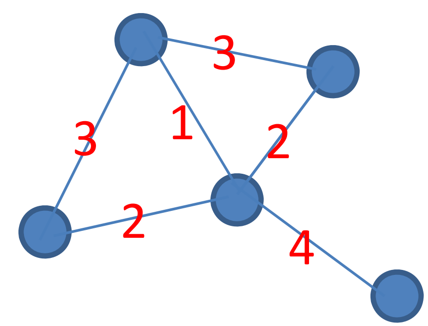
For $x_{ij} = 1$:

$\theta = 1$



$\theta = 2$





Ranking of the links remains unchanged for different θ of the same sign.

Part 2.

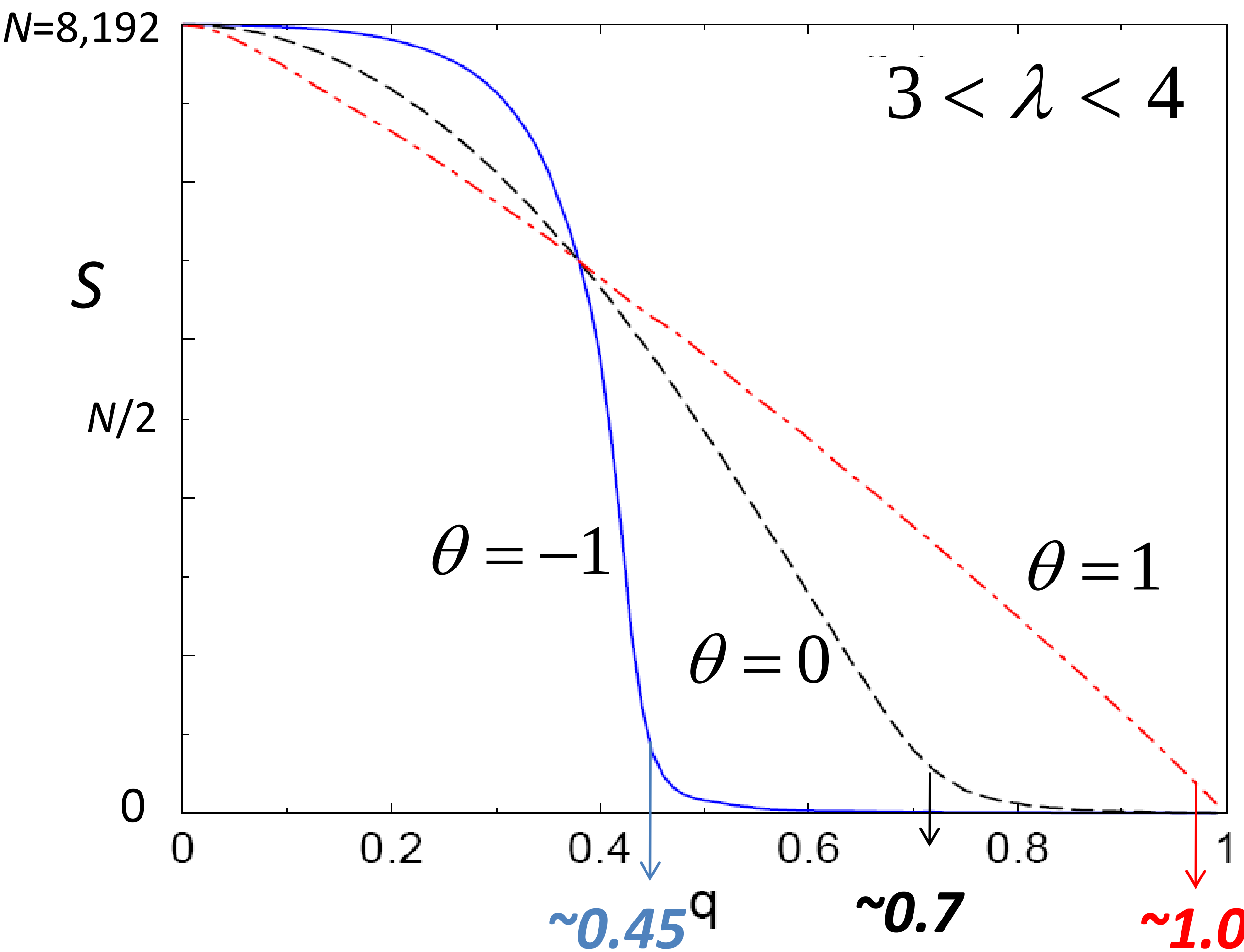
Specific questions

1. Will the θ change the critical fraction q_c of a network?

2. Will the θ change the universality class of a network?

Part 2. q_c: simulations on the model (number of nodes N = 8,192)

Comparison of q_c for different θ



q_c: critical q to break network

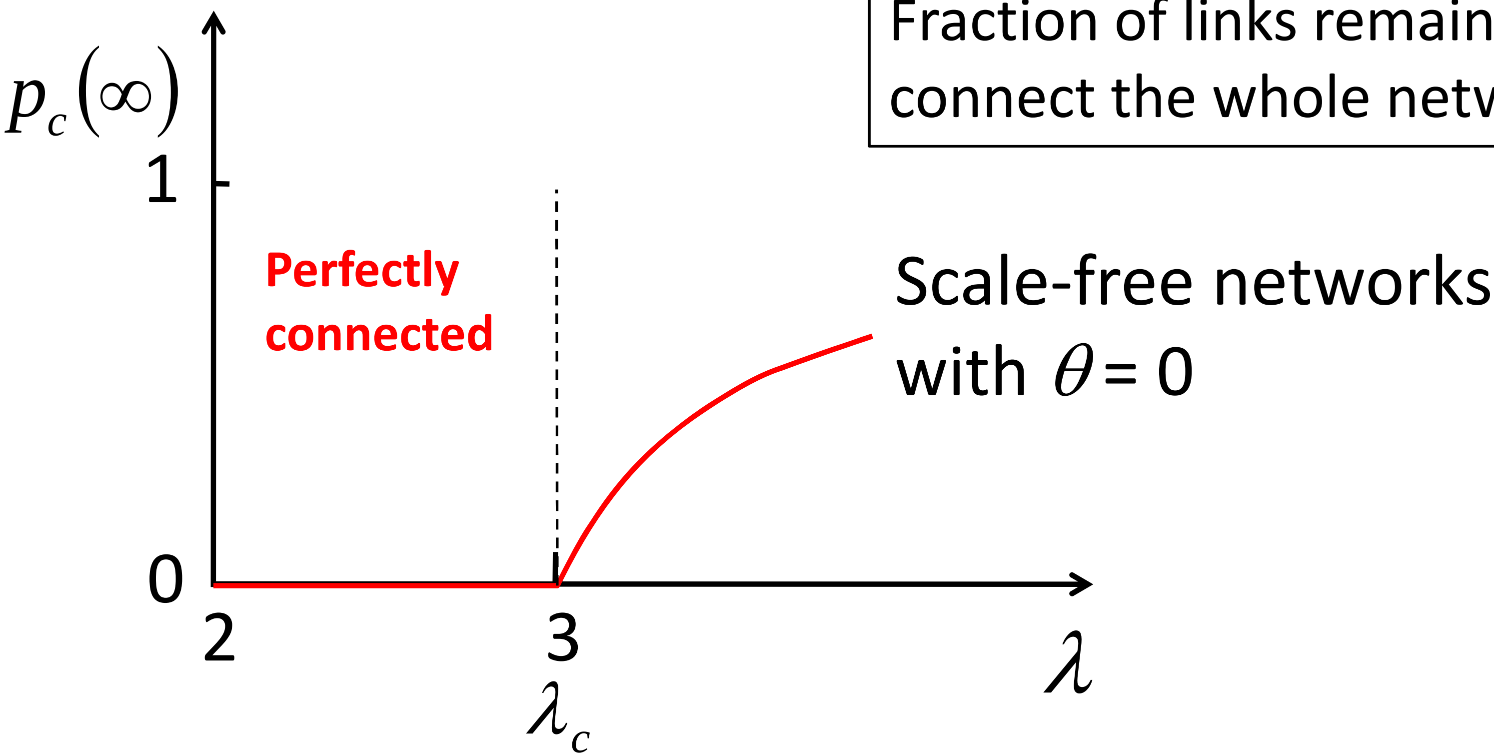
q: fraction of links removed with lowest weights

Scale-free networks with $\theta > 0$ have larger q_c than networks with $\theta \leq 0$.

Critical degree distribution exponent, λ_c

λ_c is the λ below which p_c is zero and above which p_c is finite

$p_c \equiv 1 - q_c$
Fraction of links remained to connect the whole network

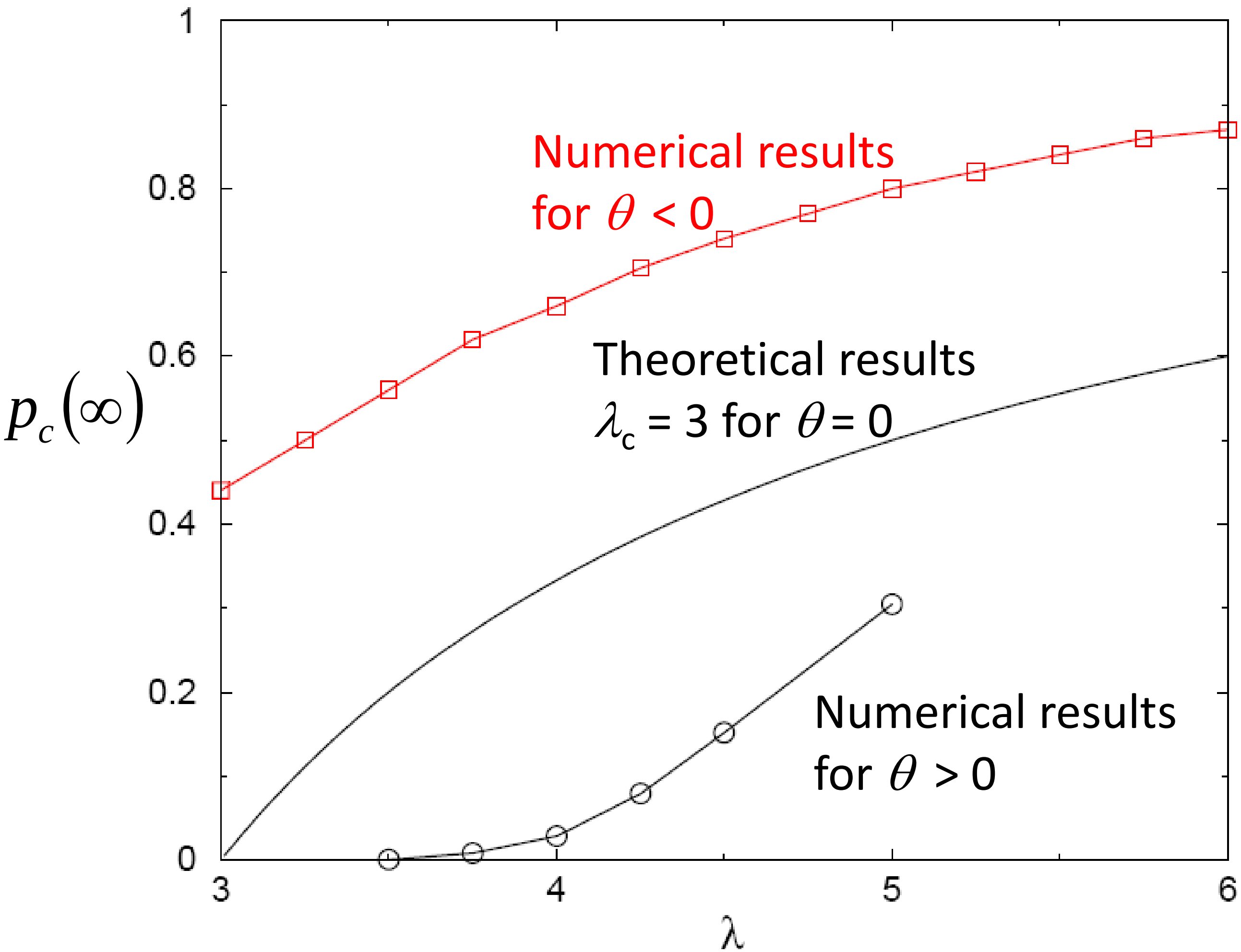


$\lambda_c = 3$ for scale-free networks with $\theta = 0$

R. Cohen, K. Erez, D. ben-Avraham, S. Havlin, *Phys. Rev. Lett.* **85**, 4626 (2000)

Part 2. λ_c: question about λ_c for $\theta > 0$

What is the λ_c for $\alpha < 0$?



$$\lambda_c \begin{cases} = 3 & ,\theta = 0 \\ = ? & ,\theta > 0 \end{cases}$$

If $\lambda_c(\theta > 0) \neq \lambda_c(\theta = 0)$ ⇒ different universality classes!

How to find out $p_c = 0$ for $\theta > 0$?

Difficulties:

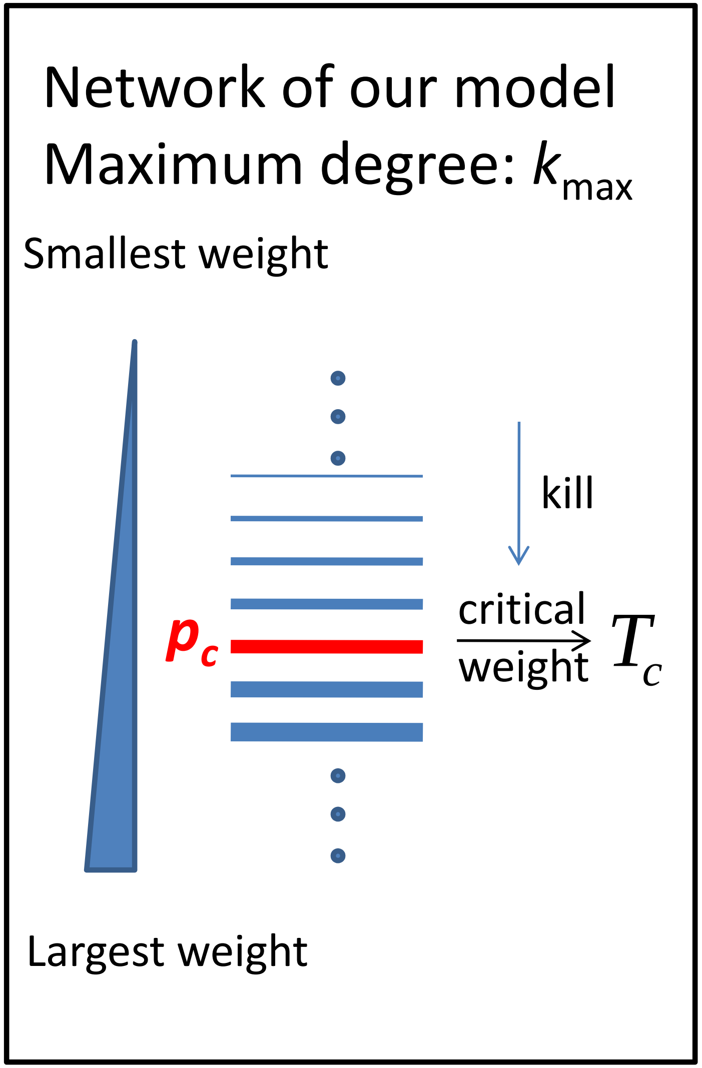
- Limit of numerical precision $\rightarrow$ hard to determine $p_c = 0$ numerically.

- Correlation $\rightarrow$ hard to find analytical solution for p_c.

Solution:

Analytical approach with numerical solution

T_c, the critical weight at p_c



T_c : the critical weight at which the system is at p_c

T_c diverge

⇓

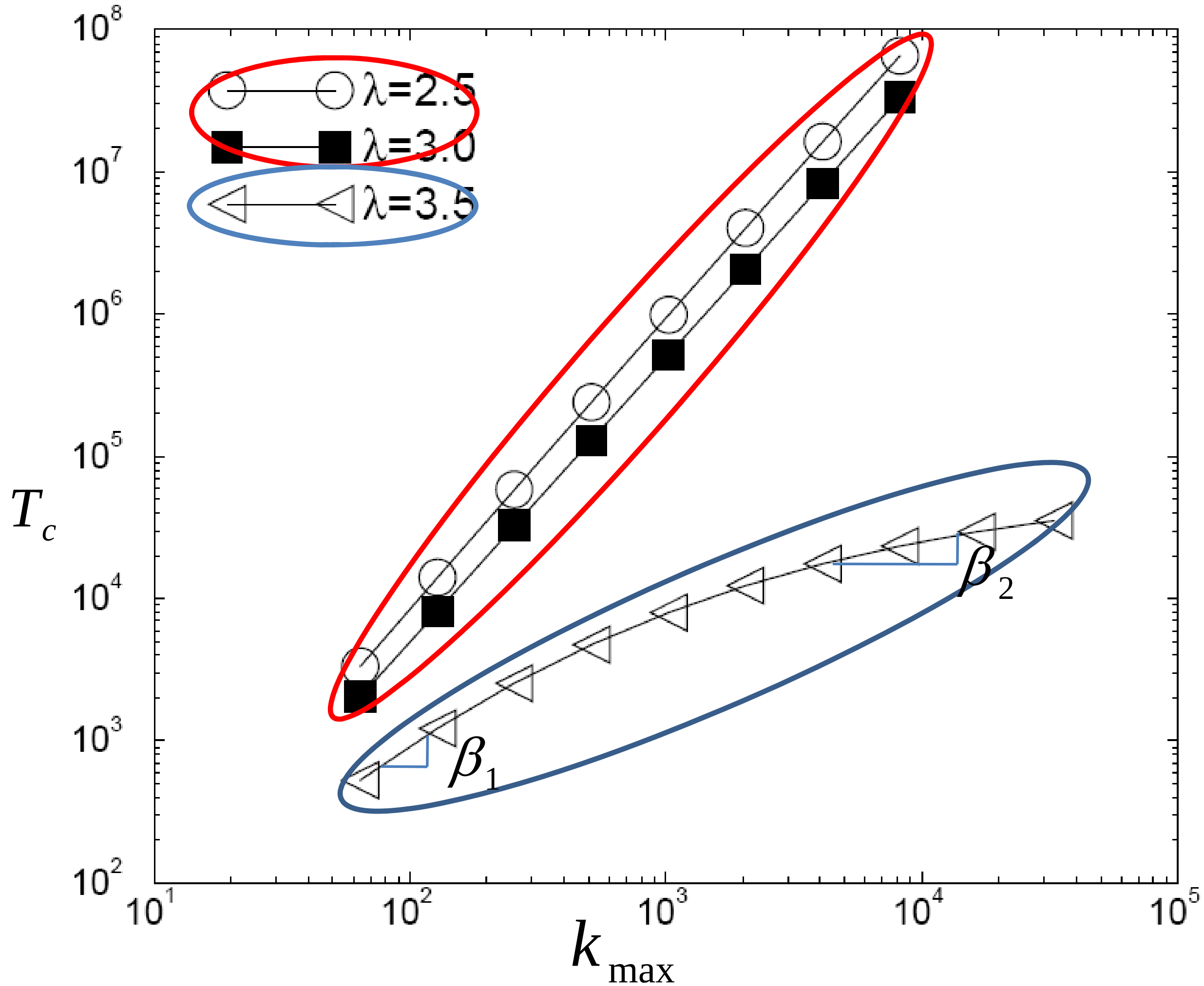
Kill all the links

⇓

$$p_c(\infty) = 0$$

The divergence of T_c tells us whether $p_c = 0$

Numerical results



T_c diverge, $\lambda \leq 3$

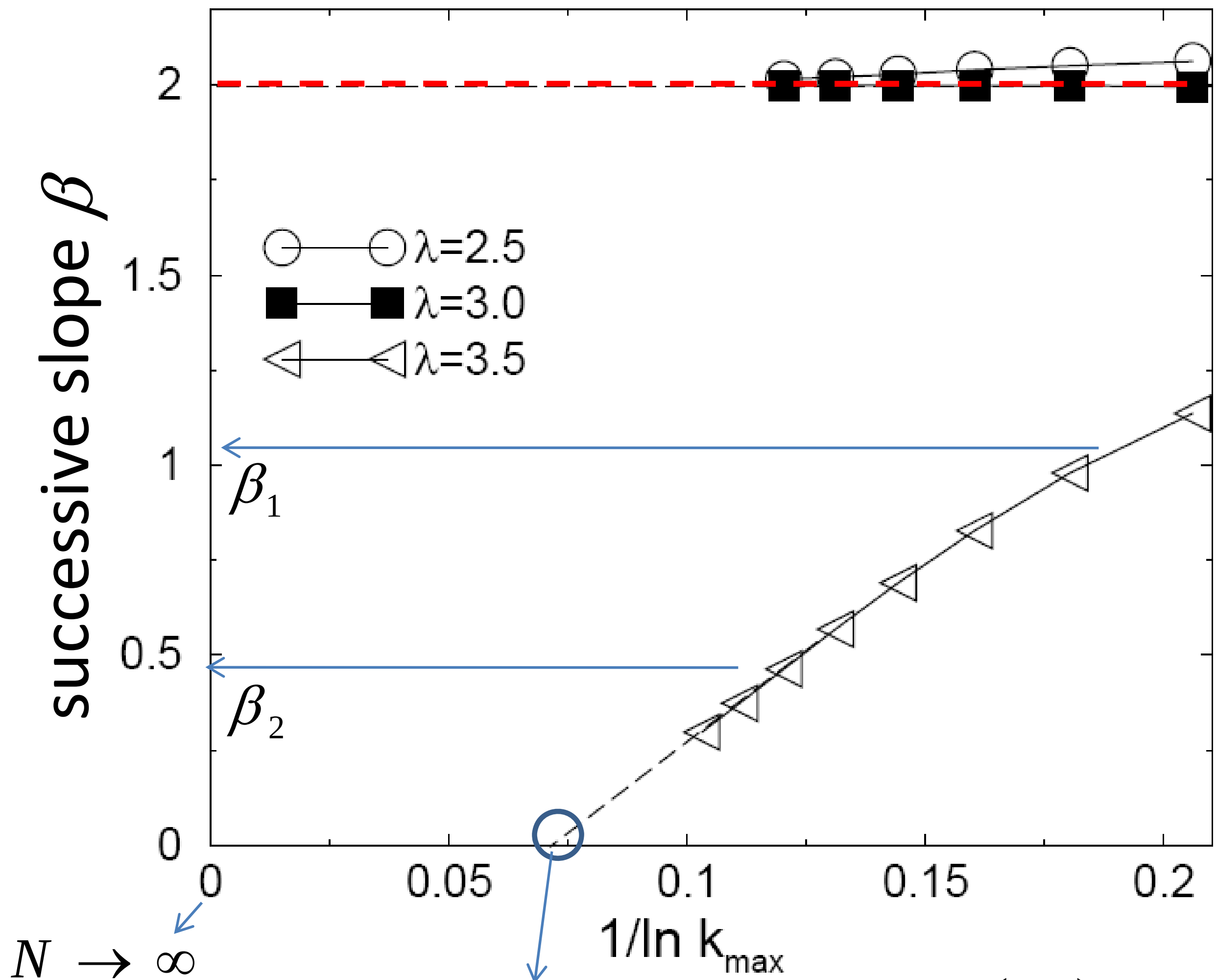
$\Rightarrow p_c(\infty) = 0$

T_c converge, $\lambda > 3$

$\Rightarrow p_c(\infty) > 0$

Result: indicates $\lambda_c = 3$ for $\theta > 0$

Strong finite size effect



$\beta \to$ finite value > 0
$\Rightarrow T_c$ diverge

$\beta \to 0$
$\Rightarrow T_c$ converge

When $N > 3\times 10^{14}$, $p_c(N) \cong p_c(\infty)$

In our simulation, we can only reach $10^6 << 10^{14}$

Part 4. Summary

- For the first time, we proposed and analyzed a model that takes into account the correlation between weights and node degrees.
- The correlation between weight T_{ij} and nodes degree k_i and k_j, which is quantified by θ, changes the properties of networks.
- Scale-free networks with $\theta > 0$, such as the WAN, have larger q_c than scale-free networks with $\theta \leq 0$.
- Scale-free networks with $\theta > 0$ and $\theta = 0$ belong to the same university class (have the same $\lambda_c = 3$)

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